Algebraic Combinatorial Optimization for Noncommutative Rank & Determinant

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Based on joint works with
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Contents

The recent development of *Noncommutative Edmonds problem*

Its applications/implications to combinatorial optimization

Part I. Edmonds problem $\approx$ cardinality maximization

Part II. Weighted Edmonds problem $\approx$ weighted maximization

Part I

Edmonds Problem

Edmonds 1967

Compute the **rank** of *linear symbolic matrix*

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

x_k : variables,

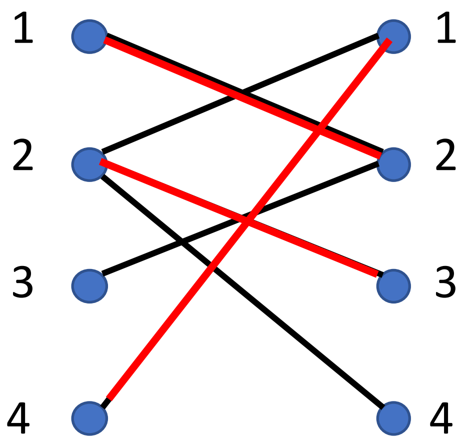
A_k: $n \times n$ matrices over field $\mathbb{K}$

A: matrix over $\mathbb{K}[x_1, x_2, \ldots, x_m] \subset \mathbb{K}(x_1, x_2, \ldots, x_m)$

- Randomized polynomial time algorithm (Lovász 1979)
- P ??
- Related to fundamental problems in diverse areas

$\sim$ combinatorial optimization, rigidity theory, TCS,…

Algebraic formulation of bipartite matching



$$G = (U, V; E)$$

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & x_{12} & & \\ 2 & x_{21} & & x_{23} & x_{24} \\ 3 & & x_{32} & & \\ 4 & & & & x_{41} \end{array} = \sum_{ij \in E} \begin{array}{c} j \\ \boxed{\begin{array}{cc} i & 1 \end{array}} \end{array} x_{ij}$$

$$= \sum_{ij \in E} e_i e_j^T x_{ij}$$

Obs. maximum matching of G = rank A

$$\because \det A = \sum_{M} \pm \prod_{ij \in M} x_{ij}$$

- min-max formula (Hall, König-Egerváry)
- polynomial time algorithm

(2x) max cardinality = rank A

0. Bipartite matching ---- $A = \sum_{ij \in E} e_i e_j^T x_{ij}$

1. Linear matroid intersection ---- $A = \sum_{k=1}^{m} a_k b_k^T \, x_k$

for $\mathbf{M}(a_1 a_2 \dots a_m)$ and $\mathbf{M}(b_1 b_2 \dots b_m)$

2. Non-bipartite matching ---- $A = \sum_{ij \in E} (e_i e_j^T - e_j e_i^T) x_{ij}$

3. Linear matroid matching ---- $A = \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T) \, x_k$

for $\mathbf{M}(a_1 b_1 a_2 b_2 \dots a_m b_m)$

∃ Polynomial time algorithm (1,2: Edmonds, 3: Lovász)

Non-commutative Edmonds Problem

Ivanyos-Qiao-Subrahmanyam 2017

Compute the rank ***(ncrank)*** of

$$A = A_1x_1 + A_2x_2 + \cdots + A_mx_m$$

x_k : noncommutative variables, i.e., $x_ix_j \neq x_jx_i$

A_k: matrices over field $\mathbb{K}$

A: matrix over $\mathbb{K}\langle x_1, x_2, \ldots, x_m\rangle$ ← noncommutative polynomial ring

$\cap$

free skew field $\mathbb{K}(\langle x_1, x_2, \ldots, x_m\rangle)$ Amitsur 1966

FACT: $\mathrm{rank}\, A \leq \mathrm{ncrank}\, A$

Nc-rank in P

- **Garg-Gurvits-Oliveira-Wigderson 2020 (FOCS 2016)**: $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$
  - Operator Sinkhorn algorithm (Gurvits algorithm)
  - Convex optimization on Hadamard manifold (symmetric space)

