Tight spans, Metric labeling, and Multicommodity flows

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Aim of talk is to discuss an interrelationship among:

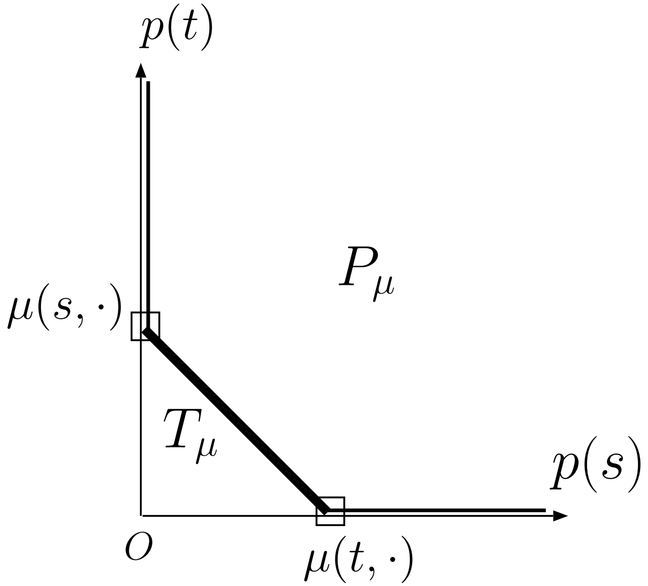
- Tight spans (Isbell 64, Dress 84)
  - Phylogenetic tree/network in biology.
- Metric labeling, and 0-extensions
  - Pattern recognitions and classifications.
  - Image restoration in computer vision.
- Multicommodity flows
  - Combinatorial optimization, network flows.

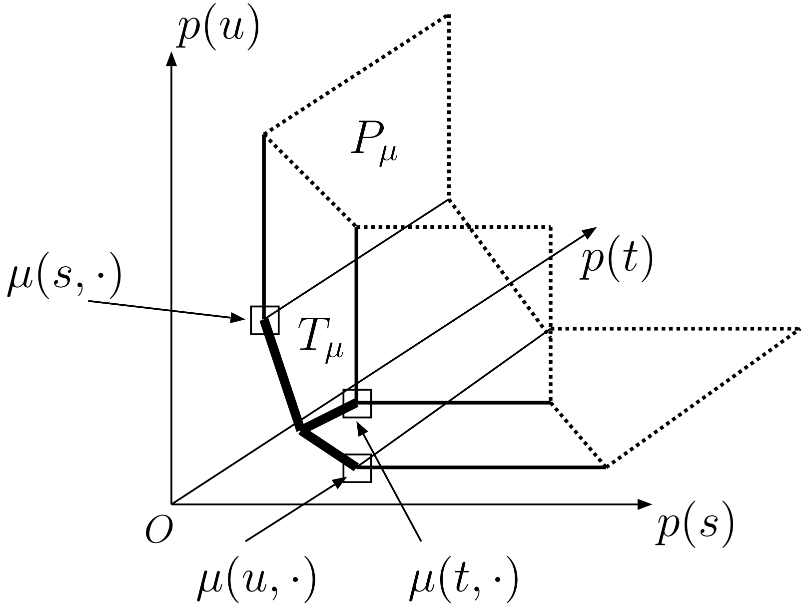
Tight spans (Isbell 64, Dress 84)

μ: a metric on a set S.

$$P_\mu := \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\},$$
$$T_\mu := \text{the set of minimal elements of } P_\mu.$$

T_μ: the *tight span* of μ.





