Metric packing for $K_3 + K_3$

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Abstract

In this paper, we consider the metric packing problem for the commodity graph of disjoint two triangles $K_3 + K_3$, which is dual to the multiflow feasibility problem for the commodity graph $K_3 + K_3$. We prove a strengthening of Karzanov's conjecture concerning quarter-integral packings by certain bipartite metrics.

1 Introduction and main result

A *metric* μ on a finite set V is a function $V \times V \to \mathbf{R}$ satisfying $\mu(i,i) = 0$, $\mu(i,j) = \mu(j,i) \geq 0$, and the triangle inequalities $\mu(i,j) + \mu(j,k) \geq \mu(i,k)$ for $i,j,k \in V$. Throughout this paper, a graph means an undirected graph. Let $G = (V,E)$ be a graph. For a nonnegative edge length function $l : E \to \mathbf{R}_+$, let $d_{G,l}$ denote the graph metric on V induced by (G,l), i.e., $d_{G,l}(i,j)$ is the shortest path length between i and j in G with respect to edge length l. Let d_G denote the graph metric on V by G with unit edge length.

Let $H = (S,R)$ be another simple graph on $S \subseteq V$, called the *commodity graph*. A finite set of metrics $\mathcal{M}$ on V together with its nonnegative weight $\lambda : \mathcal{M} \to \mathbf{R}_+$ is called a *fractional H-packing* for (G,l) if it satisfies

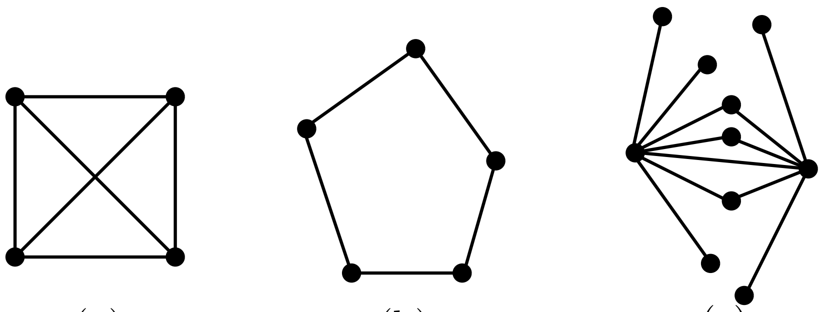
$$\begin{aligned} l(ij) \quad &\geq \quad \sum_{\mu \in \mathcal{M}} \lambda(\mu)\mu(i,j) \quad (ij \in E), \\ d_{G,l}(s,t) \quad &= \quad \sum_{\mu \in \mathcal{M}} \lambda(\mu)\mu(s,t) \quad (st \in R). \end{aligned} \tag{1.1}$$

If λ is integral, then it is called an *integral H-packing* for (G,l).

A classical theorem in the network flow theory says that if H consists of a single edge and l is integral, there is an integral H-packing by *cut metrics*. Here a metric d is called a cut metric if there is a set $X \subseteq V$ such that $d(i,j) = 1$ if $\#(X \cap \{i,j\}) = 1$ and $d(i,j) = 0$ otherwise. This is a *polar* theorem to Ford-Fulkerson's max-flow min-cut theorem [10]. As is well-known, fractional H-packing problems are polar to the multiflow feasibility problems with commodity graph H; see [23, Chapter 70]. The *multiflow feasibility problem* is: given a capacity $c : E \to \mathbf{R}_+$ and a demand $q : R \to \mathbf{R}_+$, find flows f_{st} ($st \in R$) from s to t of value $q(st)$ such that for each $e \in E$ the total flow through e does not exceed $c(e)$, or establish that no such a flow exists.

For a finite set of metrics $\mathcal{M}$ on V, an obvious necessary condition for the multiflow feasibility

$$\sum_{ij \in E} c(ij)\mu(i,j) \geq \sum_{st \in R} q(st)\mu(s,t) \quad (\mu \in \mathcal{M}) \tag{1.2}$$



(a) (b) (c)

Figure 1: (a) K_4, (b) C_5, and (c) the union of two stars

is also sufficient if and only if for any nonnegative length function $l : E \to \mathbf{R}_+$ there exists a fractional H-packing for (G, l) by $\mathcal{M}$. This is a simple consequence of the linear programming duality.

Papernov [21] has characterized the class of commodity graphs with property that the *cut condition*, i.e., (1.2) by taking $\mathcal{M}$ as cut metrics, is sufficient for the multiflow feasibility. He has shown that if H is K_4, C_5, or the union of two stars, then the cut condition is sufficient, where K_n is the complete graph on n vertices, C_m is a cycle on m vertices, and a *star* is a graph all of whose edge have a common vertex; see Figure 1. By polarity, there exists a fractional H-packing by cut metrics in this case.

Karzanov [13] has strengthened this result to a half-integral version. Here the length function l on G is said to be *cyclically even* if l is integral and $\sum_{e \in C} l(e)$ is even for any cycle C in G.

Theorem 1.1 ([13]). *Let G be a graph with cyclically even edge length l and H a commodity graph. If H is K_4, C_5, or the union of two stars, then there exists an integral H-packing for (G, l) by cut metrics.*

If H violates the condition of Theorem 1.1, the cut condition is not sufficient for the multiflow feasibility, and therefore an H-packing by cut metrics does not exist in general. Karzanov [14] has studied the multiflow feasibility problem for a five-vertex commodity graph, and shown that the $K_{2,3}$-*metric condition* is sufficient. Here, for a graph Γ on X, a metric μ on V is called a Γ-*metric* if there is a map $\phi : V \to X$ such that $\mu(i, j) = d_\Gamma(\phi(i), \phi(j))$ for $i, j \in V$. $K_{n,m}$ denotes the complete bipartite graph with parts of n and m vertices. In particular. a cut metric is nothing but a K_2-metric. The Γ-*metric condition* is (1.2) by taking $\mathcal{M}$ as the set of Γ-metrics. By this result, there is a fractional H-packing by cut metrics and $K_{2,3}$-metrics for a five-vertex commodity graph H. Again Karzanov [16] has strengthened it to:

Theorem 1.2 ([16]). *Let G be a graph with cyclically even edge length l, and H a commodity graph. If H has at most five vertices, or is the union of K_3 and a star, then there exists an integral H-packing for (G, l) by cut metrics and $K_{2,3}$-metrics.*

It is natural to ask: what is the class of commodity graphs H with the property that there exists a *finite* set $\mathcal{G}$ of graphs admitting an H-packing for any graph (G, l) by Γ-metrics over $\Gamma \in \mathcal{G}$? It is known that if H has a matching of three edges $K_2 + K_2 + K_2$, there is no such a finite set of graphs $\mathcal{G}$ [16, Section 3]. Therefore, one can expect such fractional or integral H-packings by finite types of metrics only for the class of commodity graphs H without $K_2 + K_2 + K_2$.

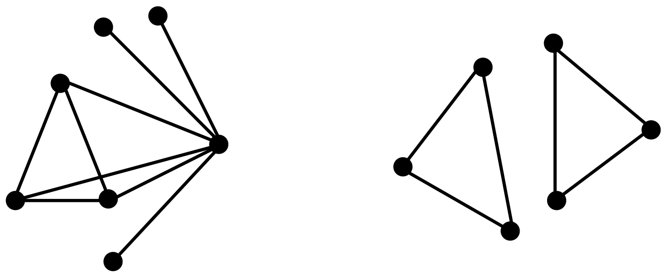


Figure 2: (d) the union of K_3 and a star, and (e) $K_3 + K_3$

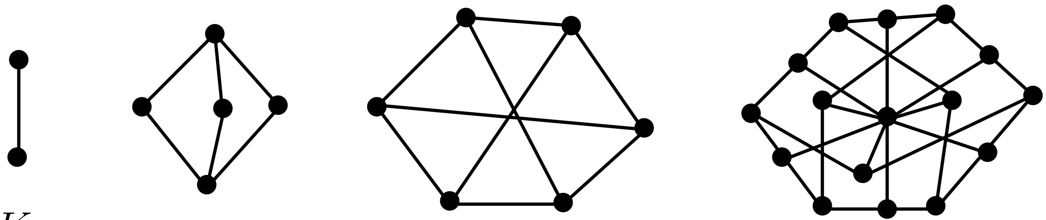


Figure 3: K_2, $K_{2,3}$, $K_{3,3}$, and $\Gamma_{3,3}$

By direct case-by-case analysis, the commodity graphs H without $K_2 + K_2 + K_2$ are classified into the following:

(1) H has at most five vertices,

(2) H is the union of two stars,

(3) H is the union of K_3 and a star, or

(4) $H = K_3 + K_3$, i.e., the sum of disjoint two triangles.

Theorems 1.1 and 1.2 above solve the first three cases (1-3). For the remaining last case (4), Karzanov [15] has shown that there exists a *fractional* H-packing by $\Gamma_{3,3}$-metrics. Here $\Gamma_{3,3}$ is the graph of 16 vertices and 27 edges obtained by subdividing each edge of $K_{3,3}$ and connecting each subdivided point to one new point; see Figure 3. In [16, Section 3], Karzanov conjectured that if $H = K_3 + K_3$ and l is cyclically even, then there exists an integral H-packing for (G, l) by $(1/2)\Gamma_{3,3}$-metrics.

Our main result solves this conjecture affirmatively in a strong form, and also completes the problem of the half or quarter integral H-packing by finite types of metrics.

Theorem 1.3. *Let G be a graph with cyclically even edge length l, and H a commodity graph. If $H = K_3 + K_3$, then there exists an integral H-packing by cut metrics, $K_{2,3}$-metrics, $K_{3,3}$-metrics, and $\Gamma_{3,3}$-metrics.*

Note that cut metrics, $K_{2,3}$-metrics, and $K_{3,3}$-metrics are submetrics of the half of $\Gamma_{3,3}$-metrics. In particular, this achieves an integral H-packing by *integral* metrics. It will turn out that a $K_{3,3}$-metric appears at most once in H-packing (1.1) and its coefficient equals 1. In a sense, a $K_{3,3}$-metric summand is a half-integral *residue* of an integral H-packing by $\varGamma_{3,3}$-metrics.

Our approach to Theorem 1.3 is based on Chepoi's striking proof [5] to Karzanov's half-integral cut and $K_{2,3}$-metric packing results above (Theorems 1.1 and 1.2). He reduced an H-packing to the problem of decomposing the *tight span* of a metric space, which has been introduced independently by Isbell [12] and Dress [9]; see also Chrobak and Larmore [7]. Since Chepoi's argument relies heavily on the classification result of tight spans of five-point metrics [9], it cannot be applied to six-vertex commodity graph $H = K_3 + K_3$. To overcome this difficulty, we introduce a notion of an *H-minimal metric*, which is essentially equivalent to a *metric whose extremal graph is H* in the sense of Karzanov [16], and show that graph H crucially affects the geometry of tight spans of H-minimal metrics. In particular, we show that for a commodity graph H without $K_2 + K_2 + K_2$, the tight span of any H-minimal metric has dimension at most 2. To obtain metric decomposition, we also develop a general decomposition theory of 2-dimensional tight spans. Our approach is independent from the classification result, and gives a geometrical interpretation to the questions why cut, $K_{2,3}$, $K_{3,3}$, and $\varGamma_{3,3}$-metrics arise, and why commodity graph H having $K_2 + K_2 + K_2$ cannot be packed by finite types of metrics.

This paper is organized as follows. In Section 2, we introduce fundamental concepts related to tight spans, and describe how an H-packing problem reduces to the problem of decomposing a tight span. In Section 3, we develop a decomposition theory for 2-dimensional tight spans. Based on this, we complete the proof of our main theorem in Section 4. In Section 5, we give several remarks including a description of an $O(n^2)$ algorithm for an integral $K_3 + K_3$-packing.

Notation. We use the following notation. Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of reals and nonnegative reals, respectively. Let $\mathbf{Z}$ and $\mathbf{Z}_+$ be the set of integers and nonnegative integers, respectively. The set of functions from a set X to $\mathbf{R}$ is denoted by $\mathbf{R}^X$. For $p, q \in \mathbf{R}^X$, the closed segment between p and q is denoted by $[p, q]$. For $p, q \in \mathbf{R}^X$, $p \leq q$ means $p(i) \leq q(i)$ for each $i \in X$. The *characteristic vector* $\chi_S \in \mathbf{R}^X$ of $S \subseteq X$ is defined as: $\chi_S(i) = 1$ for $i \in S$ and $\chi_S(i) = 0$ for $i \notin S$. We simply denote $\chi_{\{i\}}$ by χ_i, which is the i-th unit vector.

For a graph $G = (V, E)$, the edge between $i, j \in V$ is denoted by ij or ji. ii means a loop. For a graph G an subgraph G' of G is called an *isometric subgraph* if $d_G = d_{G'}$ holds on vertices of G'. A *stable set* S of G is a subset of vertices such that there is no edge both of whose endpoints belong to S. For a subset S of vertices in G, the *neighborhood* $N(S)$ of S is the set of vertices adjacent to S and not in S. G is called *bipartite* if there exists a bipartition (A, B) of vertices into two (nonempty) stable sets A, B. G is called *complete multipartite* if there is a partition of vertices into nonempty stable sets such that every pair of vertices in different parts has an edge.