    → Further developments, e.g., Bürgisser et al. FOCS2019

- **Ivanyos-Qiao-Subrahmanyam 2018 (ITCS 2017)** : $\mathbb{K}$: general
  - Wong sequence ∼ algebraic analogue of augmenting path
  - Blow-up ∼ algebraic analogue of “fractional relaxation”

- **Hamada-Hirai 2021 (JH 2017):** $\mathbb{K}$: general
  - Submodular optimization on modular lattice of vector subspaces
  - Convex optimization on nonmanifold Hadamard space (CAT(0) space)

Min-max formula for nc-rank

Thm (Fortin-Reutenauer 2004)

$$\operatorname{ncrank} \textstyle\sum_k A_k x_k = 2n - \text{Max. } r + s$$

$$\text{s.t.} \quad PA_kQ = \begin{array}{cc} & \begin{bmatrix} * & * & * \\ \boxed{0} & & * \end{bmatrix} \\ r & \\ & s \end{array} \quad (\forall k)$$

$$P, Q \in \mathrm{GL}_n(\mathbb{K})$$

$$= 2n - \text{ Max. } \dim U + \dim V$$

$$\text{s.t.} \quad A_k(U, V) = \{0\} \quad (\forall k)$$

$U, V \subseteq \mathbb{K}^n$ vector subspaces

where $A_k(u, v) := u^T A_k v$

Obs [Hamada-Hirai 2021]
Submodular minimization over the modular lattice of vector subspaces

König-Egerváry v.s. Fortin-Reutenauer

Bipartite $A = \sum_{ij \in E} e_i e_j^T x_{ij}$

$$\text{ncrank}\, A = 2n - \text{max.}\ \dim U + \dim V$$
$$\text{s.t.}\quad U\, e_i e_j^T V = \{0\}\ (\forall ij \in E),$$
$$U, V \subseteq \mathbb{K}^n$$

We can assume $U = \text{span}\{e_i\}_{i \in S}, V = \text{span}\{e_j\}_{j \in T}$

$= 2n - \max\ |S| + |T|$ s.t. $S \cup T$ has no edge

$= 2n - \text{max.}\, |\text{ stable set }|$

$= \text{max.}\, |\text{ matching }| = \text{rank}\, A$

→ $\text{rank}\, A = \text{ncrank}\, A$

Linear matroid intersection: $A = \sum_{k=1}^{m} a_k b_k^T x_k$

$\max |\text{common indep. set}| = \operatorname{rank} A = \operatorname{ncrank} A$

Non-bipartite matching: $A = \sum_{ij \in E} (e_i e_j^T - e_j e_i^T) x_{ij}$

$2 \max |\text{matching}| = \operatorname{rank} A < \operatorname{ncrank} A$

$\qquad\qquad\qquad\qquad\qquad\qquad \| \quad$

$2 \max. |\text{ fractional matching }|$

Oki-Soma SODA 2023

Linear matroid matching: $A = \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T)\, x_k$

$2 \max. |\text{ matroid matching}| = \operatorname{rank} A < \operatorname{ncrank} A$