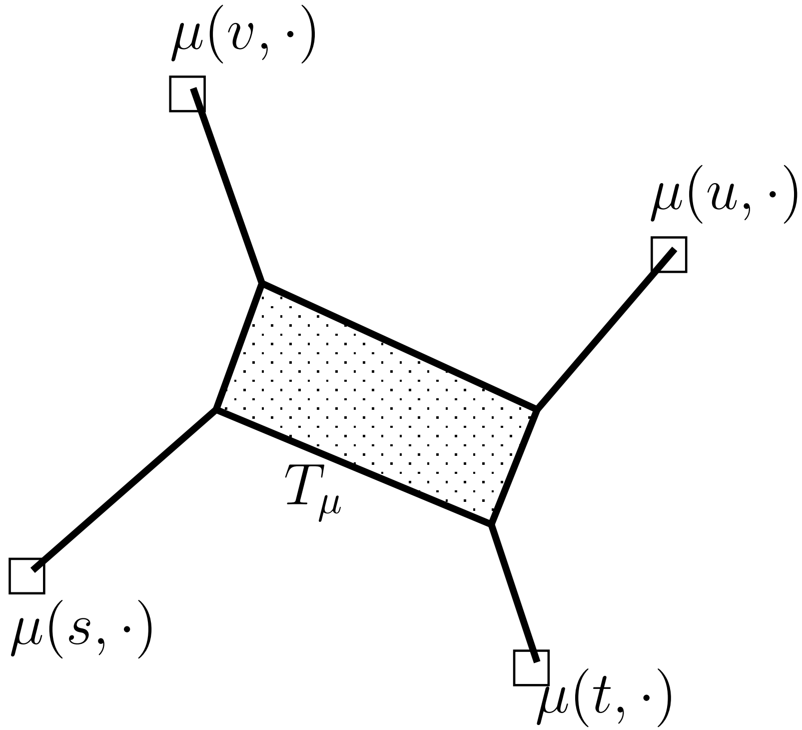
Lemma [Isbell 64, Dress 84]

(S,μ) is isometrically embeded into (T_μ, l_∞) by $s \mapsto \mu(s,\cdot) \in \mathbf{R}^S$ $(s \in S)$.

Theorem [Dress 84]
A metric μ is a tree metric if and only if T_μ is a tree.

$\longrightarrow$ T_μ is a kind of *a higher dimensional tree*.

$\longrightarrow$ phylogenetic trees in biology.



Why "tight" ?

(S,μ), (X,d): metric spaces.

(X,d): an *extension* of (S,μ) $\overset{\text{def}}{\Longleftrightarrow}$ $S \subseteq X$ and $d|_S = \mu$.

(X,d): a *tight extension* of (S,μ)
$\overset{\text{def}}{\Longleftrightarrow}$ (X,d) is an extension s.t.
$\forall$ extension (X,d') of (S,μ) with $d' \leq d \Rightarrow d' = d$.

Theorem [Isbell 64, Dress 84]

- (T_μ, l_∞) is a tight extension of (S,μ), and
- Every tight extension of (S,μ) is isometrically embeded into (T_μ, l_∞)

$\rightarrow$ T_μ is the *universal tight extension*.

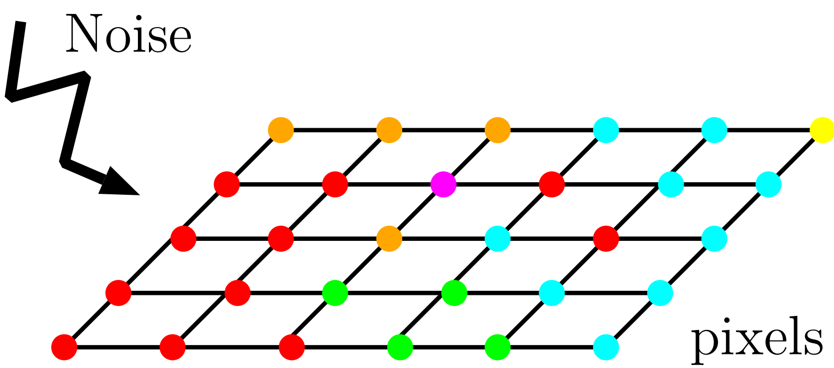
Metric labeling problem (Kleingberg & Tardos 02)

μ: a metric on a set of labels S,
$G = (V, E, c)$: a graph with edge weight $c \geq 0$,
$f : V \times S \to \mathbf{R} \cup \{+\infty\}$ (assignment cost).

$$\begin{array}{rl} \text{Minimize} & \displaystyle\sum_{x \in V} f(x, \rho(x)) + \sum_{xy \in E} c(xy)\, \mu(\rho(x), \rho(y)) \\ \text{subject to} & \rho : V \to S \text{ (assignment of labels)} \end{array}$$

Image restoration in computar vision

(Ishikawa and Geiger 99, Boykov et al. 01)



- Modeling by Markov Random Field (MRF)