We often identify a metric space (S, μ) with metric μ. We shall regard a metric as an edge length on the complete graph. A metric is called a *cyclically even* if it is cyclically even as an edge length on the complete graph. We use the standard terminology of polytope theory such as *faces*, *extreme points*, *polyhedral complex*, and so on; see [24].

2 Preliminaries

The main purpose of this section is to introduce fundamental concepts concerning tight spans, and is to describe how an H-packing problem reduces to the problem of decomposing a tight span.

Let μ be a metric on a finite set S. We define two polyhedral sets P_μ and T_μ in $\mathbf{R}^S$ as

$$\begin{aligned} P_\mu &= \{p \in \mathbf{R}^S \mid p(i) + p(j) \geq \mu(i,j) \ (i,j \in S)\}, \\ T_\mu &= \text{the set of minimal elements of } P_\mu. \end{aligned}$$

T_μ is called the *tight span* of μ [12, 9]. We immediately see the following characterization of T_μ.

Lemma 2.1 (see [9]). *For $p \in P_\mu$, the following conditions are equivalent.*

(1) *p belongs to T_μ.*

(2) *for $i \in S$, there exists $j \in S$ such that $p(i) + p(j) = \mu(i,j)$.*

(3) *p belongs to a bounded face of P_μ.*

Therefore, T_μ is the union of bounded faces of P_μ, and thus is compact. For $i \in S$, let μ_i be a vector in $\mathbf{R}^S$ defined by

$$\mu_i(j) = \mu(i,j) \quad (j \in S). \tag{2.1}$$

Namely, μ_i is the i-th column vector of the distance matrix μ.

Lemma 2.2 (see [9]). *μ_i has the following properties.*

(1) $\{\mu_i\} = T_\mu \cap \{p \in \mathbf{R}^S \mid p(i) = 0\}$ *for* $i \in S$.

(2) $\|\mu_i - \mu_j\|_\infty = \mu(i,j)$ *for* $i,j \in S$.

Proof. (1). Take $p \in T_\mu$ with $p(i) = 0$. Then we have $p(j) \geq \mu(i,j)$ for $j \in S$. For $k \in S$, by Lemma 2.1 (2), there exists $j \in S$ such that $p(k) + p(j) = \mu(k,j) \leq \mu(k,i) + \mu(i,j) \leq p(k) + p(j)$. Therefore, $p(k) = \mu(k,i)$.

(2). $\mu(i,j) = |\mu_i(i) - \mu_j(i)| \leq \|\mu_i - \mu_j\|_\infty$. Conversely, by the triangle inequality, we have $\mu(i,j) \geq |\mu(i,k) - \mu(j,k)| = |\mu_i(k) - \mu_j(k)|$ for $k \in S$. □

In particular, (S,μ) is isometrically embedded into (T_μ, l_∞) by (2). Next we introduce a lattice (a discrete subgroup) in $\mathbf{R}^S$ that behaves nicely with respect to the cyclically evenness. Let L be a lattice in $\mathbf{R}^S$ defined as

$$L = \{p \in \mathbf{R}^S \mid p(i) + p(j) \equiv 0 \pmod 2 \ (i,j \in S)\}.$$

Namely, L is the set of vectors all of whose components have the same parity. In other words, L is the union of even integer vectors and odd integer vectors.

Lemma 2.3. *If μ is cyclically even, then we have*

$$\mu_i - \mu_j \in L \quad (i,j \in S).$$

Proof. By the cyclically evenness, we have

$$\begin{aligned} (\mu_i - \mu_j)(k) + (\mu_i - \mu_j)(l) &= \mu(i,k) - \mu(j,k) + \mu(i,l) - \mu(j,l) \\ &\equiv \mu(i,k) + \mu(k,j) + \mu(j,l) + \mu(l,j) \pmod 2 \\ &\equiv 0 \pmod 2. \end{aligned}$$

□

Motivated by this fact, let A_μ be an affine lattice defined by $\mu_i + L$ for $i \in S$. By definition, it is easy to see

$$p(i) - q(i) \equiv p(j) - q(j) \equiv \|p - q\|_\infty \pmod 2, \tag{2.2}$$
$$p(i) + p(j) - \mu(i,j) \equiv 0 \pmod 2 \quad (p, q \in A_\mu, i, j \in S). \tag{2.3}$$

As was suggested in [5], the following *discrete nonexpansive retraction* plays an important role in H-packing problems. Here we give it in a more precise form than that given in [5, Section 2].

Proposition 2.4. *Suppose that μ is a cyclically even metric on S. Then there exists a map $\phi : P_\mu \cap A_\mu \to T_\mu \cap A_\mu$ such that*

(1) $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ *for $p, q \in P_\mu \cap A_\mu$, and*

(2) $\phi(p) \leq p$ *for $p \in T_\mu \cap A_\mu$, and thus ϕ is identical on $T_\mu \cap A_\mu$.*

Proof. Our proof is a slight modification of that in [9, (1.9)]. For $i \in S$, define a map $\phi_i : P_\mu \cap A_\mu \to P_\mu \cap A_\mu$ by

$$\phi_i(p)(j) = \begin{cases} p(j) & \text{if } j \neq i, \\ \max_{k \in S \setminus i} \{\mu(i,k) - p(k), 0\} & \text{if } j = i,\ p(j) \in 2\mathbf{Z}, \\ \max_{k \in S \setminus i} \{\mu(i,k) - p(k), 1\} & \text{if } j = i,\ p(j) \notin 2\mathbf{Z}, \end{cases} \qquad (j \in S).$$

Note that $p(i) - (\mu(i,k) - p(k))$ is a nonnegative even integer by $p \in P_\mu$ and (2.3). Namely, ϕ_i decreases the i-th component of p as much as possible belonging to $P_\mu \cap A_\mu$. Clearly ϕ_i satisfies (2). We show that ϕ_i satisfies (1). It suffices to show that $|\phi_i(p)(i) - \phi_i(q)(i)| \leq \|p - q\|_\infty$. Indeed, for $p \neq q$, we have

$$\begin{aligned} \phi_i(p)(i) &\leq \max_{k \in S \setminus i} \{\mu(i,k) - p(k), 1\} \\ &= \max_{k \in S \setminus i} \{\mu(i,k) - q(k) + q(k) - p(k), 1\} \\ &\leq \max_{k \in S \setminus i} \{\mu(i,k) - q(k), 1\} + \max_{k \in S \setminus i} \{q(k) - p(k), 1\} \\ &\leq \phi_i(q)(i) + 1 + \|p - q\|_\infty. \end{aligned}$$

Therefore $\phi_i(p)(i) - \phi_i(q)(i) - \|p - q\|_\infty \leq 1$. By $\phi_i(p)(i) \equiv p(i) \pmod 2$ and $\phi_i(q)(i) \equiv q(i) \pmod 2$, and (2.2), $\phi_i(p)(i) - \phi_i(q)(i) - \|p - q\|_\infty$ is an even integer. Therefore $\phi_i(p)(i) - \phi_i(q)(i) - \|p - q\|_\infty \leq 0$. Hence $|\phi_i(p)(i) - \phi_i(q)(i)| \leq \|p - q\|_\infty$ as desired.

Let $S = \{i_1, i_2, \ldots, i_n\}$. Define a map $\phi : P_\mu \cap A_\mu \to P_\mu \cap A_\mu$ by the composition

$$\phi = \phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_2} \circ \phi_{i_1}.$$

Clearly ϕ satisfies (1) and (2). We show that if $\mu \neq 0$, then $\phi(q) \in T_\mu \cap A_\mu$ for any $q \in P_\mu \cap A_\mu$. Note that if $\mu = 0$, then $T_\mu = 0$ and $P_\mu = \mathbf{R}^S_+$, and the statement is obvious. Let $p = \phi(q)$. By construction of ϕ, for each $i \in S$, either

(i) there exists $j \in S$ such that $p(i) + p(j) = \mu(i,j)$, or

(ii) $p(i) = 1$ and $1 + p(j) > \mu(i,j)$ for all $j \in S$.

If there is no $i \in S$ of case (ii), then we have $p \in T_\mu \cap A_\mu$. If every $i \in S$ is of case (ii), then $\mu = 0$ by (2.3), and a contradiction. Suppose that there exist $i, j, k \in S$ such that i is of case (ii) and $p(j) + p(k) = \mu(j,k)$. By $1 + p(j) > \mu(i,j)$, $1 + p(k) > \mu(i,k)$, and (2.3), we have $p(j) \geq \mu(i,j) + 1$ and $p(k) \geq \mu(i,k) + 1$. Then $\mu(j,k) = p(j) + p(k) \geq \mu(i,j) + \mu(i,k) + 2 \geq \mu(j,k) + 2$. A contradiction. Thus $p \in T_\mu \cap A_\mu$. $\square$

This property reduces an H-packing problem to the problem of decomposing the finite metric $(T_\mu \cap A_\mu, l_\infty)$ [5]. However, to apply this approach to the case $H = K_3 + K_3$, we need one more step.

For a simple graph $H = (S, R)$, a metric μ on S is called an *H-minimal metric* if there is no metric $\mu'(\neq \mu)$ on S such that $\mu' \leq \mu$ and $\mu'(k,l) = \mu(k,l)$ for each $kl \in R$. An H-minimal metric is essentially equivalent to a *metric having H as an extremal graph* in the sense of Karzanov [16]. Throughout this paper, we use this simpler term "H-minimal metric".

Lemma 2.5. *Let $H = (S, R)$ be a simple graph and μ a metric on S. The following two conditions are equivalent.*

(1) *μ is H-minimal.*

(2) *for each $i, j \in S$ with $\mu(i,j) > 0$, there exists $kl \in R$ such that*

$$\mu(k,l) = \mu(k,i) + \mu(i,j) + \mu(j,l).$$

Proof. (1) $\Rightarrow$ (2). Suppose that (2) fails for $i, j \in S$ with $\mu(i,j) > 0$, i.e., for any k, l with $kl \in R$,

$$\mu(k,l) < \mu(k,i) + \mu(i,j) + \mu(j,l). \tag{2.4}$$

Let μ' be defined as $\mu'(i,j) = \mu(i,j) - \epsilon$ for small $\epsilon > 0$ and $\mu'(k,l) = \mu(k,l)$ for $\{k,l\} \neq \{i,j\}$. μ' may not be a metric. Consider the *metric closure* μ'' of μ', which is defined by the shortest path metric on the complete graph on S with edge length μ'. Then $\mu'' \leq \mu' \leq \mu$, and $\mu''(k,l) = \mu(k,l)$ for each $kl \in R$ since edge ij is not used by every shortest path connecting k and l by (2.4). Thus μ is not H-minimal.

(2) $\Rightarrow$ (1). Suppose that μ is not H-minimal. There exists a metric μ' such that $\mu'(k,l) = \mu(k,l)$ for $kl \in R$, $\mu' \leq \mu$ and $\mu'(i,j) < \mu(i,j)$. Therefore $\mu(i,j) > 0$ and we can not extend a shortest path between i and j to a shortest path between k and l for any $kl \in R$ with respect to edge length μ. Thus (2) fails. □

In [16], a minimal graph H satisfying (2) is called an *extremal graph* of μ; see also [8, Section 24.5]. The following lemma and its proof are essentially appeared in [16].