$\qquad\qquad\qquad\qquad\qquad\qquad \| \quad$

$2 \max. |\text{ fractional matroid matching }|$

Oki-Soma SODA 2023

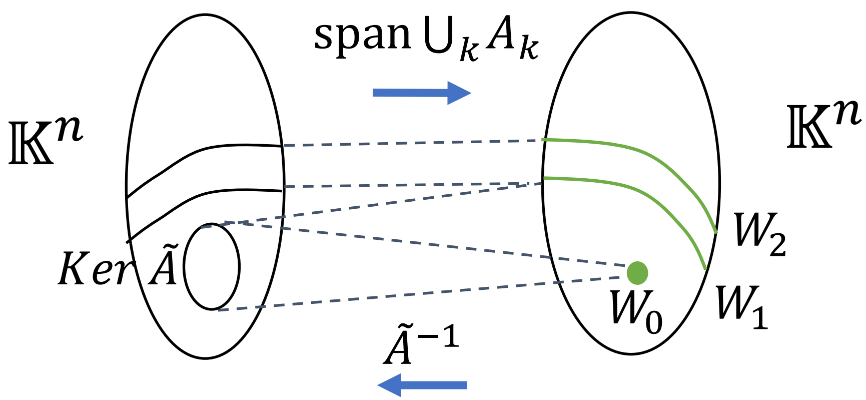
Wong sequence & Blow up

Key ideas of Ivanyos-Qiao-Subrahmanyam algorithm (**IQS-algorithm**)

Def: Wong sequence [Ivanyos-Karpinski-Qiao-Santha 2015]

For $\tilde{A} = \sum_k A_k z_k$, where $z_k \in \mathbb{K}$

$$W_0 := \{0\},\ W_i := \mathrm{span} \bigcup_k A_k \tilde{A}^{-1} W_{i-1} \ \ (i = 1,2,\ldots)$$



Lem [Ivanyos-Karpinski-Qiao-Santha 2015]

- $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_j = W_{j+1} = \cdots =: W_\infty$
- If $W_\infty \subseteq Im\ \tilde{A}$ → $\mathrm{rank}\ \tilde{A} = \mathrm{rank}\ A = \mathrm{ncrank}\ A$

For bipartite / matroid intersection $A, z = 1_M$

→ Wong sequence = alternating path / reachable node set in residual network

IQS-algorithm: For $\tilde{A} = \sum_k A_k z_k$ ($z_k \in \mathbb{K}$), compute W_∞

If $W_\infty \subseteq Im\ \tilde{A}$, optimal. Otherwise, *augment* z or *blow up* A

Def: d-th blow up of $A = \sum_k A_k x_k$

$$A^{\{d\}} := \sum_k A_k \otimes X_k \quad \text{where } X_k := \begin{pmatrix} X_{k,11} & X_{k,12} & \cdots & X_{k,1d} \\ X_{k,21} & X_{k,22} & \cdots & X_{k,2d} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k,d1} & X_{k,d2} & \cdots & X_{k,dd} \end{pmatrix}$$

Thm [Ivanyos-Qiao-Subrahmanyam 2017]

$$\mathrm{ncrank}\ A = \max_{d=1,2,\ldots} \frac{1}{d} \mathrm{rank}\ A^{\{d\}}$$

$d \geq n-1$ suffices (Derksen-Makam 2017)

2x2 generic partitioned matrix

Ito-Iwata-Murota 1994, Iwata-Murota 1995

$$A = \begin{pmatrix} A_{11}x_{11} & A_{12}x_{12} & \cdots & A_{1n}x_{1n} \\ A_{21}x_{21} & A_{22}x_{22} & \cdots & A_{2n}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}x_{n1} & A_{n2}x_{n2} & \cdots & A_{nn}x_{nn} \end{pmatrix}$$

where $A_{ij} \in \mathbb{K}^{2\times 2}$
$$= \begin{pmatrix} A_{ij,11} & A_{ij,12} \\ A_{ij,21} & A_{ij,22} \end{pmatrix}$$

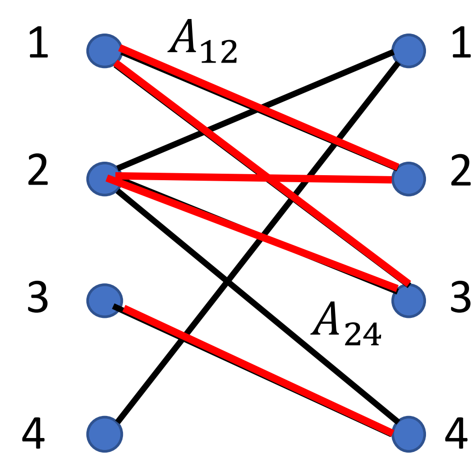
FACT: $\text{rank}\, A = \text{ncrank}\, A$ (← Iwata-Murota 1995)