In many cases, MLP reduces to O-extension problem.
Suppose $S \subseteq V$.

$$\begin{array}{rl} \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\, \mu(\rho(x), \rho(y)) \\ \text{subject to} & \rho : V \to S, \rho|_S = id_S \end{array}$$

$$\begin{array}{rl} \simeq \text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)\, d(x, y) \\ \text{subject to} & d\text{: metric on } V \text{ with } d|_S = \mu \\ & \forall x \in V \exists s \in S, d(x, s) = 0 \end{array}$$

Metric labeling and 0-extension are NP-hard

- Good heuristics (Boykov et al. 01)
- Approximation algorithms
  (Kleinberg and Tardos 02, Calinescu et al. 04, $\cdots$)
- Polynomially-solvable classes (Karzanov 98, 04)

Karzanov's LP-relaxation for 0-extension (Karzanov 98)

$$
\begin{array}{rl}
\text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)d(x,y) \\
\text{subject to} & (V,d)\text{: 0-extension of } (S,\mu) \\
\Longrightarrow & \\
\text{Minimize} & \displaystyle\sum_{xy \in E} c(xy)d(x,y) \\
\text{subject to} & (V,d)\text{: extension of } (S,\mu)
\end{array}
$$

When does this relaxation exactly solves the 0-extension ?

In the case of graph metric

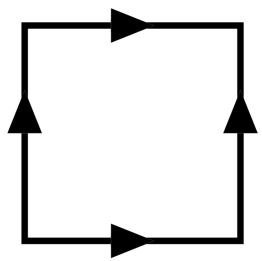
Γ: a graph

d_Γ: graph metric of Γ

Theorem [Karzanov 98]

The LP-relaxation solves 0-extension for $(\forall G; S, d_\Gamma)$ exactly

$\Leftrightarrow$ Γ is bipartite, orientable, and has no isometric k-cycle ($k \geq 6$).



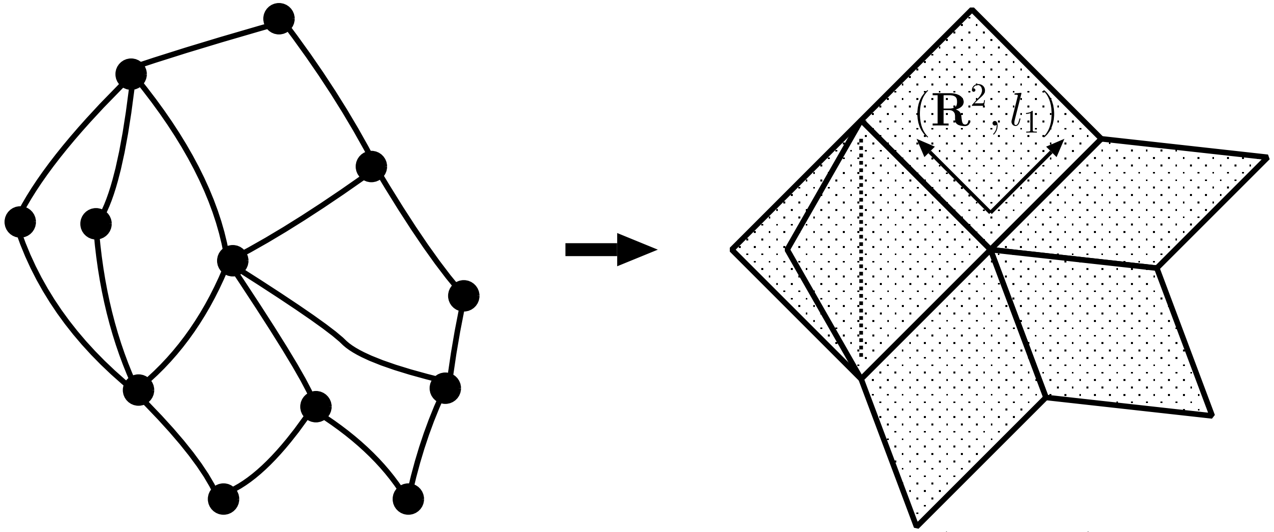
Such a graph is called a *frame* (including trees, grids, $\cdots$).