Lemma 2.6. *For a cyclically even metric μ on S and a simple graph $H = (S, R)$, if $\mu(s,t) > 0$ for some $st \in R$, then there exists a cyclically even H-minimal metric μ^* such that $\mu^* \leq \mu$, $\mu^*(k,l) = \mu(k,l)$ for each $kl \in R$, and $A_\mu = A_{\mu^*}$.*

Proof. Suppose that there are $i, j \in S$ such that (*) $\mu(i,j) \geq 2$ and $\mu(k,l) < \mu(k,i) + \mu(i,j) + \mu(j,l)$ for each $kl \in R$. By cyclically evenness, $\mu(k,i) + \mu(i,j) + \mu(j,l) - \mu(k,l)$ is a positive even integer. Let μ' be defined by $\mu'(k,l) = \mu(k,l)$ for $\{k,l\} \neq \{i,j\}$ and

$$\mu'(i,j) = \begin{cases} \max\limits_{k,l \in S: kl \in R} \{\mu(k,l) - \mu(k,i) - \mu(j,l), 0\} & \text{if } \mu(i,j) \in 2\mathbf{Z}, \\ \max\limits_{k,l \in S: kl \in R} \{\mu(k,l) - \mu(k,i) - \mu(j,l), 1\} & \text{if } \mu(i,j) \notin 2\mathbf{Z}. \end{cases}$$

The resulting μ' may not be a metric. Consider its metric closure μ''. Then the resulting metric μ'' satisfies $\mu(i,j) - \mu''(i,j) \in 2\mathbf{Z}_+$ and $\mu''(k,l) = \mu(k,l)$ for each $kl \in R$. Therefore μ'' is cyclically even and $A_\mu = A_{\mu''}$. Set $\mu \leftarrow \mu''$. Repeat this process to i, j with property (*) if exists. The number of such pairs strictly decreases, and thus this procedure terminates. We show that the resulting μ is a desired one, i.e., μ satisfies (2) in Lemma 2.5. For a pair $i, j \in S$, if $\mu(i,j) \geq 2$, then pair i, j satisfies (2) by construction. Suppose $\mu(i,j) = 1$. We may assume that there exists $u \in S$ with $\mu(i,u) > 0$ and

$\mu(j,u) > 0$. Indeed, suppose that no such u exists. Then μ is necessarily a cut metric of bipartition $(\{u \mid \mu(i,u) = 0\}, \{u \mid \mu(j,u) = 0\})$. $\mu(s,t) > 0$ implies that $\mu(s,t) = 1$ and edge st joins the different parts. So condition (2) is clearly fulfilled. We may assume that $\mu(i,u) \geq \mu(j,u) > 0$. Since $\mu(i,u) + \mu(j,u) + \mu(i,j)$ is even, we have $\mu(i,u) \geq 2$ and $\mu(i,u) > \mu(j,u)$. Then $\mu(i,j) + \mu(j,u) = \mu(i,u)$ holds. Therefore there exists $kl \in R$ such that $\mu(k,l) = \mu(k,i) + \mu(i,u) + \mu(u,l) = \mu(k,i) + \mu(i,j) + \mu(j,u) + \mu(u,l) = \mu(k,i) + \mu(i,j) + \mu(j,l)$, as required. □

The following decomposition theorem is our central subject to prove the main theorem (Theorem 1.3). The proof is given in Section 4.

Theorem 2.7. *Suppose $H = (S,R) = K_3 + K_3$. Let μ be a cyclically even H-minimal metric on S. Then the finite metric space $(T_\mu \cap A_\mu, l_\infty)$ is decomposed into the sum of cut metrics, $K_{2,3}$-metrics, $K_{3,3}$-metrics, and $\Gamma_{3,3}$-metrics with nonnegative integral coefficients.*

Now using this, we can derive our main theorem (Theorem 1.3) as follows. Let $G = (V,E)$ be a connected graph, and let $H = (S,R)$ with $S \subseteq V$ be $K_3 + K_3$. Let l be a cyclically even edge length function on E. Then, clearly, the graph metric $d_{G,l}$ is a cyclically even metric on V. Let μ be the restriction of $d_{G,l}$ to S. We may assume that $\mu(s,t) > 0$ for some $st \in R$. Otherwise the problem is trivial. By Lemma 2.6, we can take a cyclically even H-minimal metric μ^* with $\mu^* \leq \mu$, $\mu^*(k,l) = \mu(k,l)$ for each $kl \in R$, and $A_\mu = A_{\mu^*}$. Consider P_{μ^*} and T_{μ^*}. For $k \in V$, we define a vector $p_k \in \mathbf{R}^S$ as

$$p_k = \mu^*_k \quad (k \in S) \tag{2.5}$$

and

$$p_k(j) = d_{G,l}(k,j) \quad (j \in S,\ k \in V \setminus S). \tag{2.6}$$

Then we have

$$p_k \in P_{\mu^*} \cap A_{\mu^*} \quad (k \in V).$$

Indeed, we have $p_k = \mu^*_k \in T_{\mu^*} \cap A_{\mu^*}$ for $k \in S$ and

$$\begin{aligned} p_k(i) + p_k(j) &= d_{G,l}(k,i) + d_{G,l}(k,j) \\ &\geq d_{G,l}(i,j) = \mu(i,j) \geq \mu^*(i,j) \quad (k \in V \setminus S). \end{aligned}$$

Therefore $p_k \in P_{\mu^*}$ for $k \in V \setminus S$. By the cyclically evenness of $d_{G,l}$ and the construction of μ^*, we have $p_k \in A_\mu = A_{\mu^*}$. Then we have

$$\begin{aligned} l(ij) &\geq d_{G,l}(i,j) \geq \|p_i - p_j\|_\infty \quad (ij \in E), \\ d_{G,l}(i,j) &= \mu^*(i,j) = \|p_i - p_j\|_\infty \quad (ij \in R). \end{aligned}$$

Take a nonexpansive retraction $\phi : P_{\mu^*} \cap A_{\mu^*} \to T_{\mu^*} \cap A_{\mu^*}$ in Proposition 2.4. Then we obtain

$$\begin{aligned} l(ij) &\geq \|\phi(p_i) - \phi(p_j)\|_\infty \quad (ij \in E), \\ d_{G,l}(i,j) &= \|\phi(p_i) - \phi(p_j)\|_\infty \quad (ij \in R). \end{aligned}$$

Therefore, the decomposition of $(T_{\mu^*} \cap A_{\mu^*}, l_\infty)$ in Theorem 2.7 yields a required integral H-packing.

3 A decomposition theory for 2-dimensional tight spans

The goal of this section is to develop a decomposition theory for 2-dimensional tight spans, which is the basis for the proof of Theorem 2.7 in the next section. Let (S, μ) be a finite metric space. We further suppose that μ is cyclically even.

The first task is to represent the finite metric $(T_\mu \cap A_\mu, l_\infty)$ as the graph metric of a graph obtained from the lattice L. Let $\tilde{\Gamma}_\mu$ be an infinite graph on vertex set $P_\mu \cap A_\mu$ obtained by connecting $p, q \in P_\mu \cap A_\mu$ if $\|p - q\|_\infty = 1$.

Lemma 3.1. *We have the following.*

(1) *$\tilde{\Gamma}_\mu$ is bipartite.*

(2) *$d_{\tilde{\Gamma}_\mu}(p, q) = \|p - q\|_\infty$ holds for $p, q \in P_\mu \cap A_\mu$.*

Proof. (1). Lattice L is the disjoint union of odd vectors and even vectors. Then l_∞-distances on points of the same parity are even integers. So the graph on L obtained by connecting points with the unit l_∞-distance is necessarily a bipartite graph whose bipartition is given by odd vectors and even vectors. Consequently $\tilde{\Gamma}_\mu$ is also bipartite.

(2). $(\geq)$ is obvious. We show the converse by constructing a path from p to q with length $\|p - q\|_\infty$. For $p, q \in P_\mu \cap A_\mu$, let U be the set $\{i \in S \mid q(i) < p(i)\}$. Clearly, $p' := p - \chi_U + \chi_{S \setminus U}$ belongs to $P_\mu \cap A_\mu$. If $p(i) \neq q(i)$ for all $i \in S$, then $\|p - q\|_\infty = 1 + \|p' - q\|_\infty$. If $p(i) = q(i)$ for some $i \in S$, then, by $p - q \in L$, we have $\|p - q\|_\infty \geq 2$, and therefore $\|p - q\|_\infty = 1 + \|p' - q\|_\infty$. Repeating this process to p' and q, we obtain the desired path. □

Let Γ_μ be the subgraph of $\tilde{\Gamma}_\mu$ induced by $T_\mu \cap A_\mu$. Then Γ_μ is an isometric subgraph of $\tilde{\Gamma}_\mu$. Indeed, for $p, q \in T_\mu \cap A_\mu$, consider the image of a shortest path joining p and q in $\tilde{\Gamma}_\mu$ by a nonexpansive retraction in Proposition 2.4. Then this yields a shortest path in Γ_μ. In particular, $(T_\mu \cap A_\mu, l_\infty)$ coincides with the graph metric of Γ_μ. The decomposability of the graph metric d_{Γ_μ} is our central interest.

It will turn out that 2-dimensionality of T_μ is crucial for d_{Γ_μ} to have a nice decomposability property. To study the dimension of T_μ, we introduce a graph $K(p)$ associated with a point $p \in P_\mu$, which is a fundamental tool to investigate T_μ [9]. For $p \in P_\mu$, we define the graph $K(p) = (S, E(p))$ as $ij \in E(p) \Leftrightarrow p(i) + p(j) = \mu(i, j)$. Namely, $K(p)$ represents the information of facets of P_μ containing p. In particular, $p \in T_\mu$ if and only if $K(p)$ has no isolated vertices by Lemma 2.1 (2). Loop ii appears exactly when $p(i) = 0$, and in this case $p = \mu_i$ holds by Lemma 2.2 (1). Let $F(p)$ be the face of T_μ containing p as its relative interior. It should be noted that

$$F(p) \subseteq F(q) \text{ if and only if } E(p) \supseteq E(q) \quad (p, q \in P_\mu). \tag{3.1}$$

For a face F of T_μ, we denote the corresponding graph by K_F, i.e.,

$$K_F := K(p) \text{ for a relative interior point } p \text{ in a face } F. \tag{3.2}$$

The dimension of $F(p)$ is characterized by a graphical property of $K(p)$.

Lemma 3.2 ([9]). *For $p \in T_\mu$, we have*

$$\dim F(p) = \textit{the number of bipartite components of } K(p).$$

Sketch of proof. $\dim F(p)$ is given by $\#S$ minus the rank of the matrix whose columns are $\{\chi_i + \chi_j \mid ij \in E(p)\}$. The rank of this 0-1 matrix can be characterized in such a graphical way. □

It turns out in the proof of the next proposition that the graph $K(p)$ and the commodity graph H are closely related; see Section 5.1 for further discussion. This was a motivation for the H-minimality.

Proposition 3.3. *Let $H = (S, R)$ be a simple graph and μ an H-minimal metric on S. If H has no matching of size n ($n \geq 2$), then T_μ is at most $(n-1)$-dimensional.*

For the proof, we use the following lemma, which connects $K(p)$ and the H-minimality.

Lemma 3.4. *Let μ be an H-minimal metric. For $p \in T_\mu$, if $K(p)$ has no loops and $ij \in E(p)$, then there exists $kl \in R$ with $il, jk, kl \in E(p)$.*

Proof. We note the following property of a point $p \in T_\mu$ [9, Theorem 3 (iv)].

$$p(i) + \mu(i,j) \geq p(j) \quad (p \in T_\mu,\ i,j \in S). \tag{3.3}$$

Indeed, $p(j) > p(i) + \mu(i,j)$ implies $p(j) + p(k) > p(i) + \mu(i,j) + p(k) \geq \mu(i,k) + \mu(i,j) \geq \mu(j,k)$ for any k. This contradicts to Lemma 2.1 (2).

Since $K(p)$ has no loops, p is positive, and thus $\mu(i,j) = p(i) + p(j) > 0$. By Lemma 2.5, there exists an edge $kl \in R$ such that

$$\mu(k,l) = \mu(k,i) + \mu(i,j) + \mu(j,l). \tag{3.4}$$

By (3.3) and $ij \in E(p)$, we have $\mu(k,l) \geq p(k) - p(i) + p(i) + p(j) + p(l) - p(j) = p(k) + p(l) \geq \mu(k,l)$. This implies $kl \in E(p)$. By (3.4), $\mu(k,j) = \mu(k,i) + \mu(i,j)$ necessarily holds. So $\mu(k,j) \geq p(k) - p(i) + p(i) + p(j) = p(k) + p(j) \geq \mu(k,j)$, and thus $jk \in E(p)$. Similarly we have $il \in E(p)$. □

Proof of Proposition 3.3. Suppose that T_μ is n-dimensional. There exists a point $p \in T_\mu$ such that $K(p)$ has n bipartite connected components by Lemma 3.2. Then $K(p)$ has no loops. Indeed, $ii \in E(p)$ implies that $p = \mu_i$ (Lemma 2.2 (1)), and thus i is adjacent to all vertices. Thus $K(p)$ is connected. A contradiction. By Lemma 3.4, each component has at least one edge of H, and thus H has a matching of size n. □

In the case where H has no matching of size 3, T_μ is at most 2-dimensional. To concentrate on this case, we assume that T_μ is at most 2-dimensional in the sequel. Our next task is to investigate how the graph Γ_μ is drawn in T_μ. In particular, we will determine the connected components of

$$T_\mu \setminus \bigcup \{[p,q] \mid p, q \in T_\mu \cap A_\mu, \|p - q\|_\infty = 1\}.$$

In the subsequent arguments, the following moving process in T_μ is important.

Lemma 3.5. *For $p \in T_\mu$ and a stable set U in $K(p)$, we have the following.*

1. *For sufficiently small $\epsilon > 0$, point $p + \epsilon(-\chi_U + \chi_{N(U)})$ belongs to T_μ if and only if there is no vertex $i \in S \setminus U$ incident only to $N(U)$.*
2. *In addition, if $p \in T_\mu \cap A_\mu$ and U is maximal stable, then $p - \chi_U + \chi_{S \setminus U}$ belongs to $T_\mu \cap A_\mu$.*

Proof. For $p \in T_\mu$, a stable set U in $K(p)$, and small $\epsilon > 0$, let $p^{U,\epsilon} := p + \epsilon(-\chi_U + \chi_{N(U)})$. Then $p^{U,\epsilon} \in P_\mu$. Moreover we see that $K(p^{U,\epsilon})$ is obtained by deleting all edges connecting $N(U)$ and $S \setminus U$ from $K(p)$. Recall that $p^{U,\epsilon} \in T_\mu$ if and only if $K(p^{U,\epsilon})$ has no isolated vertex (Lemma 2.1). From this fact, we have (1).