||

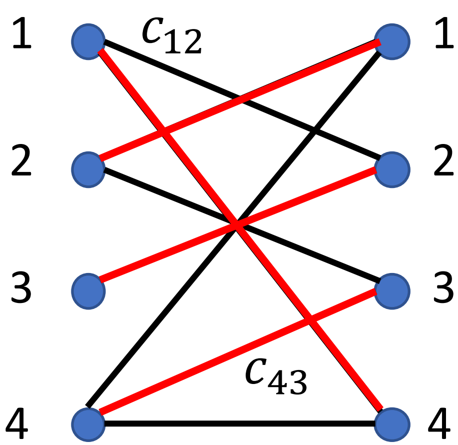
$\max. |\, \text{certain 2matching} \,|$

Blow-up free augmenting path algo combinatorializing IQS-algorithm

Hirai-Iwamasa 2022 (IPCO 2020)



Algebraic formulation of weighted matching



$$A(t) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & x_{12} t^{c_{12}} & & x_{14} t^{c_{14}} \\ x_{21} t^{c_{21}} & & x_{23} t^{c_{23}} & \\ & x_{32} t^{c_{32}} & & \\ x_{41} t^{c_{41}} & & x_{43} t^{c_{43}} & x_{44} t^{c_{44}} \end{pmatrix} \end{array}$$

$$= \sum_{ij \in E} e_i e_j^T x_{ij} t^{c_{ij}}$$

Obs. Max weight of perfect matching $=$ $\deg_t \det A(t)$

$$\because \deg \det A = \deg \sum_{M} \pm t^{c(M)} \prod_{ij \in M} x_{ij} = \max_{M} c(M)$$

Weighted Edmonds Problem

Compute $\deg_t \det$ of

$$B = B_1 x_1 + \cdots + B_m x_m$$

$x_1, x_2, \ldots, x_m$: variable

$B_1, \ldots, B_m$: matrices over $\mathbb{K}(t)$

B: matrix over $\mathbb{K}(x_1, x_2, \ldots, x_m, t)$

Important special case:

$$A[c] = A_1 x_1 t^{c_1} + \cdots + A_m x_m t^{c_m}$$

$A_1, \ldots, A_m$: matrices over $\mathbb{K}$ $c_1, \ldots, c_m$: integers

(2x) max weight $= \deg_t \det A[c]$

0. Bipartite matching ---- $A = \sum_{ij \in E} e_i e_j^T x_{ij} t^{c_{ij}}$

1. Linear matroid intersection ---- $A = \sum_{k=1}^{m} a_k b_k^T x_k t^{c_k}$

2. Non-bipartite matching ---- $A = \sum_{ij \in E} (e_i e_j^T - e_j e_i^T) x_{ij} t^{c_{ij}}$

3. Linear matroid matching ----- $A = \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T) x_k t^{c_k}$

∃ Polynomial time algorithm
(1:Edmonds, 2: Edmonds, Lawler, Iri-Tomizawa, 3: Pap, Iwata-Kobayashi)

Weighted version of noncommutative Edmonds Problem

Hirai 2019

Compute $\deg_t Det$ of

$$B = B_1 x_1 + \cdots + B_m x_m$$

$x_1, x_2, \ldots, x_m$: noncommutative variable, $x_i x_j \neq x_j x_i$

$B_1, \ldots, B_m$: matrices over $\mathbb{K}(t)$

B: matrix over $\mathbb{F}(t)$ for $\mathbb{F} := \mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$

