Multiflow Feasibility Problem for $K_3 + K_3$

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Contents

H. Hirai, Bounded fractionality of the multiflow feasibility problem for demand graph $K_3 + K_3$ and related maximization problems, *RIMS-Preprint 1645*, 2008.

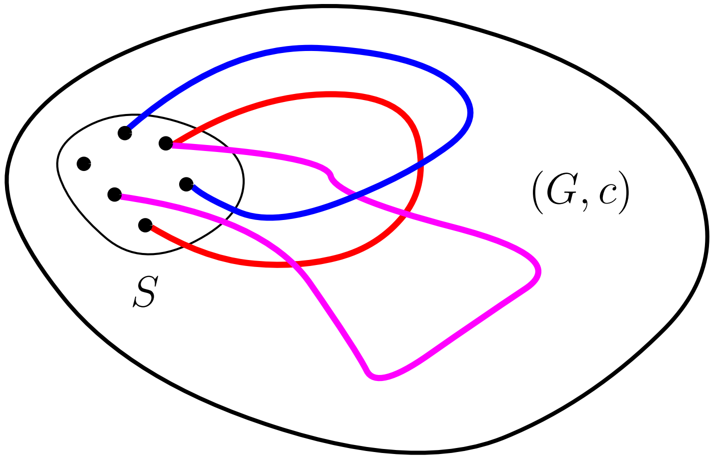
- Reviews on multiflow feasibility problems
- Karzanov's conjecture
- Main theorem
- Proof sketch

Multiflows (Multicommodity flows)

G: an undirected graph (*supply graph*)

$c : EG \to \mathbf{R}_+$: nonnegative edge capacity

$S \subseteq VG$: terminal set



A *multiflow* $f = (\mathcal{P}, \lambda) \overset{\text{def}}{\Longleftrightarrow}$

$\mathcal{P}$: set of S-paths

$\lambda : \mathcal{P} \to \mathbf{R}_+$: flow-value function satisfying capacity constraint

$$\sum\{\lambda(P) \mid P \in \mathcal{P} : e \in P\} \leq c(e) \quad (e \in EG).$$

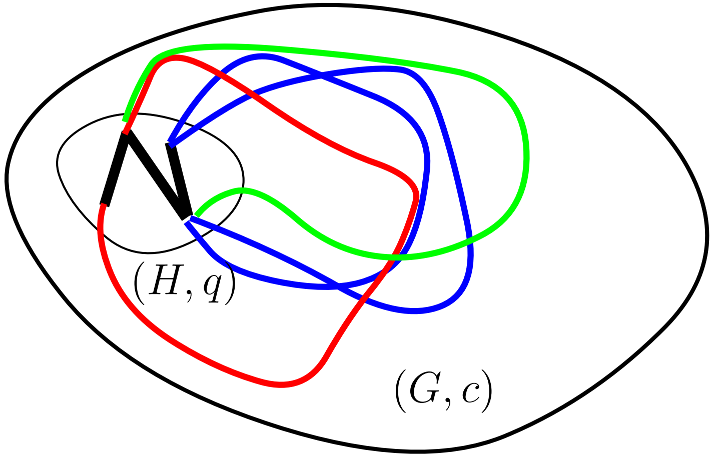
Multiflow feasibility problem

H: *demand graph* with $VH = S$
$q : EH \to \mathbf{R}_+$: demand function on edges EH.

Find a multiflow $f = (\mathcal{P}, \lambda)$ satisfying demand requirement

$$\sum\{\lambda(P) \mid P \in \mathcal{P} : P \text{ is } (s,t)\text{-path}\} = q(st) \quad (st \in EH),$$

or establish that no such a multiflow exists.



We are interested in behavior of multiflows for a fixed H and arbitrary G, c, q.

- $H = K_2$: single commodity flows

Theorem [Ford-Fulkerson 54]

c, q integral, feasible $\Rightarrow \exists$ *integral* solution.

- $H = K_2 + K_2$: 2-commodity flows

Theorem [Hu 63]

c, q integral, feasible $\Rightarrow \exists$ *half-integral* solution.

- $H = K_2 + K_2 + \cdots + K_2$: k-commodity flows

??? c, q integral, feasible $\Rightarrow \exists$ $1/p$-integral solution ($p \leq k$)???

(Jewell 67, Seymour 81)

Theorem [Lomonosov 85]

There is no integer $k > 0$ such that every feasible 3-commodity flow problem with integer capacity and demand has a $1/k$-integral solution.

Fractionality

frac(H) := the least positive integer k with the property:

$\forall c, q$ integral, feasible $\Rightarrow$ $\exists$ $1/k$-integral solution.

Problem [Karzanov 89,90]

Classify demand graphs H with frac(H) $< +\infty$.

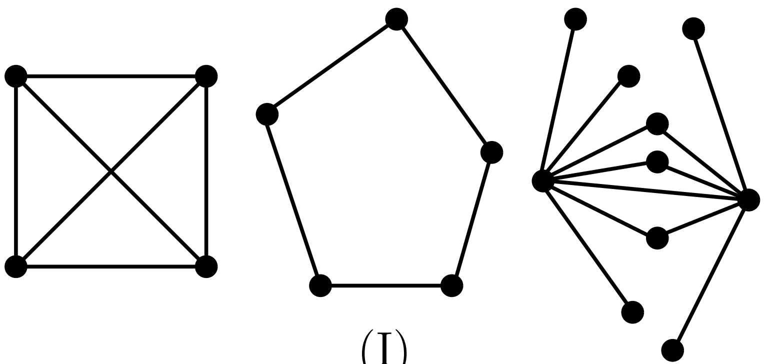
Remark $H \supseteq K_2 + K_2 + K_2 \Rightarrow$ frac(H) $= +\infty$.

Graphs without $K_2 + K_2 + K_2$

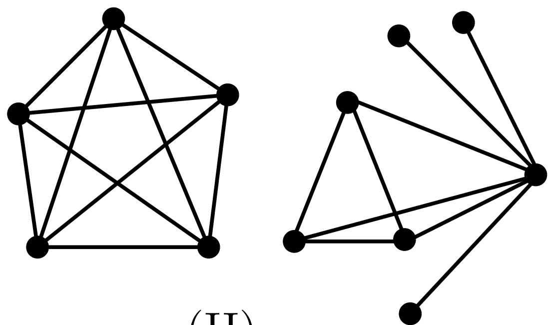
(I) K_4, C_5, or star + star,

(II) K_5 or star + K_3,

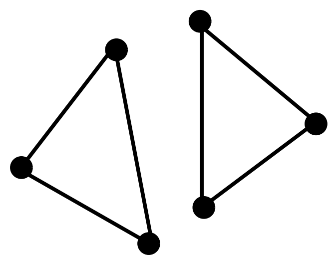
(III) $K_3 + K_3$.