Proof sketch (Karzanov 98)

$$\text{Min.} \sum_{xy \in E} c(xy) d(x,y) \qquad \simeq \qquad \text{Min.} \sum_{xy \in E} c(xy) \|\rho(x) - \rho(y)\|_\infty$$

s. t. d: a tight extension of μ $\qquad$ s. t. $\rho : V \to T_\mu,\ \rho(s) = \mu(s, \cdot)\ (s \in S)$.

- For a frame Γ, the tight span T_{d_Γ} is obtained by filling l_1-space into each 4-cycle.



Γ $\qquad$ (T_{d_Γ}, l_∞)

Multicommodity flows (multiflows)

$G = (V, E, c)$: a graph with nonnegative edge capasity c.

$S \subseteq V$: a set of terminals.

A *multiflow* $f = (\mathcal{P}, \lambda) \overset{\text{def}}{\Longleftrightarrow}$

$\mathcal{P}$: a set of S-paths,

$\lambda : \mathcal{P} \to \mathbf{R}_+$: a flow-value function satisfying capasity constraint

$$\sum_{P \in \mathcal{P}: e \in P} f(P) \leq c(e) \quad (e \in E).$$

Maximum multiflow problem

Given terminal weight $\mu : S \times S \to \mathbf{R}_+$ with $\mu(s,t) = \mu(t,s) \geq \mu(s,s) = 0$.

$$\begin{array}{ll} \text{Max.} & \sum_{P \in \mathcal{P}} \mu(s_P, t_P) f(P) \\ \text{s. t.} & f = (\mathcal{P}, \lambda)\text{: a multiflow,} \end{array}$$

where s_P, t_P: endpoints of P.

LP-dual to maximum multiflow problem
(Onaga-Kakusho 71, Iri 71, Lomonosov 85):

$$\begin{array}{rl} \text{Min.} & \sum_{xy \in E} c(xy) d(x,y) \\ \text{s. t.} & d\text{: metric on } V \text{ with } d|_S \geq \mu \end{array}$$

When μ is metric,

$$\begin{array}{rl} \text{Min.} & \sum_{xy \in E} c(xy) d(x,y) \\ \text{s.t.} & d\text{: metric on } V \text{ with } d|_S = \mu \end{array}$$

This is just Karzanov's LP-relaxation of 0-extension for μ !

$$\begin{array}{rl} \simeq \text{Min.} & \sum_{xy \in E} c(xy) \|\rho(x) - \rho(y)\|_\infty \\ \text{s.t.} & \rho : V \to T_\mu,\ \rho(s) = \mu(s, \cdot) \quad (s \in S). \end{array}$$

$\rightarrow$ *Nonmetric version ?*

Observation [H. 06]:
tight spans are definable for nonmetric distances

Theorem [H.07, to appear in JCTB]