Det := *Dieudonne determinant*

$\sim$ determinant concept for skew fields

Min-Max Formula for deg Det

$B = B_1 x_1 + \cdots + B_m x_m$

Thm [Hirai 2019]

$$\deg Det\, B = \begin{array}{ll} \text{Min.} & -\deg\det P - \deg\det Q \\ \text{s.t.} & \deg (PB_kQ)_{ij} \le 0 \quad (\forall k, \forall ij) \\ & P, Q \in \mathrm{GL}_n(\mathbb{K}(t)) \end{array}$$

Weak duality (c.f. Murota 1995)

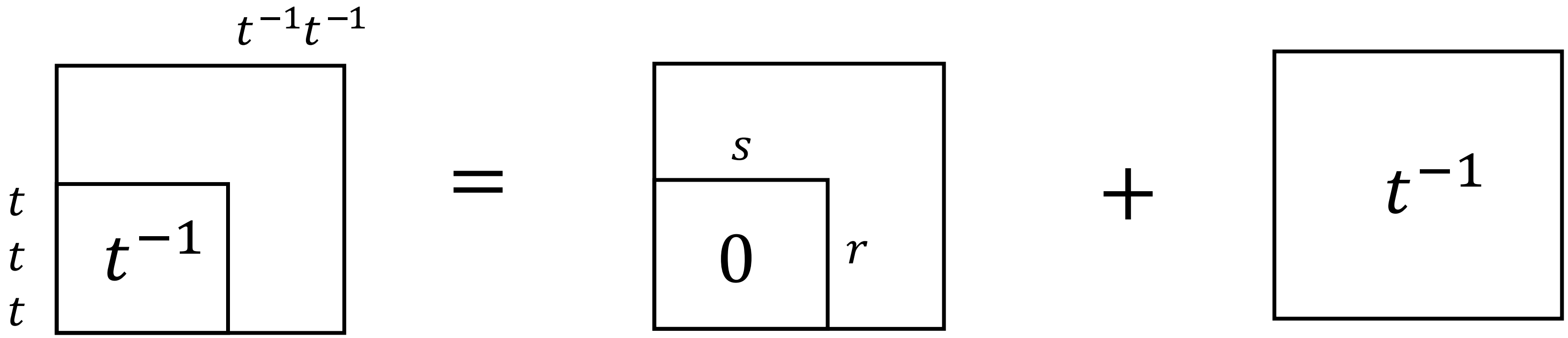
$$0 \ge \deg Det\, PBQ = \deg Det\, P + \deg Det\, B + \deg Det\, Q$$

→ $\deg\det B \le \deg Det\, B$

- This is *not* a good characterization for $\deg Det\, A[c]$

Deg-Det algorithm (Hirai 2019)

inspired by *combinatorial relaxation method* (Murota 1995)



$$S\ PBQ\ T \quad = \quad S \textstyle\sum_k (PB_kQ)^0\, x_k\, T \quad + \quad O(t^{-1})$$

over $\mathbb{K}$

$$P, Q \ \leftarrow \ \begin{pmatrix} I & O \\ O & tI_r \end{pmatrix} SP, QT \begin{pmatrix} I & O \\ O & t^{-1}I_{n-s} \end{pmatrix}$$

$$\text{obj} \ \leftarrow \ \text{obj} - (r + s - n)$$

If $\text{ncrank} \sum_k (PB_kQ)^0\, x_k = n$ → P, Q optimal

$(\because 0 = \deg Det\ PBQ)$

This algorithm is pseudo-polynomial for

$$A[c] = A_1 x_1 t^{c_1} + \cdots + A_m x_m t^{c_m}$$

But can deduce

Bipartite matching $A[c]$ $\Longrightarrow$ Hungarian method

Matroid intersection $A[c]$ $\Longrightarrow$ Weight splitting (Frank 1981)

Furue-Hirai 2020

$\deg \det = \deg Det$

Thm [Hirai-Ikeda 2022]

$\deg Det\ A[c]$ can be computed in $\mathrm{poly}(n, m, \log C)$

calls of solving ncrank

- Cost scaling for c
- $\mathbb{K} = \mathbb{Q}$: bit complexity ← modulo-p trick by Iwata-Kobayashi 2022
- Frank-Tardos method for removing $\log C$