(I)



(II)



(III)

(I) $H = K_4$, C_5, or star + star,

Theorem [Rothschild & Winston 66, Seymour 80, Lomonosov 76,85]
c, q Eulerian, feasible $\Rightarrow \exists$ integral solution ($\to$ frac(H) = 2).

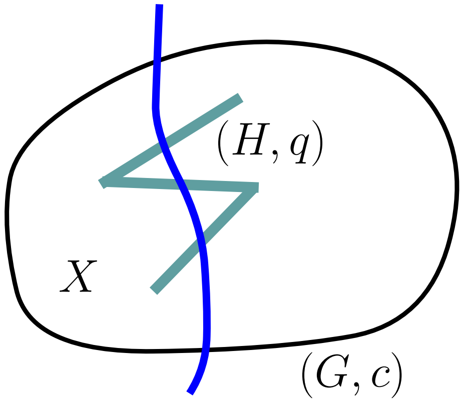
Combinatorial feasibility condition [Papernov 76]
c, q feasible $\Leftrightarrow$ *cut condition*

$$\langle c, \delta_X \rangle_{EG} \geq \langle q, \delta_X \rangle_{EH} \quad (\forall X \subseteq VG),$$

where δ_X is the *cut metric* of X.

$\delta_X =$

	X	$\overline{X}$
X	0	1
$\overline{X}$	1	0



(II) $H = K_5$ or star + K_3

Theorem [Karzanov 87]

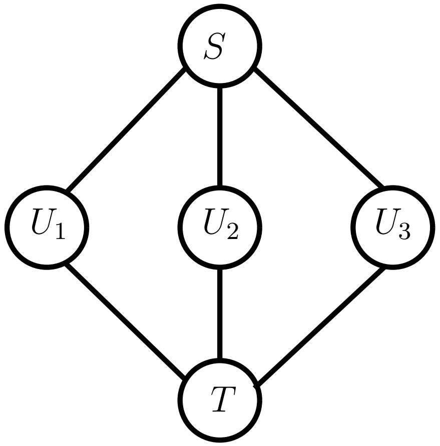
c, q Eulerian, feasible $\Rightarrow \exists$ integral solution ($\rightarrow$ frac(H) = 2).

Combinatorial feasibility condition [Karzanov 87]

c, q feasible $\Leftrightarrow$ $K_{2,3}$-*metric condition*

$$\langle c, d \rangle_{EG} \geq \langle q, d \rangle_{EH} \quad (\forall\ K_{2,3}\text{-metric } d \text{ on } VG).$$

$K_{2,3}$-metric $d \overset{\text{def}}{\Longleftrightarrow} d = d_{K_{2,3}}(\phi(\cdot), \phi(\cdot))$ for $\exists \phi : VG \rightarrow VK_{2,3}$



$d =$

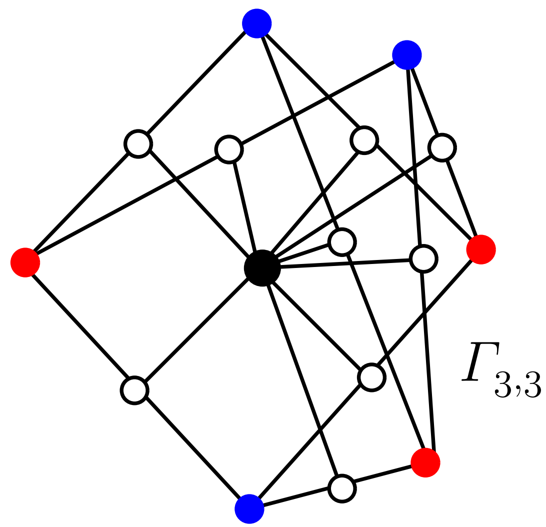
	S	T	U_1	U_2	U_3
S	0	2	1	1	1
T	2	0	1	1	1
U_1	1	1	0	2	2
U_2	1	1	2	0	2
U_3	1	1	2	2	0

(III) $H = K_3 + K_3$

Remark $\exists c, q$ integral, feasible
$\Rightarrow$ no integral, no half-integral, $\exists$ quarter-integral solution.
$\rightarrow$ frac$(K_3 + K_3) \geq 4$.

Combinatorial feasibility condition [Karzanov 89]

c, q feasible $\Leftrightarrow$ $\Gamma_{3,3}$-*metric condition*



Conjecture [Karzanov 90, ICM, Kyoto]

1. frac$(K_3 + K_3) < +\infty$.
2. c, q Eulerian, feasible $\Rightarrow$ $\exists$ half-integral solution ($\rightarrow$ frac$(K_3 + K_3) = 4$)

Cf. Problems 51, 52 in Schrijver's book "*Combinatorial Optimization*"

Main Theorem [H. 08]

$H = K_3 + K_3$, c, q Eulerian, feasible

$\Rightarrow \exists$ 1/12-integral solution.

$\rightarrow$ *the complete classification of*

demand graphs having finite fractionality

Corollary frac$(H) < +\infty \Leftrightarrow H \not\supseteq K_2 + K_2 + K_2$.

Corollary frac$(K_3 + K_3) \in \{4, 8, 12, 24\}$.

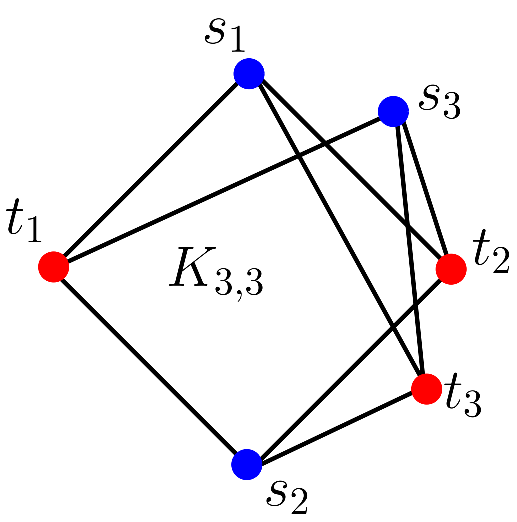
Proof Sketch

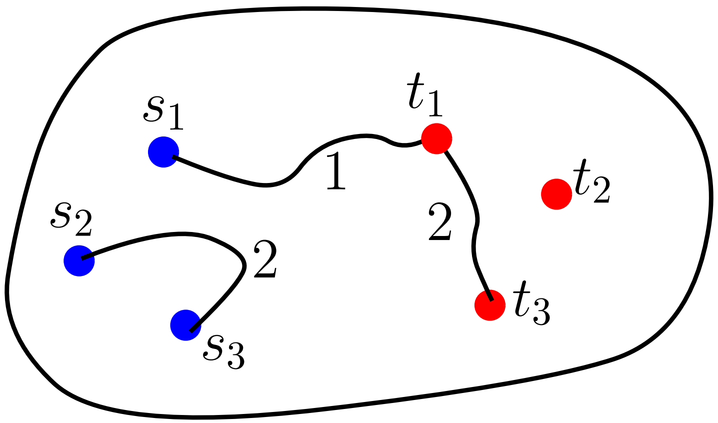
1. Reduction to
   $K_{3,3}$-metric-weighted maximum multiflow problem
2. A combinatorial dual problem
3. Its optimality criterion
4. Fractional splitting-off with potential update