Max. multiflow value for $(G; S, \mu) =$

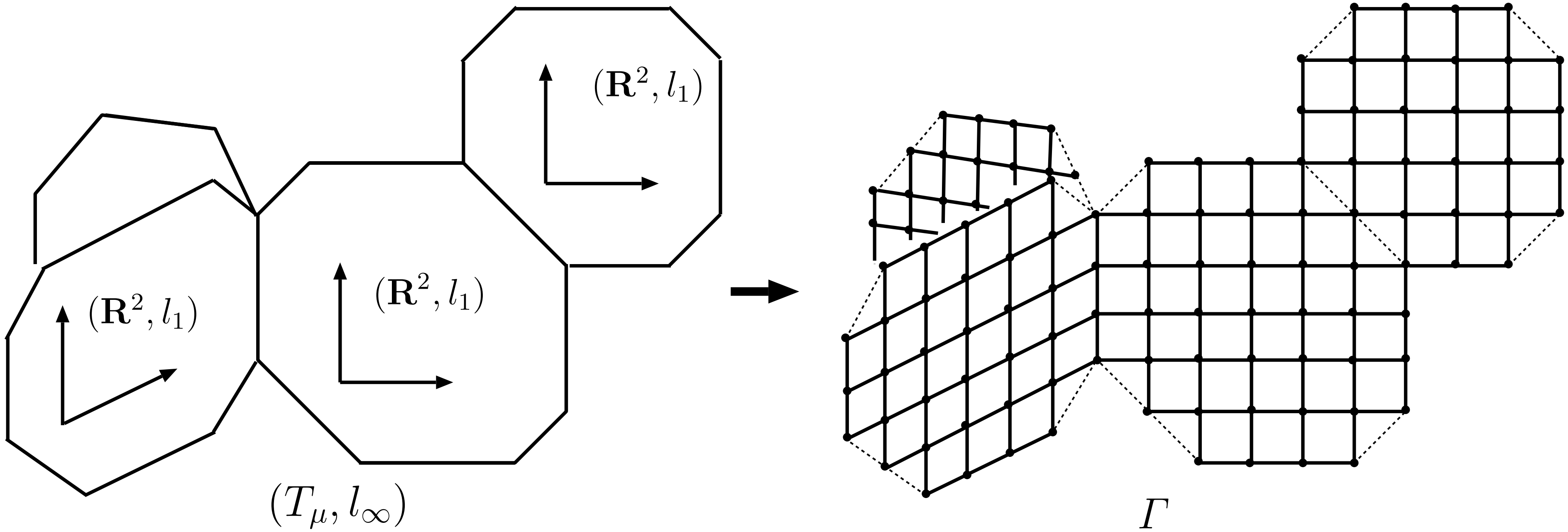
$$\begin{aligned} \text{Min.} \quad & \sum_{xy \in E} c(xy) \| \rho(x) - \rho(y) \|_\infty \\ \text{s. t.} \quad & \rho : V \to T_\mu, \ \rho(s) \in T_{\mu,s} \quad (s \in S), \\ & T_{\mu,s} := \{ p \in T_\mu \mid p(s) = 0 \} \ (s \in S). \end{aligned}$$

Theorem [H.07] μ:rational

$\dim T_\mu \leq 2 \Leftrightarrow \exists$ graph Γ on T_μ and $k \in \mathbf{Z}_{>0}$ such that

$$\begin{aligned} & \text{Max. multiflow value for } (G; S, \mu) \\ = \ \text{Min.} \quad & \frac{1}{k} \sum_{xy \in E} c(xy) \ d_\Gamma(\rho(x), \rho(y)) \\ \text{s.t.} \quad & \rho : V \to V\Gamma, \ \rho(s) \in V\Gamma \cap T_{\mu,s} \ (s \in S). \end{aligned}$$

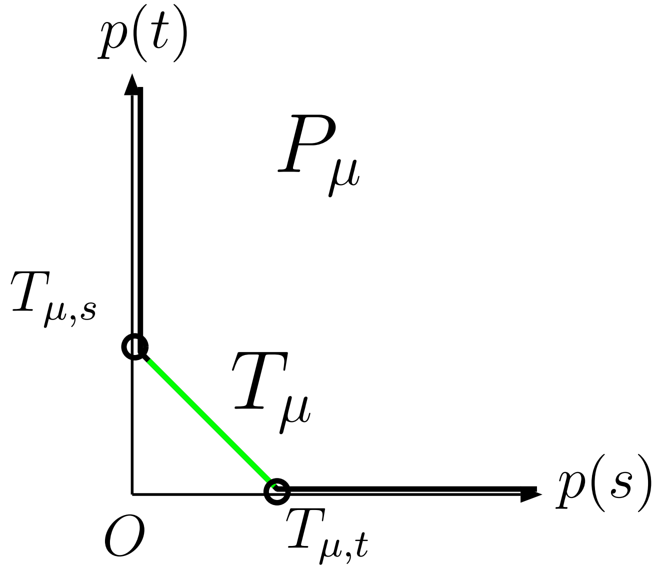
Proof sketch (H. 07)



Single-commodity flows

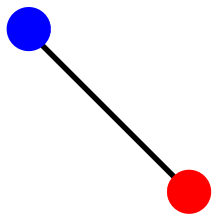
$\mu =$

	s	t
s	0	1
t	1	0



Max flow value =

$$\text{Min.} \quad \sum_{xy \in E} c(xy)\,\text{dist}\,(\rho(x), \rho(y)) \quad \text{s. t.} \quad \rho : V \to$$