The maximum step $\max\{\epsilon \geq 0 \mid p^{U,\epsilon} \in T_\mu\}$ is given by

$$\min\left\{\min_{i,j\in U}\frac{p(i)+p(j)-\mu(i,j)}{2}, \min_{i\in U, j\in S\setminus(U\cup N(U))} p(i)+p(j)-\mu(i,j)\right\}. \tag{3.5}$$

Suppose that U is maximal stable. Clearly U satisfies the condition (1) and $N(U) = S\setminus U$. If $p \in T_\mu \cap A_\mu$, then (3.5) is positive integral by (2.3), and thus $p - \chi_U + \chi_{S\setminus U} \in T_\mu \cap A_\mu$. □

The first application of this lemma is:

Lemma 3.6. *For $p, q \in T_\mu \cap A_\mu$, if $\|p-q\|_\infty = 1$, then $[p,q] \subseteq T_\mu$ and $q = p - \chi_U + \chi_{S\setminus U}$ for some maximal stable set U of $K(p)$.*

Proof. We show that the set $U = \{i \in S \mid p(i) - q(i) = 1\}$ is a nonempty maximal stable set in $K(p)$. If U or $S \setminus U$ is empty, then $p < q$ or $q > p$ and this contradicts to the minimality of p, q in P_μ. Thus both U and $S\setminus U$ are nonempty. If there exist $i, j \in U$ with $ij \in E(p)$, then $1 = p(i) - q(i) = \mu(i,j) - p(j) - q(i) \leq q(j) - p(j) = -1$. This is a contradiction. Therefore U is stable in $K(p)$. Suppose that U is not maximal stable. Then there exists $j \in S \setminus U$ such that j is not incident to U. Since $q = p - \chi_S + \chi_{U\setminus S}$, the vertex j is isolated in $K(q)$. This is a contradiction to $q \in T_\mu$. □

The second application reveals the structure of graph $K(p)$.

Lemma 3.7. *For $p \in T_\mu$, graph $K(p)$ has at most two connected components. In addition, if $K(p)$ has two connected components, then $K(p)$ has no loops and both components are complete multipartite.*

Proof. Note that if $K(p)$ has a loop, then $K(p)$ is connected; see the proof of Proposition 3.3. Let U be a maximal stable set of $K(p)$. Then $p' := p + \epsilon(-\chi_U + \chi_{S\setminus U})$ belongs to T_μ for small $\epsilon > 0$. If $K(p)$ has at least three components, then U meets all components (since there is no loop) and thus $K(p')$ has at least three bipartite components. This is a contradiction to Lemma 3.2 and the assumption $\dim T_\mu \leq 2$. Suppose that $K(p)$ has two components K^1, K^2. Suppose further that K^1 has two intersecting maximal stable sets U, U'. For small $\epsilon > 0$, $p' := p + \epsilon(-\chi_{U\cap U'} + \chi_{N(U\cap U')})$ belongs to T_μ. Indeed, the maximality of U and U' in K^1 implies that every vertex in $U \setminus U'$ is connected to a vertex in $U' \setminus U$, and apply Lemma 3.5 (1). $K(p')$ is obtained by deleting all edges connecting $N(U \cap U')$ and $S \setminus U$. Therefore $U \cap U'$ and $U \setminus U'$ belong to different components. Hence $K(p')$ has at least three components. A contradiction. Therefore, maximal stable sets in K^1 are pairwise disjoint, and this implies that K^1 is complete multipartite. So does K^2. □

Then, from Lemmas 3.2 and 3.7, we see:

(1) $F(p)$ is an extreme point of T_μ if and only if

  (1-1) $K(p)$ is connected nonbipartite or

  (1-2) $K(p)$ consists of two nonbipartite complete multipartite components.

(2) $F(p)$ is an edge of T_μ if and only if

  (2-1) $K(p)$ is connected bipartite or

  (2-2) $K(p)$ consists of one complete bipartite component and one nonbipartite complete multipartite component.

(3) $F(p)$ is a 2-dimensional face of T_μ if and only if $K(p)$ consists of two complete bipartite components.

In particular, there are two types of edges and extreme points. An edge e of T_μ is called an l_1-*edge* if K_e is connected bipartite; recall the notation (3.2). Other edge e is called an l_∞-*edge*. An extreme point p of T_μ is called a *core* if $K(p)$ has two nonbipartite components. These concepts have been introduced in [11]. Relationship among l_1-edges, l_∞-edges, cores, and the graph Γ_μ is important for us.

Lemma 3.8. *Let p be an extreme point of T_μ. Then p is integral. In addition, if p is not a core, then $p \in A_\mu$.*

Proof. For $i \in S$, then there exists a nonbipartite component containing i. Let C be an odd cycle of this component. We order vertices in C cyclically as $(j_0, j_1, \ldots, j_{m-1})$. Then $p(j_0)$ is given by $(\sum_{k=0}^{m-1}(-1)^k \mu(j_k, j_{k+1}))/2$, where the index is taken modulo m. By cyclically evenness, $p(j_0)$ is integral. There exists a path from i to j_0 in $K(p)$. Substituting the relation $p(i') + p(i'') = \mu(i', i'')$ along this path, we obtain $p(i)$ which is integral.

Next we suppose that p is not a core. Fix an odd cycle C in $K(p)$ ordered cyclically as above. For any $i, j \in S$, there exist paths connecting from C to i and j, respectively. By calculation, $p(i) + p(j)$ is given by $\sum_{e \in P} \pm\mu(e)$ for some (possibly nonsimple) path P joining i and j. Take $k \in S$. Then we have

$$(\mu_k - p)(i) + (\mu_k - p)(j) = \mu(k, i) + \mu(k, j) + \sum_{e \in P} \pm\mu(e).$$

The right hand side is the sum of μ along some (possibly nonsimple) cycle in the complete graph on S, and thus even. □

For an l_1-edge e, if the corresponding bipartite graph K_e has a bipartition (A, B), then the direction of e is parallel to $\chi_A - \chi_B \in \{1, -1\}^S$. Furthermore, we easily see that neither endpoint of e is core (by (3.1)). Therefore each l_1-edge e is a series of edges in Γ_μ.

Next we study the shape of a 2-dimensional face. We simply call a 2-dimensional face *a 2-face*. By calculation, we have the following; see [11] for details.

Lemma 3.9. *Let F be a 2-face of T_μ. Let K^1 and K^2 be two bipartite components of K_F with bipartitions (A_1, B_1) and (A_2, B_2), respectively. For $i \in A_1$ and $j \in A_2$, the projection map $(\cdot)|_{\{i,j\}} : \mathbf{R}^S \to \mathbf{R}^{\{i,j\}}$ is an isometry between (F, l_∞) and $(F|_{\{i,j\}}, l_\infty)$. Moreover $F|_{\{i,j\}}$ is represented as*

$$F|_{\{i,j\}} = \left\{ (p(i), p(j)) \in \mathbf{R}^2 \;\middle|\; \begin{array}{ll} a \le p(i) \le a', & c \le p(i) + p(j) \le c', \\ b \le p(j) \le b', & d \le p(i) - p(j) \le d' \end{array} \right\}$$

for some $a, a', b, b', c, c', d, d' \in \mathbf{Z}$.

Sketch of proof. Fix $p \in F$. Then any point q in F is uniquely represented as $p + \alpha(\chi_{A_1} - \chi_{B_1}) + \beta(\chi_{A_2} - \chi_{B_2})$ for some $\alpha, \beta \in \mathbf{R}$. From this, we easily see that the projection is an isometry. The coordinate $p(k)$ for $k \in A_1 \cup B_1$ is obtained by substituting relations $p(i') + p(i'') = \mu(i', i'')$ along a path in $K(p)$ connecting i and k in K^1. From this, we see the desired linear inequality description of $F|_{\{i,j\}}$. □

Therefore, a 2-face is isomorphic to a polygon in the l_∞-plane $\mathbf{R}^2$ each of whose edges is parallel to one of four vectors χ_1, χ_2, $\chi_1 - \chi_2$, and $\chi_1 + \chi_2$; see Figure 4 (a).

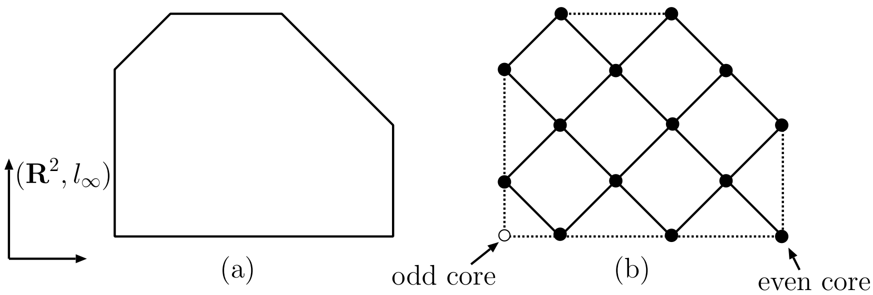


Figure 4: (a) a 2-face and (b) decomposing the 2-face by Γ_μ

Then l_∞-edges in F are parallel to χ_1 or χ_2, and l_1-edges in F are parallel to $\chi_1 - \chi_2$ or $\chi_1 + \chi_2$. As is well-known, the l_∞-plane is isomorphic to the l_1-plane by the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$; see [8, p. 31]. Then l_1-edges in F are parallel to the coordinate axes in the l_1-plane. In particular, T_μ is obtained by gluing such polygons along *the same type of edges*.

Next we study the local structure around a core. In general, an edge vector of T_μ is parallel to $\chi_A - \chi_B$ for some disjoint nonempty subsets $A, B \subseteq S$; consider the orthogonal space of vectors $\{\chi_i + \chi_j \mid ij \in E(p)\}$ having codimension 1. By combining Lemmas 3.5 and 3.2, we see that the edge e adjacent to an extreme point p is given by $[p, p + \alpha(-\chi_A + \chi_{N(A)})]$ for a maximal stable set in some connected component of $K(p)$ and a positive integer α. For a core p, if two complete multipartite components in $K(p)$ have partitions $\{A_1, A_2, \ldots, A_m\}$ and $\{B_1, B_2, \ldots, B_n\}$ $(n, m \geq 3)$, then we say that p *has the type* $(A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_n)$. We denote edges adjacent to p parallel to $-\chi_{A_i} + \chi_{\cup_{k \neq i} A_k}$ and $-\chi_{B_j} + \chi_{\cup_{k \neq j} B_k}$ by $e(p, A_i)$ and $e(p, B_j)$, respectively. By the above argument and a routine verification, we have:

Lemma 3.10. *Let p be a core of type $(A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_n)$.*

(1) *An edge e is adjacent to p if and only if e is $e(p, A_i)$ or $e(p, B_j)$ for some i, j.*

(2) *Two edges e', e'' adjacent to p belong to the common 2-face if and only if $\{e', e''\} = \{e(p, A_i), e(p, B_j)\}$ for some i, j.*

We call a core p *even* if $p \in A_\mu$, and *odd* if $p \notin A_\mu$.

Lemma 3.11. *Let p be an odd core of type $(A_1, A_2, \ldots, A_m; B_1, B_2, \ldots, B_n)$. For any i, j, both $p - \chi_{A_i} + \chi_{\cup_{k \neq i} A_k}$ and $p - \chi_{B_j} + \chi_{\cup_{k \neq j} B_k}$ belong to $T_\mu \cap A_\mu$.*

Proof. By the argument similar to the proof of Lemma 3.8, $(\mu_l - p)(i) + (\mu_l - p)(j)$ is even for $i, j \in \cup_k A_k$ or $i, j \in \cup_k B_k$. Therefore, as $p \notin A_\mu$, $(\mu_l - p)(i) + (\mu_l - p)(j)$ is odd for $i \in \cup_k A_k$ and $j \in \cup_k B_k$. □

Summarizing these arguments, a 2-face is decomposed by Γ_μ as in Figure 4 (b), where the black points are vertices of Γ_μ, the white point is an odd core, the broken lines represent l_∞-edges, and other black lines are edges of Γ_μ.