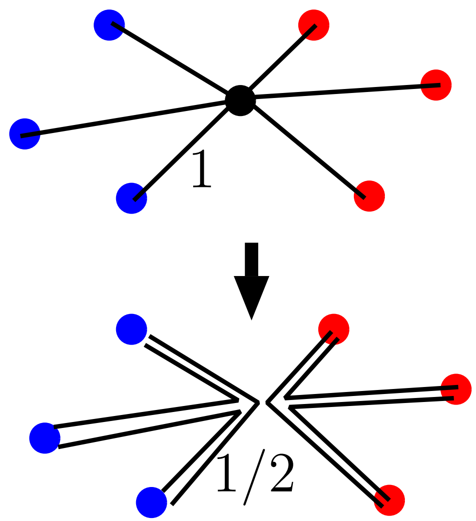
$K_{3,3}$-metric weighted maximum multiflow problem

(G, c): an undirected graph with edge-capacity
$S \subseteq VG$: 6-element terminal set with $S = VK_{3,3}$

$$\text{Max.} \quad \sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P)\lambda(P)$$
$$\text{s. t.} \quad f = (\mathcal{P}, \lambda) : \text{multiflow for } (G, c; S)$$







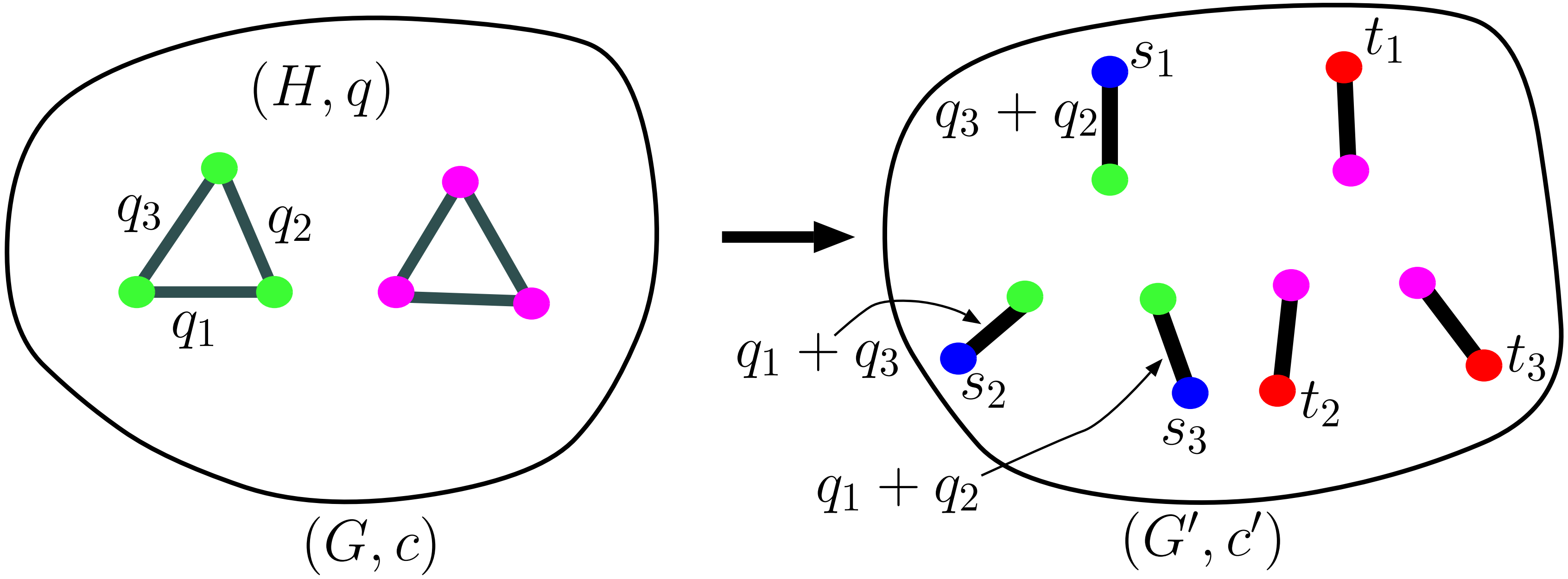
Remark no integral optimum even if (G, c) inner Eulerian.

Remark

$\exists$ $1/k$-integral optimum in inner Eulerian $K_{3,3}$-max problem

$\Longrightarrow$

$\exists$ $1/k$-integral solution in Eulerian $K_3 + K_3$-feasibility problem

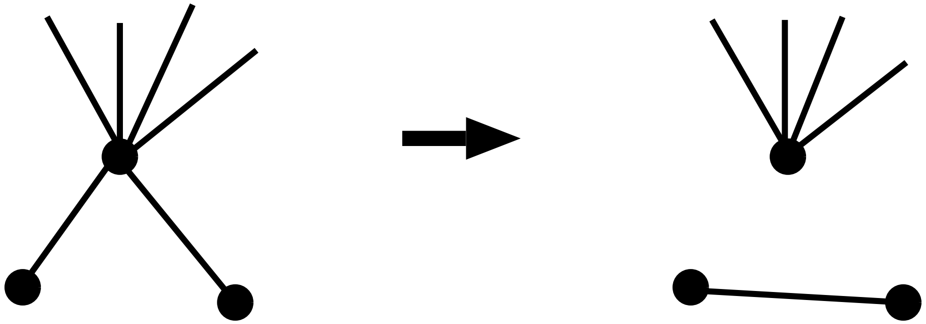


Theorem [H. 08]

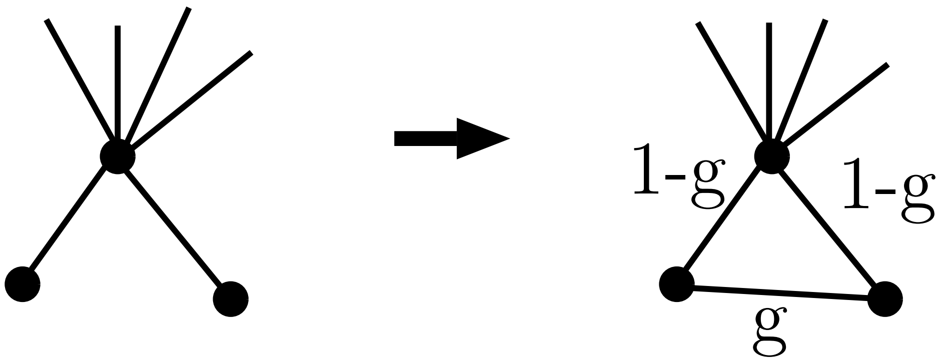
$\exists$ $1/12$-integral optimum in every inner Eulerian $K_{3,3}$-max problem.