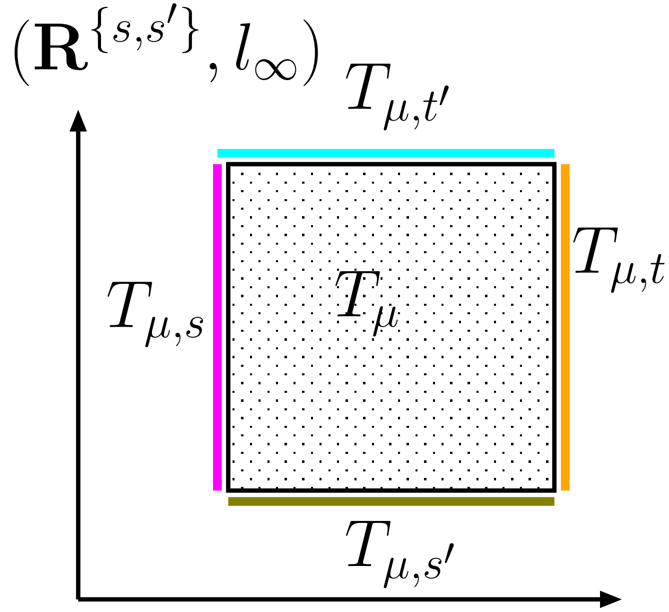
$$\rho(s) = \bullet \qquad \rho(t) = \bullet$$

$\to$ Max-flow Min-cut theorem by Ford-Fulkerson (1954)

Two-commodity flows

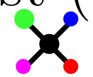
$\mu =$

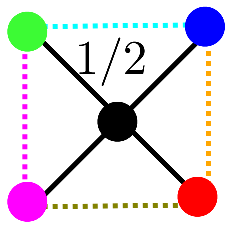
	s	s'	t	t'
s	0	0	1	0
s'	0	0	0	1
t	1	0	0	0
t'	0	1	0	0



Max flow value =

$$\text{Min.} \quad \sum_{xy \in E} c(xy)\, \text{dist} \left(\rho(x), \rho(y)\right) \quad \text{s. t.} \quad \rho : V \to$$





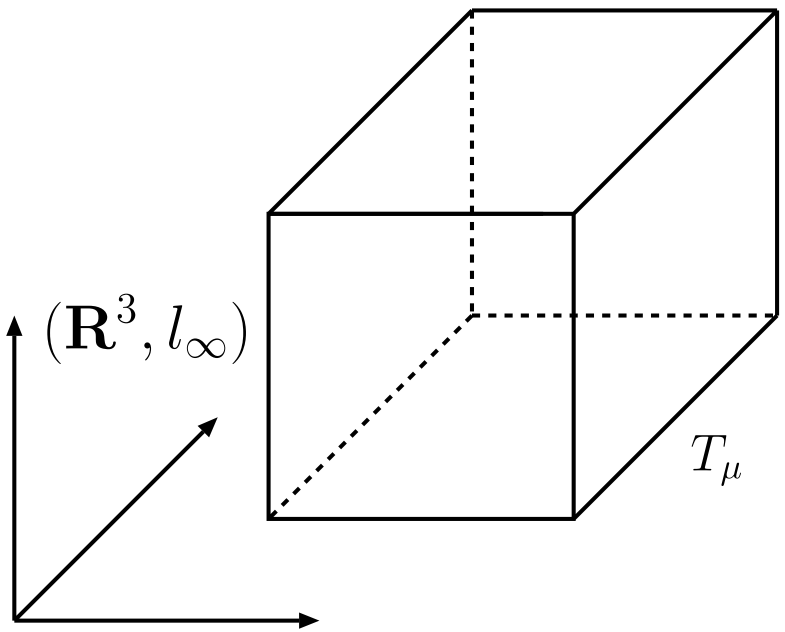
$\rho(s) \in \{$●, ●$\}$
$\rho(t) \in \{$●, ●$\}$
$\rho(s') \in \{$●, ●$\}$
$\rho(t') \in \{$●, ●$\}$

→ Max-biflow Min-cut theorem by Hu (1963)