Lemma 3.12. *For an l_∞-edge e, there exist at least three 2-faces containing e.*

Proof. Let p be a relative interior point in e. Then $K(p)$ consists of one bipartite graph and one nonbipartite graph K. The graph K is complete multipartite with partition $\{A_1, A_2, \ldots, A_m\}$ $(m \geq 3)$. For each A_i, a point $p' := p + \epsilon(-\chi_{A_i} + \chi_{\cup_{k \neq i} A_k})$ for small $\epsilon > 0$ belongs to T_μ by Lemma 3.5 (1). Then $K(p')$ has two bipartite components, and $F(p')$ is a 2-face. □

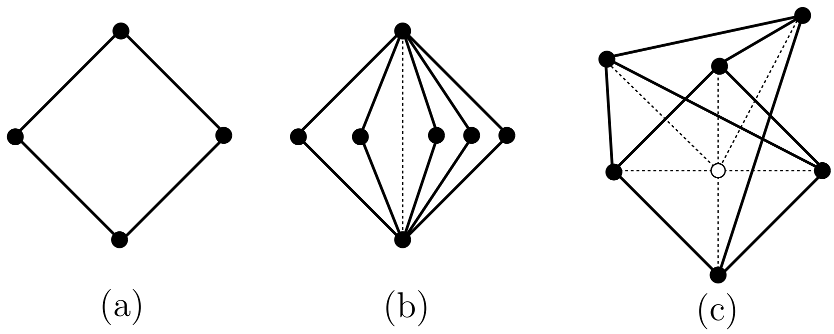


Figure 5: (a) square, (b) $K_{2,5}$-folder, and (c) $K_{3,3}$-folder

Let us return back to the original problem to determine the closure of connected components of the set

$$T_\mu \setminus \bigcup \{[p, q] \mid p, q \in T_\mu \cap A_\mu, \|p - q\|_\infty = 1\}. \tag{3.6}$$

Recall that the graph Γ_μ decomposes each 2-face as in Figure 4 and T_μ is obtained by gluing polygons along the same type of edges.

There are unit squares with its edges parallel to $\chi_{A_1 \cup A_2} - \chi_{B_1 \cup B_2}$ and $\chi_{B_1 \cup A_2} - \chi_{A_1 \cup B_2}$ for some four-partition $\{A_1, A_2, B_1, B_2\}$ of S. We call it a *square*. Suppose that there exists an l_∞-edge e having two points p, q of A_μ. Then $q - p$ is parallel to $\chi_A - \chi_{N(A)}$ for *nonmaximal* stable set A of $K(p)$. Therefore $q - p$ is necessarily an even vector. Then we can take two points $p, q \in e$ with $\|p - q\|_\infty = 2$. By Lemma 3.12, there exist $m (\geq 3)$ 2-faces containing segment $[p, q]$. Therefore, the closure of the component meeting this segment is a *folder* obtained by gluing m right-angled isosceles triangles along their longer edge. The graph of boundary edges is the complete bipartite graph $K_{2,m}$. We call it a $K_{2,m}$*-folder*. Suppose that there exists an odd core p of type $(A_1, \ldots, A_m, B_1, \ldots, B_n)$ $(m, n \geq 3)$. The closure of the component containing p is the union of triangles whose vertices p, $p - \chi_{A_i} + \chi_{\cup_{k \neq i} A_k}$, and $p - \chi_{B_j} + \chi_{\cup_{k \neq j} B_k}$ over all pairs (i, j) of $1 \leq i \leq m$ and $1 \leq j \leq n$. Its boundary graph is the complete bipartite graph $K_{n,m}$. We call it a $K_{n,m}$*-folder*. In other words, a $K_{n,m}$-folder is isomorphic to the complex of the join of $K_{n,m}$ and one point. Therefore, the graph Γ_μ decomposes T_μ into squares, $K_{2,l}$-folders, and $K_{n,m}$-folders. See Figure 5 (a), (b), and (c). Such a folder decomposition of a 2-dimensional tight span has already been obtained by [17, 18, 4] via different approach; see Section 5.3 for further discussion. Also this type of polyhedral complexes has been studied by Chepoi [6, Section 7]. A new point here is a relation among the lattice L, odd cores, and l_1/l_∞-edges.

Remark 3.13. An even core p of type $(A_1, \ldots, A_m, B_1, \ldots, B_n)$ $(m, n \geq 3)$ belongs to n $K_{2,m}$-folders and m $K_{2,n}$-folders. The union of their boundary graphs is $\Gamma_{n,m}$; see Figure 6. Here $\Gamma_{n,m}$ is the graph obtained by subdividing $K_{n,m}$ and connecting each subdivided point to one new point. This will turn out to be a reason why $\Gamma_{3,3}$-metrics appear in the $K_3 + K_3$-packing.

Next we discuss the decomposition of the graph metric d_{Γ_μ} of Γ_μ. In fact, this is a special case of the decomposition of a modular graph into its orbit graphs, which is discussed in [3, 18, 19]. We use some of terminology in [19, Section 2] with a slight modification. Two edges e, e' in Γ_μ are called *mates* if there is a square containing e, e' as its parallel edges, or there is a $K_{2,l}$-folder or a $K_{n,m}$-folder containing e, e' as its edges.

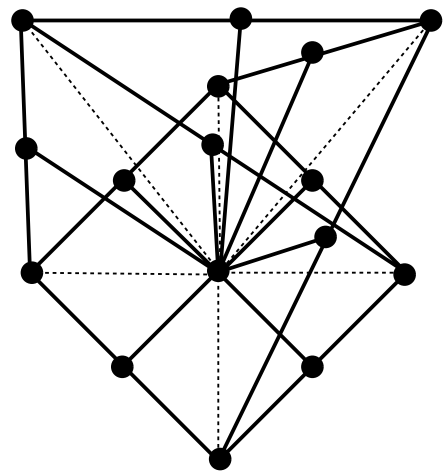


Figure 6: The folder structure around an even core

Two edges e, e' in Γ_μ are said to be *projective* if there is a sequence $e = e_1, e_2, \ldots, e_k = e'$ such that e_i and e_{i+1} are mates. The projectivity defines an equivalence relation on edges of Γ_μ. An equivalence class is called an *orbit*. For an orbit O, the *orbit graph* Γ_μ^O of Γ_μ is the graph obtained by contracting all edges not in O and then identifying parallel edges appeared. By the construction, we naturally obtain a map ϕ^O from vertices of Γ_μ to vertices of Γ_μ^O by defining $\phi^O(p)$ to be the contracted point. Then the graph metric d_{Γ_μ} is decomposed into the graph metrics of the orbit graphs as follows.

Proposition 3.14. *Let $\mathcal{O}$ be the set of all orbits of Γ_μ. Then we have*

$$d_{\Gamma_\mu}(p, q) = \sum_{O \in \mathcal{O}} d_{\Gamma_\mu^O}(\phi^O(p), \phi^O(q)) \quad (p, q \in T_\mu \cap A_\mu).$$

We will derive Theorem 2.7 from this decomposition principle, which is a simple consequence of the following property of shortest paths in Γ_μ.

Proposition 3.15. *Let $p, q \in T_\mu \cap A_\mu$. Let P be a shortest path connecting p and q, and Q an arbitrary path connecting p and q. For any orbit O, we have*

$$\#(P \cap O) \leq \#(Q \cap O).$$

Let P be a shortest path connecting p and q. The image of P by ϕ^O induces a path connecting $\phi^O(p)$ and $\phi^O(q)$ in Γ_μ^O whose length is $\#(P \cap O)$. Then $\phi^O(P)$ is shortest in Γ_μ^O. Suppose not. Then there exists another (not necessarily shortest) path P' connecting p and q in Γ_μ with $\#(P' \cap O) < \#(P \cap O)$. This contradicts to Proposition 3.15. Thus we have $d_{\Gamma_\mu}(p, q) = \sum_{O \in \mathcal{O}} \#(P \cap O) = \sum_{O \in \mathcal{O}} d_{\Gamma_\mu^O}(\phi^O(p), \phi^O(q))$.

Propositions 3.14 and 3.15 hold under a more general setting [2, 18, 19, 20]; also see [8, Section 20.3]. Although one can verify that our graph Γ_μ belongs to the class of the graphs treated by [2, 18, 19], we give a direct proof of Proposition 3.15. In the sequel, we often use the following distance property of Γ_μ:

$$d_{\Gamma_\mu}(u, q) - d_{\Gamma_\mu}(u', q) \in \{-1, 1\} \text{ if } u \text{ and } u' \text{ are adjacent.} \tag{3.7}$$

This immediately follows from the bipartiteness of Γ_μ (Lemma 3.1 (1)).

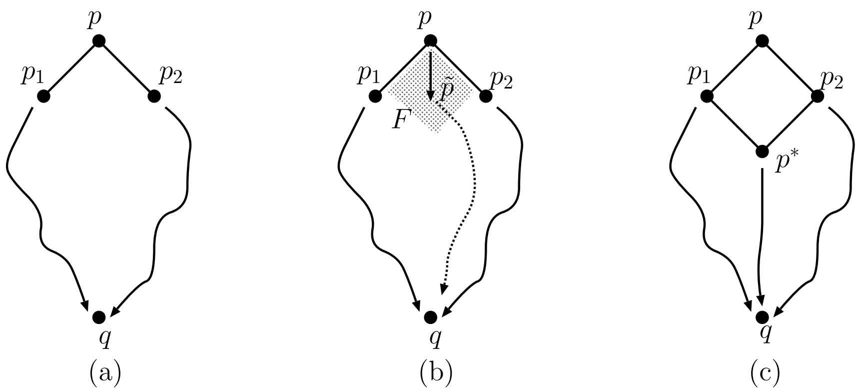


Figure 7: The quadrangle condition

Lemma 3.16. *Suppose that distinct four points $p, p_1, p_2, q \in T_\mu \cap A_\mu$ satisfy*

$$d_{\Gamma_\mu}(p, p_1) = d_{\Gamma_\mu}(p, p_2) = 1,$$
$$d_{\Gamma_\mu}(p, q) = d_{\Gamma_\mu}(p, p_1) + d_{\Gamma_\mu}(p_1, q) = d_{\Gamma_\mu}(p, p_2) + d_{\Gamma_\mu}(p_2, q).$$

Then there exist a folder F containing p, p_1, p_2 and a common neighbor $p^* \in F \cap A_\mu$ of p_1, p_2 with $d_{\Gamma_\mu}(p, q) = d_{\Gamma_\mu}(p^*, q) + 2$.

In [6], this property is called the *quadrangle condition*; see Figure 7.

Proof. By Lemma 3.6, p_k is represented by $p - \chi_{A_k} + \chi_{N(A_k)}$ for some maximal stable set A_k of $K(p)$ for $k = 1, 2$. Let $S_{p,q} := \{i \in S \mid p(i) - q(i) = \|p - q\|_\infty\}$. The condition $d_{\Gamma_\mu}(p, q) = d_{\Gamma_\mu}(p, p_i) + d_{\Gamma_\mu}(p_i, q)$ implies

$$S_{p,q} \subseteq A_1 \cap A_2. \tag{3.8}$$

Moreover $i \in S_{p,q}$ and $p(i)+p(j) = \mu(i, j)$ imply $j \in S_{q,p}$ and $q(i)+q(j) = \mu(i, j)$. Indeed, this follows from $\|p - q\|_\infty = p(i) - q(i) = \mu(i, j) - p(j) - q(i) \leq q(j) - p(j) \leq \|p - q\|_\infty$. Therefore we have

$$S_{q,p} \subseteq N(A_1 \cap A_2). \tag{3.9}$$

Let $p^\epsilon := p + \epsilon(-\chi_{A_1 \cap A_2} + \chi_{N(A_1 \cap A_2)})$. Then p^ϵ belongs to T_μ for $\epsilon \in [0, 1]$. Indeed, we have already seen it in the proof of Lemma 3.7 for small ϵ. By (3.5), ϵ can be taken as an integer. For small $\epsilon > 0$, $K(p^\epsilon)$ has two connected components, and p and p^ϵ belong to some folder F. Both segments $[p, p_1]$ and $[p, p_2]$ belong to F as boundary edges. One can see this fact by verifying $p^\epsilon + \epsilon'(-\chi_{A_k} + \chi_{N(A_k)})$ still belongs to T_μ for small $\epsilon, \epsilon' > 0$. In particular F contains p, p_1, p_2.