Splitting-off for multiflows

[Rothschild-Winston 66, Lovász 76, Seymour 80, Karzanov 87]



- Very powerful for showing an integral optimum in Eulerian problems.
- *How about showing a $1/k$-integral optimum ($k \geq 2$) ?*
- A naive fractional variant violates Eulerianess, and induction fails.



Three key ingredients

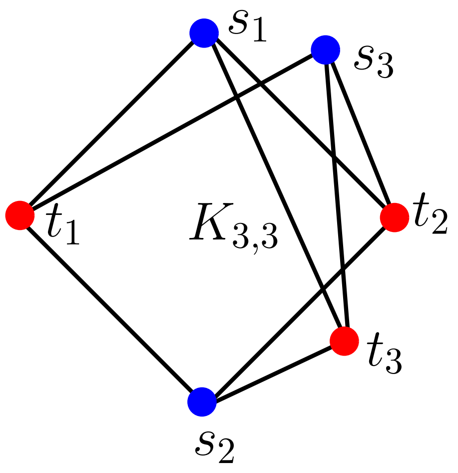
- Combinatorial dual problem [Karzanov 89, 98]
- Its optimality criterion [H. 08]
- Fractional splitting-off with potential update [H. 08]

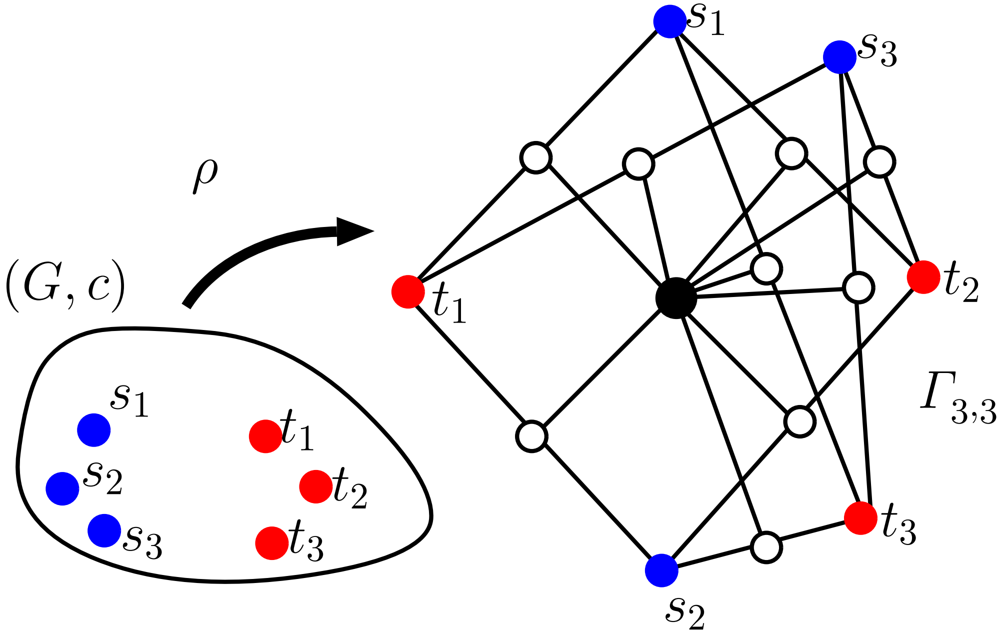
Combinatorial duality relation [Karzanov 89, 98]

$$\text{Max.} \quad \sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P)\lambda(P) \quad \text{s.t.} \quad f = (\mathcal{P}, \lambda) \text{ for } (G, c; S)$$

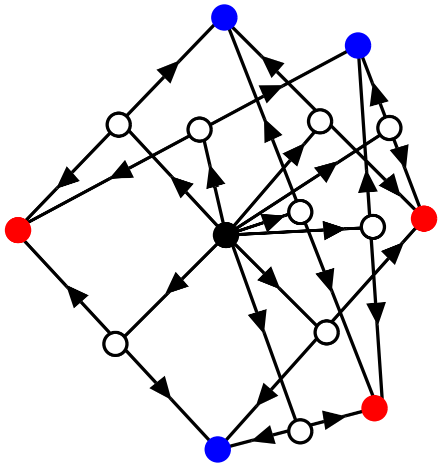
$$= \text{Min.} \quad \frac{1}{2} \sum_{xy \in EG} c(xy) d_{\Gamma_{3,3}}(\rho(x), \rho(y))$$

$$\text{s. t.} \quad \rho : VG \to V\Gamma_{3,3}, \; \rho|_S = id,$$ ← *potential*

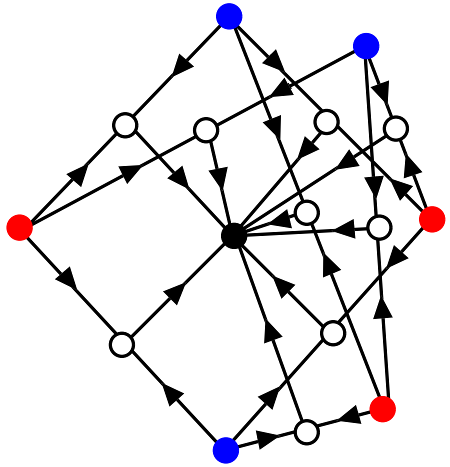




Optimality criterion [H. 08]



forward



backward

ρ': a *forward neighbor* of $\rho \overset{\text{def}}{\iff}$ in forward orientation $\overrightarrow{\Gamma_{3,3}}$,

$$\rho'(x) \neq \rho(x) \Longrightarrow \overrightarrow{\rho(x)\rho'(x)} \in \overrightarrow{E\Gamma_{3,3}} \text{ or } (\rho(x), \rho'(x)) = (\bullet, {\color{blue}\bullet}) \text{ or } (\bullet, {\color{red}\bullet})$$

ρ': a *backward neighbor* of $\rho \overset{\text{def}}{\iff}$ in backward orientation ...