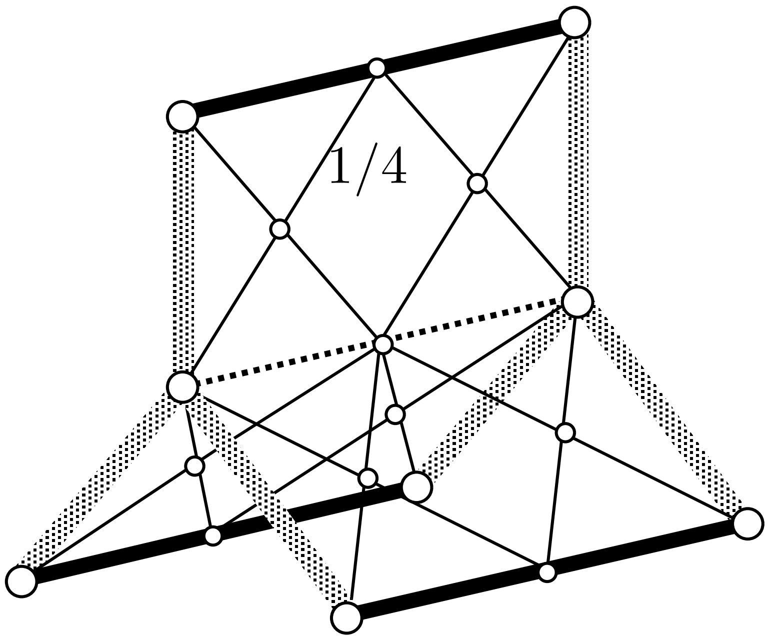
Three-commodity flow
has no combinatorial duality theorem since $\dim T_\mu \geq 3$.

$\mu =$

	s_1	s_2	s_3	t_1	t_2	t_3
s_1				1		
s_2					1	
s_3						1
t_1	1					
t_2		1				
t_3			1			



More examples



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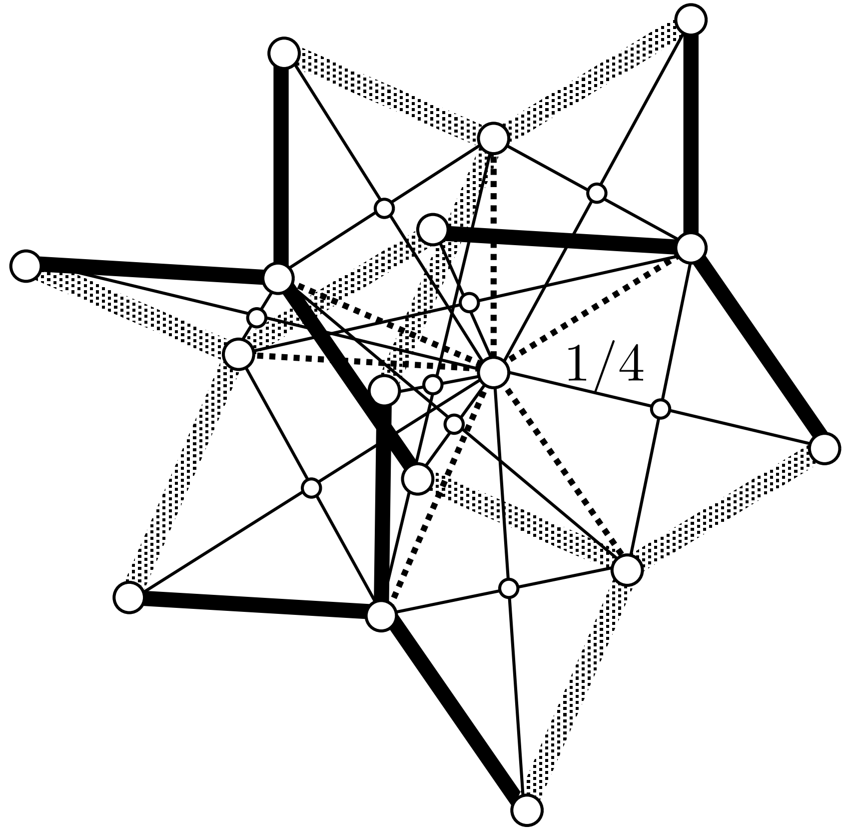

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Summery

Tight spans have a potential to provide a unified framework for metric labeling, 0-extensions, and multicommodity flows.

Future works

- Design of heuristics/approximation algorithms for metric labeling and 0-extension based on tight spans.
- Design of efficient/practical algorithms for multiflows based on tight spans.
- Fractionality problems in the multiflow theory (Karzanov 90, H. 08).