It suffices to show that F has a vertex p^* with $d_{\Gamma_\mu}(p^*, q) < d_{\Gamma_\mu}(p, q) - 1$. Then $d_{\Gamma_\mu}(p^*, q) = d_{\Gamma_\mu}(p, q) - 2$ necessarily holds. By bipartiteness, p^* and p must belong to the same part of bipartite graph Γ_μ. Since the boundary graph of folder F is complete bipartite, p^* is necessarily a common neighbor of p_1 and p_2. Let $\tilde{p} := p - \chi_{A_1 \cap A_2} + \chi_{N(A_1 \cap A_2)}$. Then $\tilde{p}$ is the center point of folder F. By (3.8)-(3.9), $d_{\Gamma_\mu}(p, q) = \|p - q\|_\infty = \|p - \tilde{p}\|_\infty + \|\tilde{p} - q\|_\infty = 1 + \|\tilde{p} - q\|_\infty$. Since (T_μ, l_∞) is geodesic (see [9]), we can take a (continuous) path P in T_μ connecting $\tilde{p}$ and q with length $\|\tilde{p} - q\|_\infty$; see Figure 7 (b). This path P have to cross a boundary point p' of F. Then $\|p' - q\|_\infty < \|\tilde{p} - q\|_\infty = \|p - q\|_\infty - 1$ must hold. If p' is a vertex, then it is a desired one. If p' lies on a boundary edge of F, then one of its ends is a desired one (by (3.7)). □

Proof of Proposition 3.15. Suppose $P = (p = u_0, u_1, u_2, \ldots, u_n = q)$ and $Q = (p = v_0, v_1, v_2, \ldots, v_k = q)$. We use the induction on k. We may assume $u_1 \neq v_1$. By (3.7), we have $d_{\Gamma_\mu}(v_1, q) - d_{\Gamma_\mu}(p, q) \in \{-1, 1\}$. Suppose $d_{\Gamma_\mu}(v_1, q) = d_{\Gamma_\mu}(p, q) + 1$. Then $P \cup \{pv_1\}$ is a shortest path connecting v_1 and q, and $Q \setminus \{pv_1\}$ is a path connecting v_1 and q of length $k-1$. By induction, we have $\#(Q \cap O) \geq \#((Q \setminus \{pv_1\}) \cap O) \geq \#((P \cup \{pv_1\}) \cap O) \geq \#(P \cap O)$ for any orbit O. So suppose $d_{\Gamma_\mu}(v_1, q) = d_{\Gamma_\mu}(p, q) - 1 = d_{\Gamma_\mu}(u_1, q)$. Apply Lemma 3.16 for quadrangle p, u_1, v_1, q. There exist a folder F containing p, u_1, v_1 and a common neighbor p^* of u_1, v_1 with $d_{\Gamma_\mu}(p, q) = d_{\Gamma_\mu}(p^*, q) + 2$. Let P^* be a shortest path connecting p^* and q. Then both $P \setminus \{u_1 p\}$ and $P^* \cup \{p^* u_1\}$ are shortest paths connecting q and u_1. Also $P^* \cup \{p^* v_1\}$ is a shortest path connecting q and v_1, and $Q \setminus \{v_1 p\}$ is a path connecting q and v_1 of length $k - 1$. By induction, $\#((P \setminus \{u_1 p\}) \cap O) = \#((P^* \cup \{p^* u_1\}) \cap O)$ and $\#((Q \setminus \{v_1 p\}) \cap O) \geq \#((P^* \cup \{p^* v_1\}) \cap O)$. Therefore $\#(P \cap O) = \#((P^* \cup \{p^* u_1, u_1 p\}) \cap O)$ and $\#(Q \cap O) \geq \#((P^* \cup \{p^* v_1, v_1 p\}) \cap O)$. Since (p, u_1, p^*, v_1) forms a cycle in folder F, by definition of orbits, $p^* u_1 \in O \Leftrightarrow v_1 p \in O$ and $p^* v_1 \in O \Leftrightarrow v_1 p \in O$. Hence we have the desired equality $\#(Q \cap O) \geq \#(P \cap O)$. □

4 Proof of Theorem 2.7

Let us start the proof of Theorem 2.7. Let $H = (S, R) = K_3 + K_3$. We suppose $S = \{1, 2, 3, 1', 2', 3'\}$ and $R = \{12, 23, 31, 1'2', 2'3', 3'1'\}$; see Figure 8 (a). Let μ be a cyclically even H-minimal metric on six-point set S. Our goal is to show that possible orbit graphs of Γ_μ are K_2, $K_{2,3}$, $K_{3,3}$, and isometric subgraphs of $\Gamma_{3,3}$. Then the orbit graph decomposition (Proposition 3.14) yields a desired decomposition.

Suppose that T_μ has no l_∞-edges. Then T_μ is decomposed into squares. Therefore, each orbit consists of edges with parallel direction, and its orbit graph is K_2. In this case, d_{Γ_μ} is an integral sum of cut metrics, and we obtain an integral H-packing by cut metrics; also see Section 5.1 for further discussion.

We concentrate on the case where T_μ has an l_∞-edge. Recall that there exists a $K_{2,l}$-folder or a $K_{m,n}$-folder meeting this l_∞-edge. We determine possible l, m, n. Recall the definition (3.2) of the graph K_F for a face F (and K_e for an edge e) of T_μ.

Lemma 4.1. *For an l_∞-edge e, either*

(case 1) *K_e equals H minus one edge, or*

(case 2) *K_e is the disjoint sum of one edge in H and K_4 minus one edge containing one triangle of H.*

Proof. K_e consists of one bipartite component and one nonbipartite component. Recall Lemma 3.7 that both components are complete multipartite and have no loops. By Lemma 3.4, any pair of different parts in each component is joined by an edge in H. Since the nonbipartite component (without loops) has at least three vertices, the bipartite component has at least two vertices, and $\#S = 6$, the nonbipartite component has three or four vertices. Therefore K_e must be (case 1) or (case 2). □

See Figure 8 (b), (c) for examples of the above two cases. By the same argument in the proof of previous lemma, we have the following.

Lemma 4.2. *If there exists a core p, then $K(p) = H$, and therefore p is a unique core.*

Therefore, the connected components of (3.6) consist of squares, $K_{2,3}$-folders, and one $K_{3,3}$-folder (if an odd core exists).

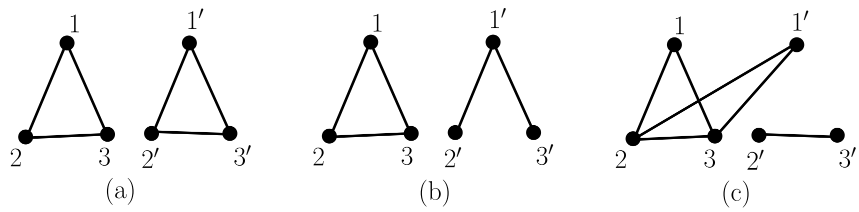


Figure 8: (a) $H = K_3 + K_3$, (b) the case 1, and (c) the case 2

To investigate the orbit graph, we need to *trace* the orbit started from some edge. For the trace, we use the following lemma that characterizes a *boundary* l_1-edge (corresponding to the case $k = 1$).

Lemma 4.3. *An l_1-edge e belongs to exactly k 2-faces if and only if K_e has exactly $k+2$ maximal stable sets.*

Proof. K_e is connected bipartite. Let (A, B) be the bipartition of K_e. Both A and B are maximal stable. Suppose that there exists another maximal stable set C. Take $p \in e$ in the relative interior. Then $p^{C,\epsilon} := p + \epsilon(-\chi_C + \chi_{N(C)})$ belongs to T_μ for small $\epsilon > 0$, and $K(p^{C,\epsilon})$ has two bipartite components. Therefore the 2-face $F(p')$ contains e. Conversely, if there exists 2-face F containing e, then there exists a maximal stable set $C(\neq A, B)$ in $K(p)$ such that $p^{C,\epsilon}$ belongs to F. For distinct maximal stable sets $C', C''(\neq A, B)$, the graphs $K(p^{C',\epsilon})$ and $K(p^{C'',\epsilon})$ are distinct, and hence $F(p^{C',\epsilon})$ and $F(p^{C'',\epsilon})$ are distinct. Thus the proof is done. □

The proof of Theorem 2.7 is completed by showing:

(1) the orbit graph generated by a $K_{3,3}$-folder containing odd core p is $K_{3,3}$.

(2) the orbit graph generated by a $K_{2,3}$-folder meeting l_∞-edge e of (case 2) is $K_{2,3}$.

(3) the orbit graph generated by a $K_{2,3}$-folder meeting l_∞-edge e of (case 1) is an isometric subgraph of $\Gamma_{3,3}$.

Note that if an orbit meet none of $K_{l,m}$-folders, then its orbit graph is K_2. In the sequel, for distinct $i, j, k, l, \cdots$, the vector $-\chi_{\{i,j,\cdots\}} + \chi_{\{k,l,\cdots\}}$ is denoted by $\chi^{ij\cdots}_{kl\cdots}$, such as $\chi^{11'}_{232'3'}(= -\chi_{\{1,1'\}} + \chi_{\{2,3,2',3'\}})$, and the graph obtained by deleting edge set E' from H is denoted by $H - E'$.

(1). Suppose that there exists an odd core p. We trace the orbit started from a boundary edge of $K_{3,3}$-folder containing p. The type of p is given by $(1, 2, 3; 1', 2', 3')$. Let $F_{11'}$ be the 2-face containing edges $e(p, 1)$ and $e(p, 1')$, where we use the notation of Lemma 3.10 and singleton set $\{i\}$ is simply denoted by i. The graph $K_{F_{11'}}$ is $H - \{23, 2'3'\}$. Since the graph K_e for an edge e of $F_{11'}$ contains $K_{F_{11'}}$ as a subgraph by (3.1), it follows from Lemma 4.1 that $F_{11'}$ has no l_∞-edges except $e(p, 1)$ and $e(p, 1')$. Therefore, the orbit started from an edge $[p+\chi^1_{23}, p+\chi^{1'}_{2'3'}]$ hits the edge e of $F_{11'}$ having direction $\chi^{12'3'}_{231'}$. The graph K_e has an edge $11'$. By Lemma 4.3, this edge e is a boundary l_1-edge. Hence the orbit started from $[p+\chi^1_{23}, p+\chi^{1'}_{2'3'}]$ escapes into the boundary of T_μ; see Figure 9 (a). The same holds for any $i \in \{1, 2, 3\}$ and $i' \in \{1', 2', 3'\}$. Therefore, the orbit started from this $K_{3,3}$-folder does not meet any other $K_{2,3}$-folders, and its orbit graph is $K_{3,3}$.

(2). Next we consider the orbit of a $K_{2,3}$-folder meeting l_∞-edge e of (case 2) in Lemma 4.1. We may assume that the graph K_e is $(S, \{12, 23, 31, 1'2, 1'3, 2'3'\})$; see Figure 8 (c). There exist three 2-faces F_1, F_2, F_3 containing e. The edge sets of K_{F_1}, K_{F_2},

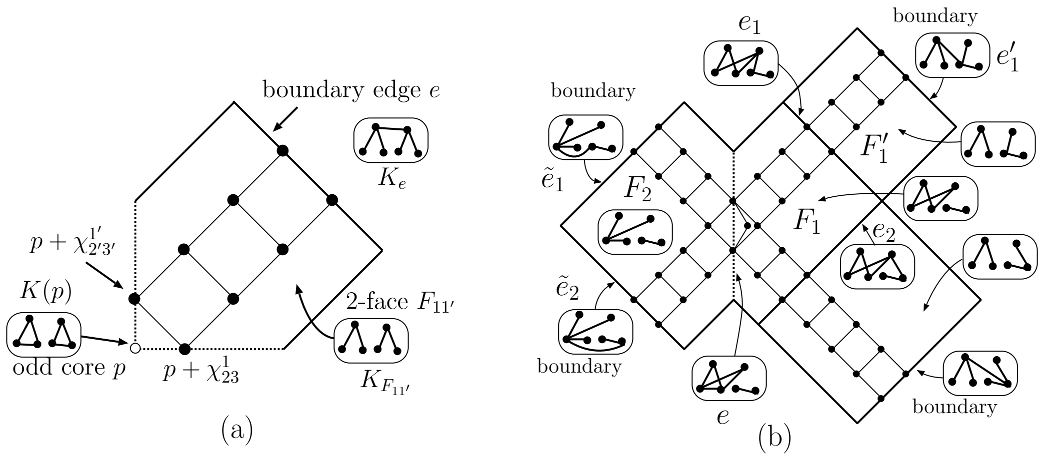


Figure 9: The orbits started from (a) $K_{3,3}$-folder and (b) $K_{2,3}$-folder of (case 2)

K_{F_3} are given by $\{12, 13, 1'2, 1'3, 2'3'\}$, $\{12, 1'2, 23, 2'3'\}$, and $\{13, 23, 1'3, 2'3'\}$, respectively. For F_1, F_2, F_3, the common edge e is a unique l_∞-edge (by Lemma 4.1). In F_2, the orbit started from this $K_{2,3}$-folder hits two edges $\tilde{e}_1$ and $\tilde{e}_2$ of F_2. Then $K_{\tilde{e}_1}$ has an edge $22'$ and $K_{\tilde{e}_2}$ has $23'$. Both $\tilde{e}_1$ and $\tilde{e}_2$ are in the boundary of T_μ. The same holds for F_3. For F_1, the orbit hits two edges e_1 and e_2 of F_1 such that K_{e_1} has $12'$ or $1'2'$ (or both), and K_{e_2} has $13'$ or $1'3'$ (or both). If K_{e_1} has both $12'$ and $1'2'$, then e_1 is in the boundary. Suppose that K_{e_1} has only $1'2'$. Then the edge e_1 belongs to one more 2-face F'_1 with $K_{F'_1} = (S, \{13, 12, 1'2', 2'3'\})$. The orbit hits an edge e'_1 of F'_1. This edge e'_1 is in the boundary; see Figure 9 (b). The case where K_{e_1} has only $12'$ does not occur since $K_{F'_1}$ for another 2-face F'_1 containing e_1 has two components one of which has no edge of H, which contradicts to Lemma 3.4. For e_2, the argument is the same. Consequently, the orbit started from the $K_{2,3}$-folder containing l_∞-edge e of (case 2) does not meet any other $K_{2,3}$-folders, and its orbit graph is $K_{2,3}$.