Proposition [H. 08]

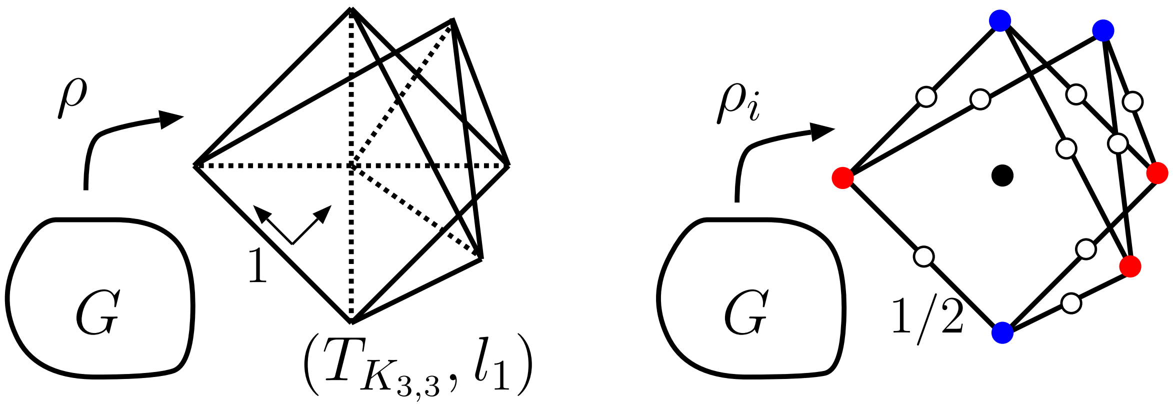
ρ is not optimal $\Rightarrow \exists$ neighbor ρ' of ρ having smaller obj. value.

Proof sketch (of duality relation and opt. criterion)

LP-dual to $K_{3,3}$-max problem

$$\begin{array}{ll} \text{Min.} & \displaystyle\sum_{xy \in EG} c(xy)d(x,y) \\ \text{s.t.} & d\text{: metric on } VG, \quad d|_S = d_{K_{3,3}} \end{array}$$

Proposition[Karzanov 98]
every minimal metric is embedded into $(T_{K_{3,3}}, l_1)$.

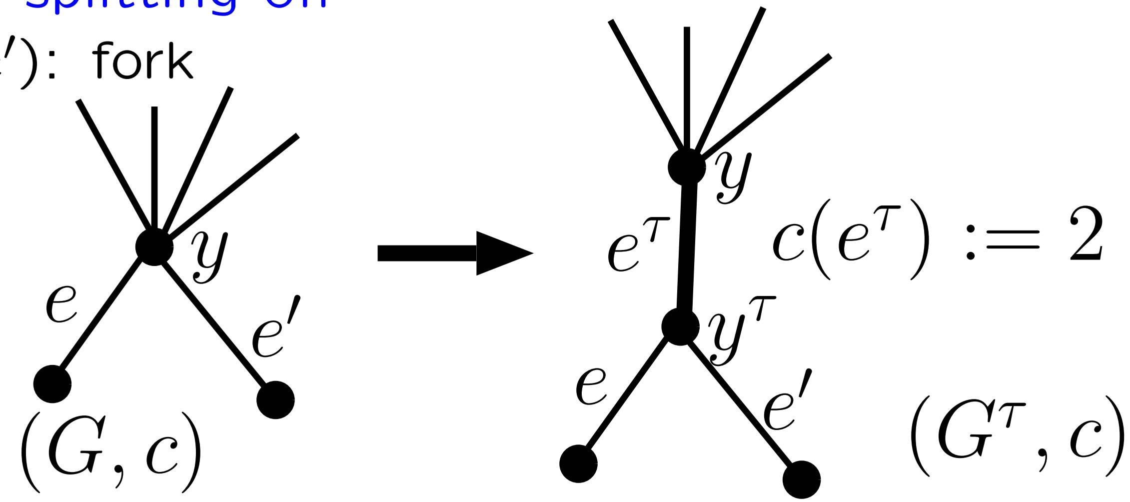


Lemma $d_{l_1} \circ \rho = \sum_i \lambda_i \, (d_{l_1} \circ \rho_i)$ for $\exists \rho_i$ with Im $\rho_i = \{$●, ●, ○, ●$\}$

cf. *tight spans* (Isbell 64, Dress 84)

Fractional splitting-off

$\tau = (e, y, e')$: fork



Splitting capacity: $\alpha(\tau) := \max\{0 \leq \alpha \leq 2 \mid \mathrm{opt}(G, c) = \mathrm{opt}(G^\tau, c - \alpha\chi_{e^\tau})\}$

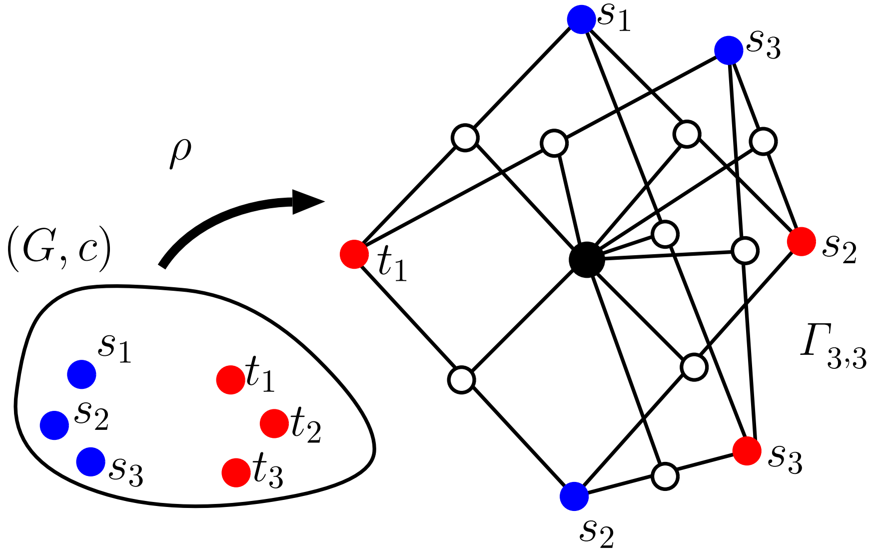
Corollary (G, c): inner Eulerian, ρ: optimal potential

$$\alpha(\tau) \; = \; \min\left\{ \frac{\langle c, d_{\Gamma_{3,3}} \circ \rho' \rangle - \langle c, d_{\Gamma_{3,3}} \circ \rho \rangle}{d_{\Gamma_{3,3}}(\rho'(y), \rho'(y^\tau))} \;\middle|\; \rho' : \text{neighbor of } \rho \text{ with } \rho'(y) \neq \rho'(y^\tau) \right\}$$

$$\in \; \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2\right\}. \qquad \langle c, d_{\Gamma_{3,3}} \circ \rho \rangle := \sum_{xy \in EG} c(xy) d_{\Gamma_{3,3}}(\rho(x), \rho(y))$$

A neighbor attaining $\alpha(\tau)$ is called a *critical neighbor*.