Based on *polyhedral interpretation* of $\deg Det\ A[c]$

Polyhedral interpretation of deg $Det\ A[c]$

$A = \sum_k A_k x_k$, $A[c] = \sum_k A_k x_k t^{c_k}$

Def: Newton polytope of $\det A$

$$P(A) \coloneqq \mathrm{conv}\{\ u \in \mathbb{Z}^m \mid \det A \text{ has term } x_1^{u_1} x_2^{u_2} \cdots x_m^{u_2}\ \}$$

Obs: $\deg \det A[c] \ = \max\ \{c^T x \mid\ x \in P(A)\}$

Def: *Nc-Newton polytope* of $\mathrm{Det}\, A$ [Hirai-Ikeda 2022]

$$Q(A) \coloneqq \bigcup_{d=1,2,\ldots} \frac{1}{d} \,\mathrm{proj}\, P(A^{\{d\}}), \quad \text{where } A^{\{d\}} = \textstyle\sum_k A_k \otimes X_k$$

Thm [Hirai-Ikeda 2022]

$$\deg Det\ A[c] = \max\{c^T x \mid x \in Q(A)\}$$

→ $Q(A)$ is an integral polytope

Rational polytope Q with $\max\{c^T x \mid x \in Q\} \in \mathbb{Z}$ $(\forall c \in \mathbb{Z})$ is integral (Edmonds-Giles)

Good characterization for deg $Det\ A[c]$

Thm [Hirai-Iwamasa-Oki-Soma 2023]

$$\deg Det\ A[c] = \max\{c^T x \mid x \in Q(A)\}$$

LP-duality ??

$$= \text{min.} \quad -\sum_{i=1}^{n} \alpha_i - \sum_{j=1}^{n} \beta_j$$

$$\text{s.t.} \quad \alpha_i + \beta_j \geq -c_k \;\; (i, j, k: A_k(U_i, V_j) \neq \{0\})$$

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n, \;\; \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$$

$$U_1 \subset U_2 \subset \cdots \subset U_n, \;\; V_1 \subset V_2 \subset \cdots \subset V_n$$

$$\alpha, \beta \in \mathbb{Z}^n, U_i, V_j \subseteq \mathbb{K}^n\text{: vector subspaces}$$

Euclidean building

- Generalize Iwamasa 2021 (IPCO2021) for 2x2 partitioned $A[c]$

Fractional matroid matching

$H_k = \mathrm{span}\{a_k, b_k\} \subseteq \mathbb{K}^n$ $(k = 1, \ldots, m)$: 2-dim vector subspaces

Def [Vande Vate 1992] fractional matroid matching $y \in \mathbb{R}_+^m$:

$$\sum_{k=1}^{m} y_k \dim X \cap H_k \leq \dim X \quad (X \subseteq \mathbb{K}^n\text{: vector subspace})$$

$A := \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T)\, x_k$ where $H_k = \mathrm{span}\{a_k, b_k\}$

Thm [Oki-Soma SODA 2023]

$$\mathrm{ncrank}\, A = 2 \max\{1^T y \mid \text{frac. mat. matching } y\}$$

Thm [Hirai-Iwamasa-Oki-Soma 2023]

$$\deg Det\ A[c] = 2 \max\{c^T y \mid \underset{2 1^T y = n}{\text{perfect}} \text{ frac. mat. matching } y\}$$

→ $Q(A) = 2 \times$ polytope of perfect frac. mat. matchings

Summary

Edmonds problem $\supset$ algebraic formulation of some class of combinatorial optimization problems

Noncommutative Edmonds problem sheds new insights on and new directions of combinatorial optimization beyond *Polyhedra and Efficiency*

Future work:

- Develop a faster & simpler algorithm for ncrank
- Develop a noncommutative algebraic framework to nonbipartite matching & generalizations, and *rigidity* problems

Thank you for your attention !