(3). Finally, we consider a $K_{2,3}$-folder meeting l_∞-edge of (case 1). We use some notation. For $(\{i, j, k\}, \{i', j', k'\}) = (\{1, 2, 3\}, \{1', 2', 3'\})$, a 2-face F with $K_F = H - \{jk, j'k'\}$ is denoted by $F_{ii'}$. For $\{i, j, k\} = \{1, 2, 3\}$ or $\{1', 2', 3'\}$, an l_∞-edge e of (case 1) with $K_e = H - \{jk\}$ is denoted by e_i. By (3.1), e_i is the common l_∞-edge of $F_{i1'}$, $F_{i2'}$, $F_{i3'}$ $(i = 1, 2, 3)$, and $e_{i'}$ is the common l_∞-edge of $F_{1i'}$, $F_{2i'}$, $F_{3i'}$ $(i' = 1', 2', 3')$. By Lemma 4.1, $F_{ii'}$ has no l_∞-edge except e_i and $e_{i'}$.

Now let us trace the orbit started by a $K_{2,3}$-folder $\mathcal{F}_{1'}$ meeting l_∞-edge $e_{1'}$. We show that the orbit escapes into the boundary, or meets other $K_{2,3}$-folders of (case 1) and returns to $\mathcal{F}_{1'}$ itself. Then five vertices of this $K_{2,3}$-folder $\mathcal{F}_{1'}$ are given by $p_{1'}$, $p_{1'} + 2\chi^{1'}_{2'3'}$, $p_{1'} + \chi^{i1'}_{jk2'3'}$ $(\{i, j, k\} = \{1, 2, 3\})$ for some $p_{1'} \in e_{1'} \cap A_\mu$. The edge $e_{1'}$ belongs to three 2-faces $F_{11'}, F_{21'}, F_{31'}$. The orbit started from $[p_{1'} + 2\chi^{1'}_{2'3'}, p_{1'} + \chi^{11'}_{231'2'}]$ through 2-face $F_{11'}$ with direction $\chi^{11'}_{232'3'}$ escapes into a boundary edge by the same argument as in (1). However, the orbit started from $[p_{1'}, p_{1'} + \chi^{11'}_{232'3'}]$ with direction $\chi^{12'3'}_{231'}$ may meet another nonboundary edge. Suppose that this orbit meets an l_1-edge. Then this edge is in the boundary of T_μ, or belongs to another 2-face F'. For the latter case, the orbit further goes through F' and escapes into the boundary; see Figure 10 (c-2). Suppose that this orbit meets l_∞-edge e_1, and thus meets a $K_{2,3}$-folder, which is denoted by $\mathcal{F}_1$. Then five vertices of this $K_{2,3}$-folder $\mathcal{F}_1$ are given by p_1, $p_1 + 2\chi^1_{23}$, $p_1 + \chi^{1i'}_{23j'k'}$ $(\{i', j', k'\} = \{1', 2', 3'\})$.

Similarly, this $K_{2,3}$-folder $\mathcal{F}_1$ meets three 2-faces $F_{11'}$, $F_{12'}$, and $F_{13'}$. Again, the orbit started from $[p_1 + 2\chi^1_{23}, p_1 + \chi^{1i'}_{23j'k'}]$ for $\{i', j', k'\} = \{1', 2', 3'\}$ escapes into the boundary. In the 2-face $F_{12'}$, the orbit started from $[p_1, p_1 + \chi^{12'}_{231'3'}]$ escapes into the boundary, or meets l_∞-edge $e_{2'}$ and thus a $K_{2,3}$-folder $\mathcal{F}_{2'}$. Consider the latter case. Then five vertices of this $K_{2,3}$-folder $\mathcal{F}_{2'}$ are given by $p_{2'}$, $p_{2'} + 2\chi^{2'}_{1'3'}$, $p_{2'} + \chi^{i2'}_{jk1'3'}$ ($\{i,j,k\} = \{1,2,3\}$). Similarly, $\mathcal{F}_{2'}$ meets three 2-faces $F_{12'}$, $F_{22'}$, and $F_{32'}$. Again the orbit started from $[p_{2'} + 2\chi^{2'}_{1'3'}, p_{2'} + \chi^{i2'}_{jk1'3'}]$ ($\{i,j,k\} = \{1,2,3\}$) escapes into the boundary. In the 2-face $F_{22'}$, the orbit started from $[p_{2'}, p_{2'} + \chi^{22'}_{131'3'}]$ escapes into the boundary, or meets l_∞-edge e_2 and thus meets a $K_{2,3}$-folder $\mathcal{F}_2$. Suppose the latter case. This $K_{2,3}$-folder $\mathcal{F}_2$ meets three 2-faces $F_{21'}$, $F_{22'}$, and $F_{23'}$. Therefore, $\mathcal{F}_{1'}$ and $\mathcal{F}_2$ share the common 2-face $F_{21'}$. Project four 2-faces $F_{11'}$, $F_{12'}$, $F_{22'}$, $F_{21'}$ by the restriction map $(\cdot)|_{\{1,1'\}} : \mathbf{R}^S \to \mathbf{R}^{\{1,1'\}}$. By Lemma 3.9, this is an injection, and we obtain a tiling by these four 2-faces in the plane. Then edges $e_{1'}$ and $e_{2'}$ have the same 1-th coordinate, and edges e_1 and e_2 have the same $1'$-th coordinate in the plane $\mathbf{R}^{\{1,1'\}}$. Indeed, since $\{p(i) + p(j) = \mu(i,j), (1 \leq i < j \leq 3)\}$ has full rank, the value $p(1)$ is constant in $e_{1'} \cup e_{2'}$. Therefore, in $F_{21'}$, the orbit started from $[p_2, p_2 + \chi^{21'}_{132'3'}]$ returns back to the first $K_{2,3}$-folder $\mathcal{F}_{1'}$; see Figure 10 (c-1).

By the same argument, the orbits started from $\mathcal{F}_{1'}$, $\mathcal{F}_2$, $\mathcal{F}_1$, $\mathcal{F}'_2$ toward other directions escape to the boundary, or meet other $\mathcal{F}_j$ and return to themselves.

Summarizing these arguments, this orbit O meets a subset of six $K_{2,3}$-folders $\{\mathcal{F}_j\}_{j \in S}$. Suppose that O meets all six $K_{2,3}$-folders. Glue six $K_{2,3}$-folders according to map ϕ^O (the contraction of all edges of Γ_μ not belonging to O). The resulting polyhedral complex is nothing but Figure 6. Thus, the corresponding orbit graph is $\Gamma_{3,3}$. Suppose that some of the orbits escape into the boundary instead of meeting other $K_{2,3}$-folders $\mathcal{F}_j$. The resulting polyhedral complex consists of a proper subset of $K_{2,3}$-folders $\{\mathcal{F}_j\}_{j \in S}$ and squares. A square appears as in the case of Figure 10 (c-2). Namely, the orbit started from $[p_{1'}, p_{1'} + \chi^{11'}_{232'3'}]$ hits an l_1-edge in $F_{11'}$ with direction $\chi^{12'3'}_{231'}$, goes through the adjacent 2-face F', and escapes into the boundary, and the orbit started from $[p_{1'}, p_{1'} + \chi^{21'}_{131'2'}]$ goes through 2-faces $F_{21'}$, $F_{22'}$, $F_{1'2}$, and F', crosses the above orbit in F' and escapes into the boundary. The metric space obtained by gluing these three $K_{2,3}$-folders $\mathcal{F}_{1'}, \mathcal{F}_2, \mathcal{F}_{2'}$ and one square is a submetric of the metric space obtained by gluing four $K_{2,3}$-folders $\mathcal{F}_{1'}, \mathcal{F}_2, \mathcal{F}_{2'}, \mathcal{F}_1$.

Consequently, the orbit graph is an isometric subgraph of $\Gamma_{3,3}$. We complete the proof of Theorem 2.7.

5 Remarks

In this section, we give several remarks.

5.1 H-packing by cut and $K_{2,3}$-metrics

Recall Proposition 3.3 that for a commodity graph H without matching of size n, the tight span of an arbitrary H-minimal metric is at most $(n-1)$-dimensional. So it would be valuable to point out a further connection between commodity graph H and tight spans of H-minimal metrics.

The graphs K_4, C_5, and the union of two stars are exactly graphs without $K_2 + K_3$ and $K_2 + K_2 + K_2$ [22]; also see [23, Theorem 72.1]. How does this condition reflect the tight span of an H-minimal metric ? The answer is:

Proposition 5.1. *Let $H = (S, R)$ be a simple graph having no $K_2 + K_3$ and $K_2 + K_2 + K_2$, and let μ be an H-minimal metric on S. Then T_μ has no l_∞-edges. Consequently, every orbit graph of Γ_μ is K_2.*

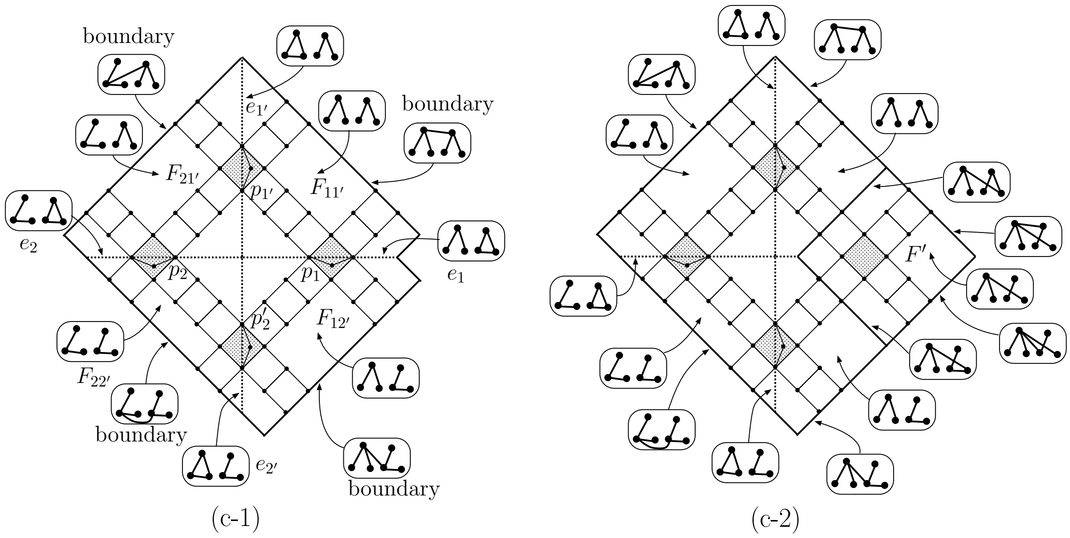


Figure 10: The orbits started from $K_{2,3}$-folder of (case 1)

.

Proof. Use Lemma 3.4 as in proof of Proposition 3.3 or Lemma 4.1. □

From this, we obtain Karzanov's half-integral cut packing theorem (Theorem 1.1). This is a tight-span interpretation of the shape of a commodity graph H admitting cut packing.

The next ask is: what happens for the case that H has at most five vertices or is the union of K_3 and a star ? In this case, the tight span of any H-minimal metric has no core; use Lemma 3.4. Therefore, $K_{n,m}$-folders for $n, m \geq 3$ do not appear. However, l_∞-edge e may exist. Then K_e consists of one bipartite component and one complete multipartite component having three parts; the proof is again similar to that of Lemma 4.1. Therefore, the tight span is the union of squares and $K_{2,3}$-folders. By tracing the orbit of $K_{2,3}$-folders as in the proof of Theorem 2.7, one can show that the orbit graph is $K_{2,3}$. From this, we obtain Karzanov's half-integral $K_{2,3}$-metric packing theorem (Theorem 1.2).

5.2 The case $\dim T_\mu \geq 3$

Here, we explain why metric spaces $(T_\mu \cap A_\mu, l_\infty)$ arising from 3-dimensional tight spans cannot be decomposed into finite types of metrics.

Let Γ_L be the graph of L obtained by connecting a pair of points having the unit l_∞-distance. If L is in the plane, then Γ_L is a grid graph, and every submetric of d_{Γ_L} can be decomposed into cut metrics. On the other hand, if the lattice L in 3-dimensional space, then there are infinitely many extreme submetrics in d_{Γ_L}, where a metric is called *extreme* if it lies on an extreme ray of the metric cone. For example, consider the subgraph $\Gamma_{Q_k \cap L}$ of Γ_L induced by the lattice points $Q_k \cap L$ in the affine 3-cube

$$Q_k = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid 0 \leq x_i + x_j \leq 2k \ (1 \leq i < j \leq 3)\}$$

for a positive integer k. One can show that $\Gamma_{Q_k \cap L}$ is an isometric subgraph of Γ_L, and the corresponding graph metric $d_{\Gamma_{Q_k \cap L}}$ is extreme. Indeed, there are eight points in $Q_1 \cap L$

$$\{(0,0,0), (-1,1,1), (1,-1,1), (1,1,-1), (2,0,0), (0,2,0), (0,0,2), (1,1,1)\},$$

and therefore the graph $\Gamma_{Q_1 \cap L}$ is the cube plus one diagonal edge; this graph appears in [14, Fig 4 (b)]. It is extreme by Avis' criterion [1]. Since Q_k is obtained by piling Q_1's, by Avis' criterion again, $d_{Q_k \cap L}$ is also extreme. Moreover one can show that for each k there is a metric μ with $\dim T_\mu \geq 3$ such that Γ_μ has $Q_k \cap L$ as an isometric subgraph. Therefore, the graph metric d_{Γ_μ} arising from 3-dimensional tight spans cannot be decomposed into finite types of metrics. Consequently, the commodity graph H having $K_2 + K_2 + K_2$ cannot be packed by finite types of metrics.