ρ: optimal potential



$$S_\rho = \{x \in VG \mid \rho(x) = \text{blue } \bullet \text{ or red } \bullet\}$$
$$M_\rho = \{x \in VG \mid \rho(x) = \circ\}$$
$$C_\rho = \{x \in VG \mid \rho(x) = \bullet\}$$

Proposition [H. 08]
(G, c) inner Eulerian, ρ optimal potential, $y \in S_\rho$ inner node
$\Rightarrow y$ has a splittable fork.

Corollary $M_\rho \cup C_\rho = \emptyset \Rightarrow \exists$ integral optimum.

cf. splitting-off idea for 5-terminus flows $H = K_5$ in [Karzanov 87]

Splitting-off with Potential UPdate (SPUP)

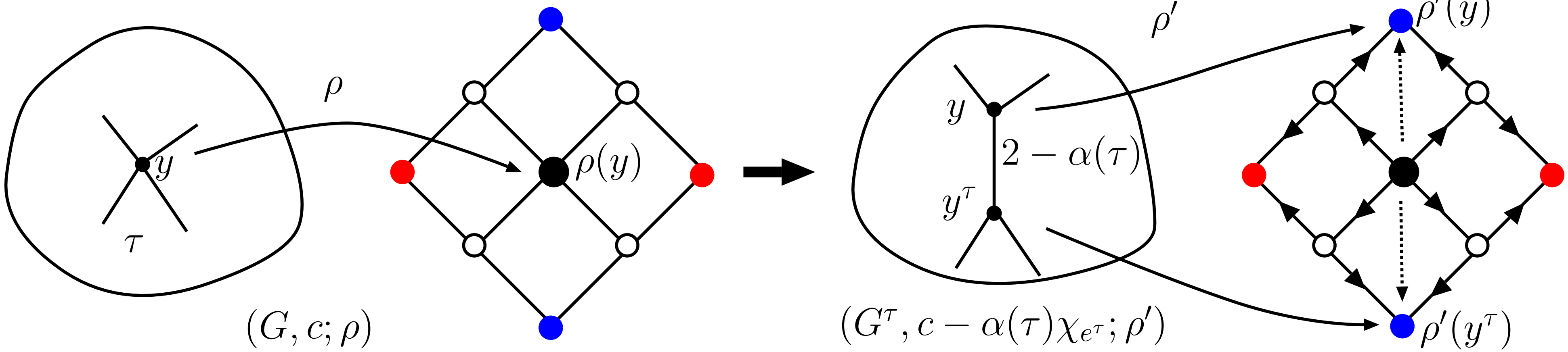
ρ: an optimal potential

τ: a fork (unsplittable) at $M_\rho \cup C_\rho$

ρ': a critical neighbor of ρ w.r.t. τ

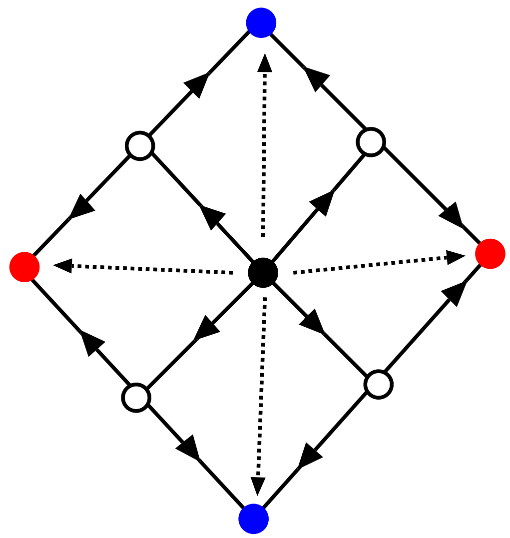
$$\left(\alpha(\tau) = \min_{\rho'} \frac{\langle c, d_{\Gamma_{3,3}} \circ \rho' \rangle - \langle c, d_{\Gamma_{3,3}} \circ \rho \rangle}{d_{\Gamma_{3,3}}(\rho'(y), \rho'(y^\tau))} \in \{0, \tfrac{1}{2}, \tfrac{2}{3}, 1, \tfrac{4}{3}, \tfrac{3}{2}, 2\} \right)$$

SPUP: $(G, c; \rho) \rightarrow (G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$



This SPUP does not keep (G, c) Eulerian, but C_ρ decreases, and still $\alpha(\tau) \in \{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2\}$ in the next forward SPUP

In forward SPUP, C_ρ is nonincreasing, and M_ρ is nonincreasing if $C_\rho = \emptyset$.



These observations suggest us a possibility to repeat forward SPUPs until $M_\rho \cup C_\rho = \emptyset$ with keeping (G, kc) inner Eulerian for a fixed integer k. ($\rightarrow \exists\ 1/k$-integral optimum)

Proposition [H. 08] We can do it for $k = 12$.

The proof is lengthy and complicated.

Concluding remarks

- We do not know whether $1/12$ is tight.
- The bounded fractionality conjecture for $K_3 + K_3$ is a very special case of the conjecture (see Proceedings);

  For a terminal weight μ, $\dim T_\mu \leq 2$ if and only if there exists $k > 0$ such that every Eulerian μ-max problem has a $1/k$-integral optimum.

$$T_\mu := \text{Minimal } \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s, t \in S)\}$$