5.3 The folder decomposition, modular closures, and $T_\mu \cap A_\mu$

The folder decomposition of tight spans has already been obtained by Karzanov [18] via the method of modular closures. Here, we explain some relation among the folder decomposition, modular closures, and the point set $T_\mu \cap A_\mu$. We need some terminology related to modular metrics. The *least generating graph* (LG-graph) of a metric (S, μ) is the graph on S obtained by connecting a pair $i, j \in S$ if there is no $k \in S \setminus \{i, j\}$ such that $\mu(i, j) = \mu(i, k) + \mu(k, j)$. A metric μ is called *modular* if for any triple $k_1, k_2, k_3 \in S$ there exists $k^* \in S$, called a *median*, such that $\mu(k_i, k_j) = \mu(k_i, k^*) + \mu(k^*, k_j)$ for $1 \leq i < j \leq 3$. A *modular closure* $(V, \tilde{\mu})$ of a metric (S, μ) is a certain minimal modular metric containing μ as a submetric. It is constructed by the following procedure. Initially, set $V := S$ and $\tilde{\mu} := \mu$. Choose a triple $s_1, s_2, s_3 \in V$ without a median, add a new point s^* to V and define the (unique) distances from s^* to the s_k's by

$$\tilde{\mu}(s^*, s_k) = (\tilde{\mu}(s_k, s_i) + \tilde{\mu}(s_k, s_j) - \tilde{\mu}(s_i, s_j))/2 \qquad (5.1)$$

for $\{i, j, k\} = \{1, 2, 3\}$. Then define distances from s^* to other points $V \setminus \{s_1, s_2, s_3\} = \{s_4, s_5, \ldots, s_n\}$ by

$$\tilde{\mu}(s^*, s_k) = \max_{1 \leq i < k} \{\tilde{\mu}(s_k, s_i) - \tilde{\mu}(s^*, s_i)\} \quad (4 \leq k \leq n). \qquad (5.2)$$

Repeat this process for another medianless triple in the current $(V, \tilde{\mu})$ until there is no medianless triple, i.e., $\tilde{\mu}$ is modular. Note that a modular closure $\tilde{\mu}$ depends on the choice of a medianless triple $\{s_1, s_2, s_3\}$ and the ordering of $V \setminus \{s_1, s_2, s_3\}$. Karzanov [18] has shown that T_μ is 2-dimensional if and only if the LG-graph of a modular closure of metric μ is hereditary modular having no $K_{3,3}^-$ as an isometric subgraph. Here a graph is called *hereditary modular* if all isometric subgraphs are modular, and $K_{3,3}^-$ is $K_{3,3}$ minus one edge. It is known that a graph is hereditary modular if and only if it is bipartite, and has no isometric cycles of length greater than four [3]. Furthermore, Karzanov [18] has shown that if $\dim T_\mu \leq 2$, then T_μ is obtained by filling *folders* appropriately into isometric subgraphs $K_{n,m}$ $(n, m \geq 2)$ of the LG-graph of a modular closure of μ as in Figure 5. Interestingly, a modular closure of μ is unique if $\dim T_\mu \leq 2$.

Our approach to obtain the folder decomposition might be regarded as a converse to this modular closure approach. In fact, one can show that if $\dim T_\mu \leq 2$, then Γ_μ is hereditary modular without $K_{3,3}^-$. Moreover, a modular closure of a cyclically even metric μ is a submetric of $(T_\mu \cap A_\mu, l_\infty)$. By construction, the restriction $(\tilde{\mu}_i)|_S$ of $\tilde{\mu}_i \in \mathbf{R}^V$ belongs to $P_\mu \cap A_\mu$ for $i \in V$, and a modular closure $(V, \tilde{\mu})$ is a *tight extension* of (S, μ), i.e., there is no metric $\tilde{\mu}'(\neq \tilde{\mu})$ on V such that $\tilde{\mu}' \leq \tilde{\mu}$ and $\tilde{\mu}'|_S = \mu|_S$; see [9]. Therefore $(\tilde{\mu}_i)|_S \in T_\mu$ and $\|(\tilde{\mu}_i)|_S - (\tilde{\mu}_j)|_S\|_\infty = \tilde{\mu}(i, j)$ for $i, j \in V$ necessarily hold [9, Theorem 3], and thus $(V, \tilde{\mu}) = (\{(\tilde{\mu}_i)|_S\}_{i \in V}, l_\infty)$ is a submetric of $(T_\mu \cap A_\mu, l_\infty)$.

5.4 An $O(n^2)$ algorithm for $K_3 + K_3$-packings

The proof of the main theorem is constructive, and therefore yields a strongly polynomial time for $K_3 + K_3$-packing problems by careful modifications. Here we give an $O(n^2)$

algorithm, where n is the cardinality of vertices of graph $G = (V, E)$. The essential idea is the same as Chepoi's $O(n^2)$ algorithm for cut and $K_{2,3}$-metric packings [5].

Let $G = (V, E)$ be a graph, $H = (S, R) = K_3 + K_3$ a commodity graph on $S \subseteq V$, and l a cyclically even length function. An algorithm of H-packing of (G, l) by $\mathit{\Gamma}_{3,3}$-metrics is the following:

(s1) Calculate $d_{G,l}(i, j)$ for $(i, j) \in S \times V$. Let μ be the restriction of $d_{G,l}$ to S.

(s2) Take a cyclically even H-minimal metric μ^* on S such that $\mu^* \leq \mu$, $\mu^*(k, l) = \mu(k, l)$ for each $kl \in R$, and $A_\mu = A_{\mu^*}$; see Lemma 2.6.

(s3) Construct T_{μ^*}.

(s4) Define vectors $U := \{p_k\}_{k \in V} \subseteq P_{\mu^*}$ by (2.5-2.6). Calculate $\phi(U)$ for a nonexpansive retraction $\phi : P_{\mu^*} \cap A_{\mu^*} \to T_{\mu^*} \cap A_{\mu^*}$ in Proposition 2.4.

(s5) Decompose finite metric $(\phi(U), l_\infty)$ into $\mathit{\Gamma}_{3,3}$-metrics.

Note that this algorithm works for any commodity graph H without $K_2 + K_2 + K_2$.

Let us consider the complexity. Note that the size of H is constant. (s1) can be done in $O(n \log n)$ time by Dijkstra algorithm. (s2) can be done in the constant time; see the proof of Lemma 2.6. Also (s3) can be done in the constant time. Indeed, since P_{μ^*} is a 6-dimensional polyhedron defined by 21 inequalities, the number of faces is constant, and thus we can calculate all extreme points, all edges, all 2-faces, and their incidence structure in the constant time. For (s4), the proof of Proposition 2.4 gives an $O(n)$ algorithm. Indeed the calculation of 6-dimensional vector $\phi_i(p_k)$ can be done in constant time. Consider (s5). The size of the graph $\mathit{\Gamma}_{\mu^*}$ is not polynomially bounded by $\log(\sum_{i,j} \mu(i, j))$. Therefore, a naive approach to retain $\mathit{\Gamma}_{\mu^*}$ does not work. Instead, we retain the incidence structure of all faces of T_{μ^*} and the local 2-dimensional coordinate of each point $\phi(p_k)$ in $F(\phi(p_k))$. For each $\phi(p_k)$, we can identify $F(\phi(p_k))$ in constant time; this is a membership problem in 6-dimensional space. So this can be done in $O(n)$ time.

To identify orbit graphs of $\mathit{\Gamma}_{\mu^*}$, we trace orbits with their *width*. First, we consider the simplest case where T_{μ^*} has no l_∞-edges. Note that the existence of an l_∞-edge or a core can be checked in (s3). Take an arbitrary l_1-edge $e = [p, q]$. Take an endpoint p and a point $p^\epsilon := p + \epsilon(q - p)$ for small width $\epsilon > 0$. Draw two lines from p and p^ϵ with l_1-direction orthogonal to e until escaping into the boundary of T_{μ^*}. Increase width ϵ until the line started from p^ϵ meets a point in $\phi(U)$ or an extreme point of T_{μ^*}. This can be done in $O(n)$ time by operating it in each face. Note that such ϵ is integral. Consider the strip sandwiched by the lines. The relative interior of this strip has no points in $\phi(U)$. Delete the relative interior of this strip from T_{μ^*}. Then the resulting set consists of two connected components, which yields a bipartition of $\phi(U)$ and a cut-metric summand of an H-packing with integral coefficient ϵ. We can determine this bipartition in $O(n)$ time (by using the incidence information of faces). Next glue this polyhedral set along the boundary of this deleted strip by translating two components together with $\phi(U)$. This can be done in $O(n)$ time (by operating it in each face). Then we obtain a 2-dimensional polyhedral set smaller than T_{μ^*}. Repeat the same process to this set until it becomes one point. Then we obtain an integral H-packing (by cut metrics). The number of the strip-deletion steps will be analyzed later.

Second, we consider the case where T_{μ^*} has l_∞-edges and has no odd core. The idea is the same as above. Take an l_∞-edge $e = [p, q]$. Take an endpoint p of e and a point $p^\epsilon := p + \epsilon(q - p)$ for small $\epsilon > 0$. Draw lines having l_1-direction started from p and p^ϵ as in Figure 11 (a). Increase width ϵ until lines started from p^ϵ meets a point in $\phi(U)$

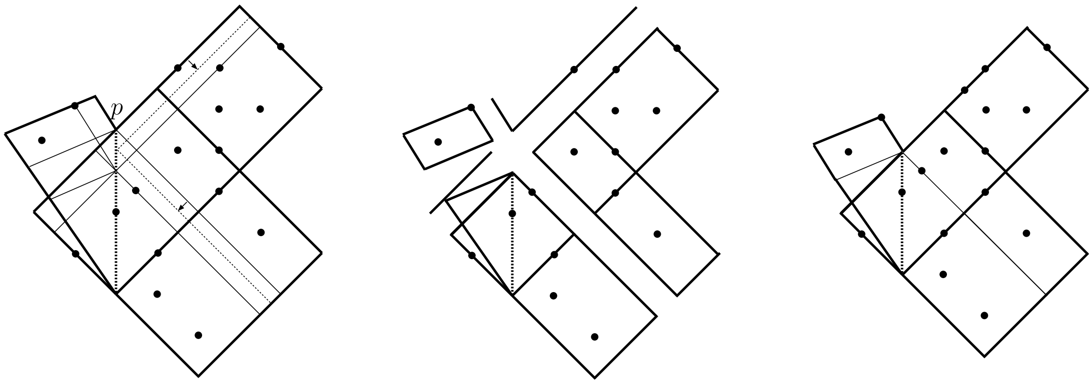


Figure 11: (a) the strip generated by $K_{2,3}$-folder, (b) deleting the strip, and (c) gluing the components

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or an extreme point of T_{μ^*}. Note that such ϵ is an even integer. Then delete the strip sandwiched by these lines (Figure 11 (b)). The resulting components yields a partition of $\phi(U)$, Again this partition can be determined in $O(n)$ time. Then we obtain a summand of an H-packing, which is a $K_{2,3}$-metric or a submetric of a $\Gamma_{3,3}$-metric, with integral coefficient $\epsilon/2$. Gluing these components along the strip (Figure 11 (b)). Repeat this process until all l_∞-edge vanish. The remaining argument reduces to the first case.

Finally, we consider the case where T_{μ^*} has an odd core p. Note that this odd core is a unique core (Lemma 4.2). Consider the $K_{3,3}$-folder containing p, delete the strip generated by this $K_{3,3}$-folder; recall Figure 9 (a). The resulting set consists of six connected components, which gives a unique $K_{3,3}$-metric summand with coefficient 1. Gluing these connected components, the remaining argument reduces to the second case.

The number of the strip-deletion steps is bounded by $O(n)$ times. Indeed, consider all lines having l_1-directions started from $\phi(U)$ and extreme points of T_{μ^*}. The number of such lines is $O(n)$. Indeed, each face has at most two lines started from each point $\phi(p_k)$; one can verify this fact by the tracing from a point in T_{μ^*} as in Section 4. Each strip-deletion step decreases number of such lines. Consequently, we can conclude that (s5) can be done in $O(n^2)$ time, and that a desired integral H-packing can be obtained by $O(n^2)$ time.

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