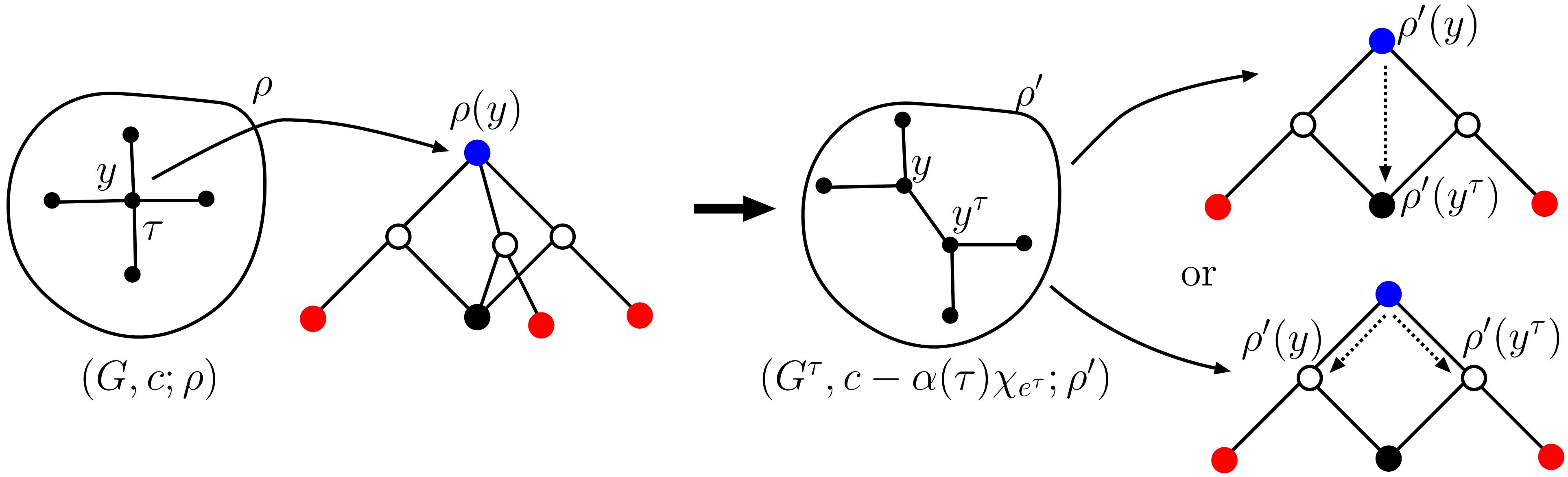
- Recently we proved it for $k = 12$ [H. 09, in preparation]
- Half-integral $\Gamma_{3,3}$-metric packing [H. 07, *Combinatorica*, to appear]

Future works

- Improving the bound $1/12$ ($\rightarrow 1/2$?).
- Augmenting path algorithms for multiflows ?

Appendix I

Proof sketch (based on Karzanov's splitting-off idea for 5-terminus flows)



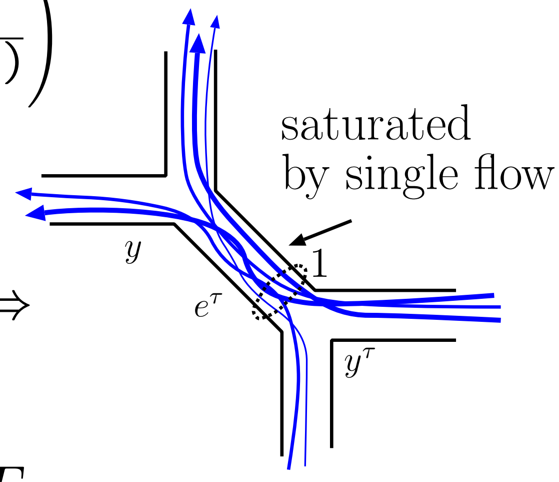
$\Rightarrow \alpha(\tau) \in \{0, 1, 2\}$. $\quad \left(\leftarrow \alpha(\tau) = 2 \min_{\rho'} \frac{\text{obj}(\rho') - \text{obj}(\rho)}{\text{dist}_{\Gamma_{3,3}}(\rho'(y), \rho'(y^\tau))} \right)$

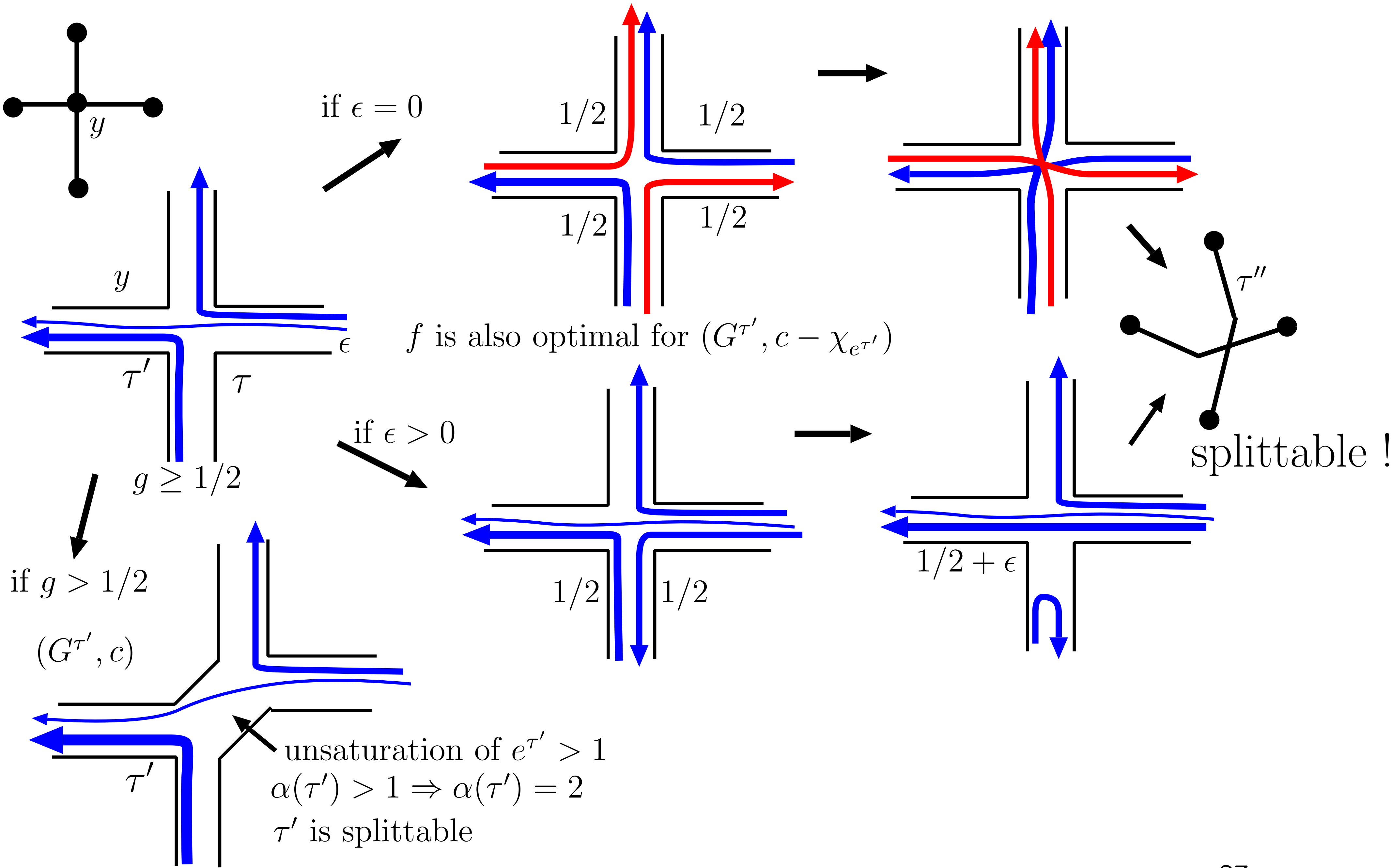
Suppose $\alpha(\tau) = 1$.

Take an optimal flow $f = (\mathcal{P}, \lambda)$ for $(G^\tau, c - \alpha(\tau)\chi_{e^\tau})$.

Complementary slackness: both $f = (\mathcal{P}, \lambda)$, ρ' optimal $\Leftrightarrow$

$$\rho'(x) \neq \rho'(y) \;\Rightarrow\; xy \text{ is saturated by } f,$$
$$P \in \mathcal{P} : \lambda(P) > 0 \;\Rightarrow\; \rho'(P) \text{ is geodesic in } \Gamma_{3,3}.$$




y
y
ϵ
τ'
τ
$g \geq 1/2$
if $\epsilon = 0$
$1/2$
$1/2$
$1/2$
$1/2$
f is also optimal for $(G^{\tau'}, c - \chi_{e^{\tau'}})$
τ''
splittable !
if $\epsilon > 0$
$1/2$
$1/2$
$1/2 + \epsilon$
if $g > 1/2$
$(G^{\tau'}, c)$
τ'
unsaturation of $e^{\tau'} > 1$
$\alpha(\tau') > 1 \Rightarrow \alpha(\tau') = 2$
τ' is splittable

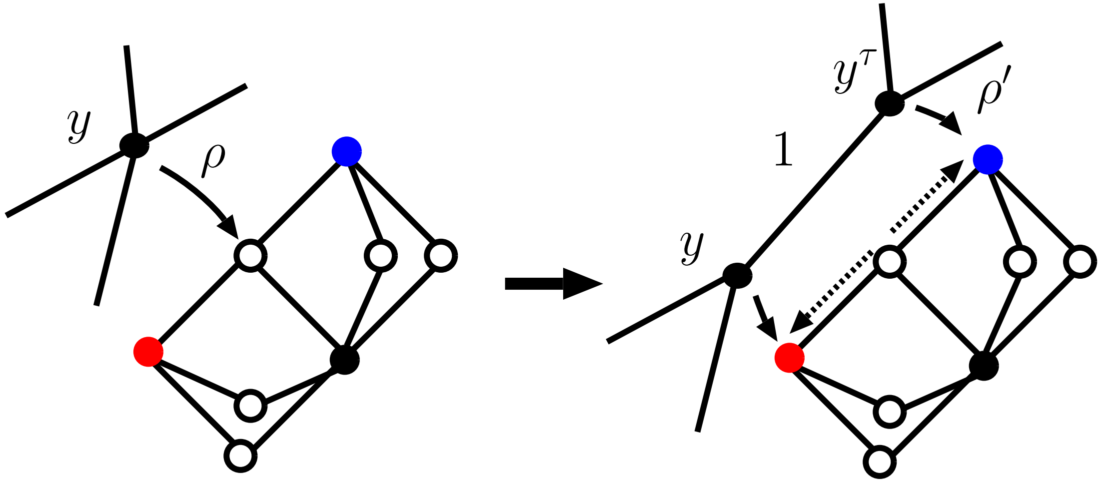
Appendix II

$(G,c;\rho)$:*restricted Eulerian* if c integer, and $\forall y \in M_\rho \cup C_\rho$ has even degree.

Lemma $(G,c;\rho)$:restricted Eulerian, τ: a fork, ρ': a critical neighbor.
ρ' is forward $\Rightarrow \alpha(\tau) \in \{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, 2\}$.

Proposition $(G,c;\rho)$:restricted Eulerian, $y \in M_\rho$ (unsplittable),

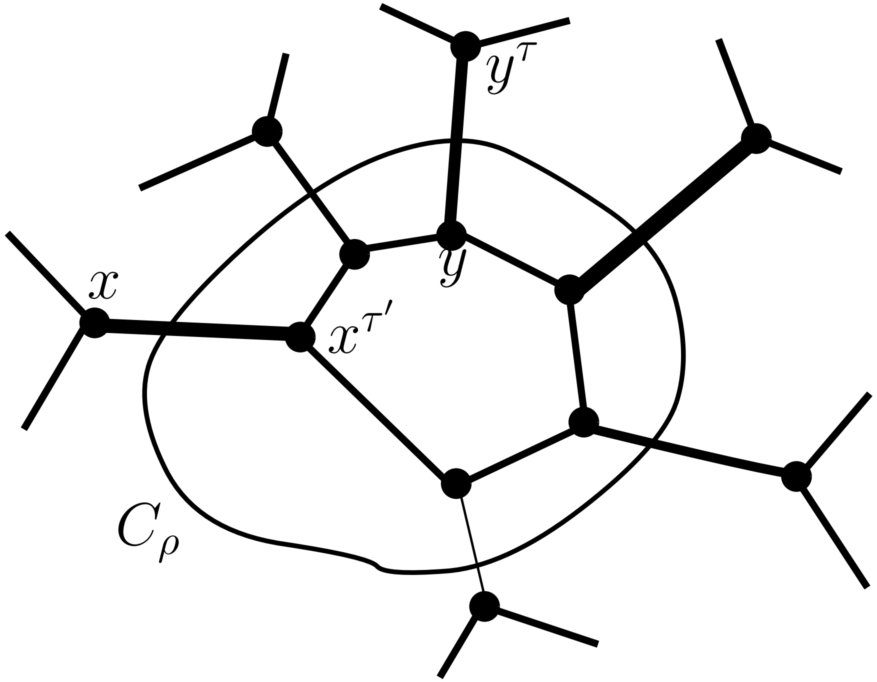
1. $\exists$ an optimal forward neighbor ρ' with $\rho'(y) \in S_\rho$, or
2. $\exists$ a fork τ s.t. a critical neighbor ρ' is forward
   ($\rightarrow \rho'(y), \rho'(y^\tau) \in S_\rho$, $\alpha(\tau) = 1$, SPUP keeps $(G,c;\rho)$ restrict Eulerian)



Corollary
$(G,c;\rho)$ restricted Eulerian with $C_\rho = \emptyset \Rightarrow \exists$ half-integral optimum.

Starting from inner Eulerian (G, c) with all inner node having degree four.

Proposition We can apply forward SPUPs to all degree four nodes in C_ρ with keeping $(G, 6c; \rho)$ restricted Eulerian.



The ring condition:

the subgraph of G induced by C_ρ consists of paths and cycles.

Proposition $(G, c; \rho)$ restricted Eulerian and the ring condition $\Rightarrow \exists$ half-integral optimum.