Discrete Convexity for Multiflows and 0-extensions

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This talk is about:

An attempt to enlarge the scope of Discrete Convex Analysis (Murota 1996~)

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Targets:

multicommodity flows, 0-extensions, discrete metrics, tight spans, Valued CSP, modular lattices, CAT(0) complexes,...

Contents

I. Motivation

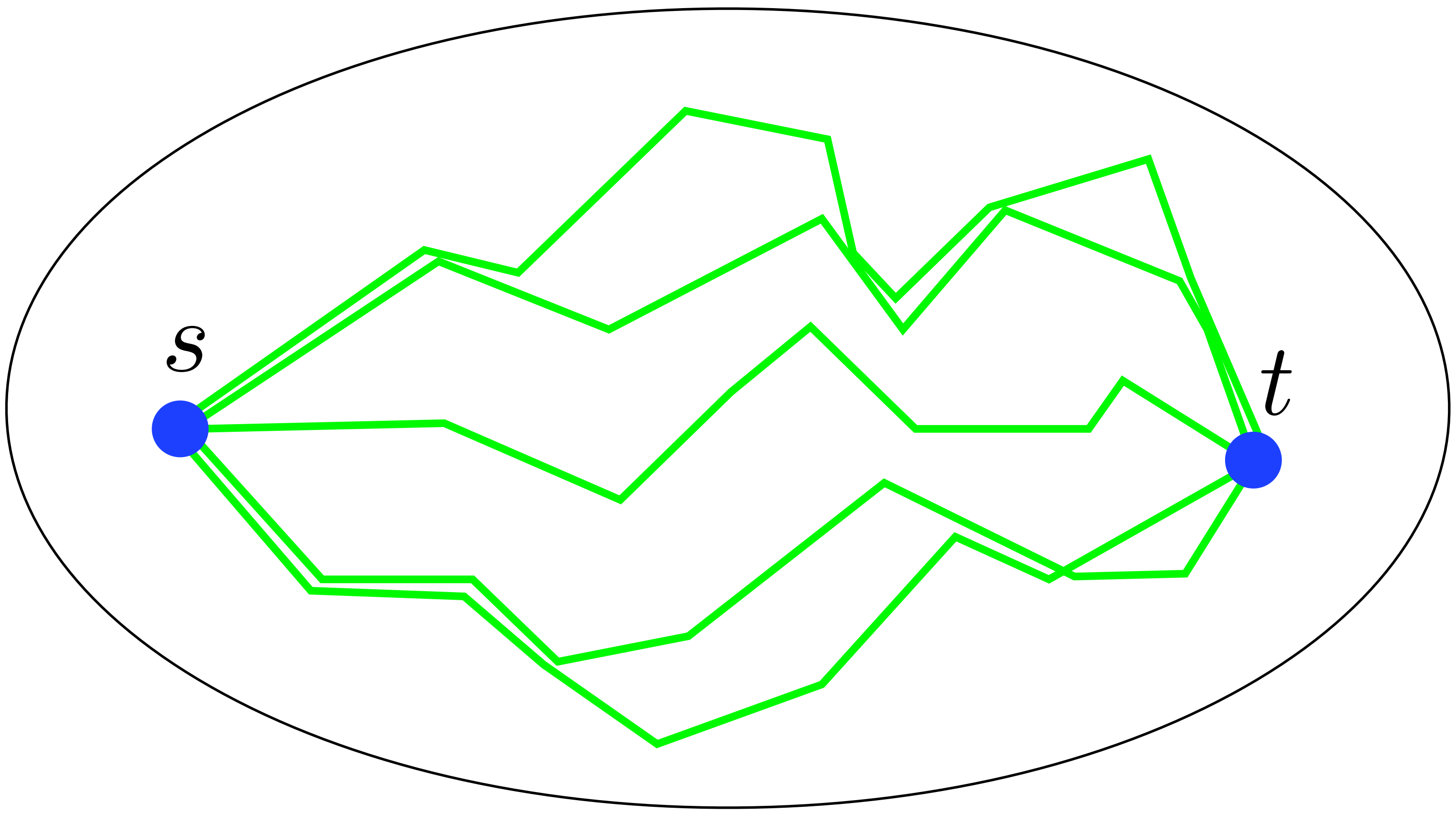
- multiflow combinatorial dualities
- minimum 0-extension problems

II. Framework

- Submodular functions on modular semilattices
- L-convex functions on modular complexes

Ford-Fulkerson 56

Maximum flow = Minimum cut



$$(V, E, c, \{s, t\})$$

Ford-Fulkerson 56

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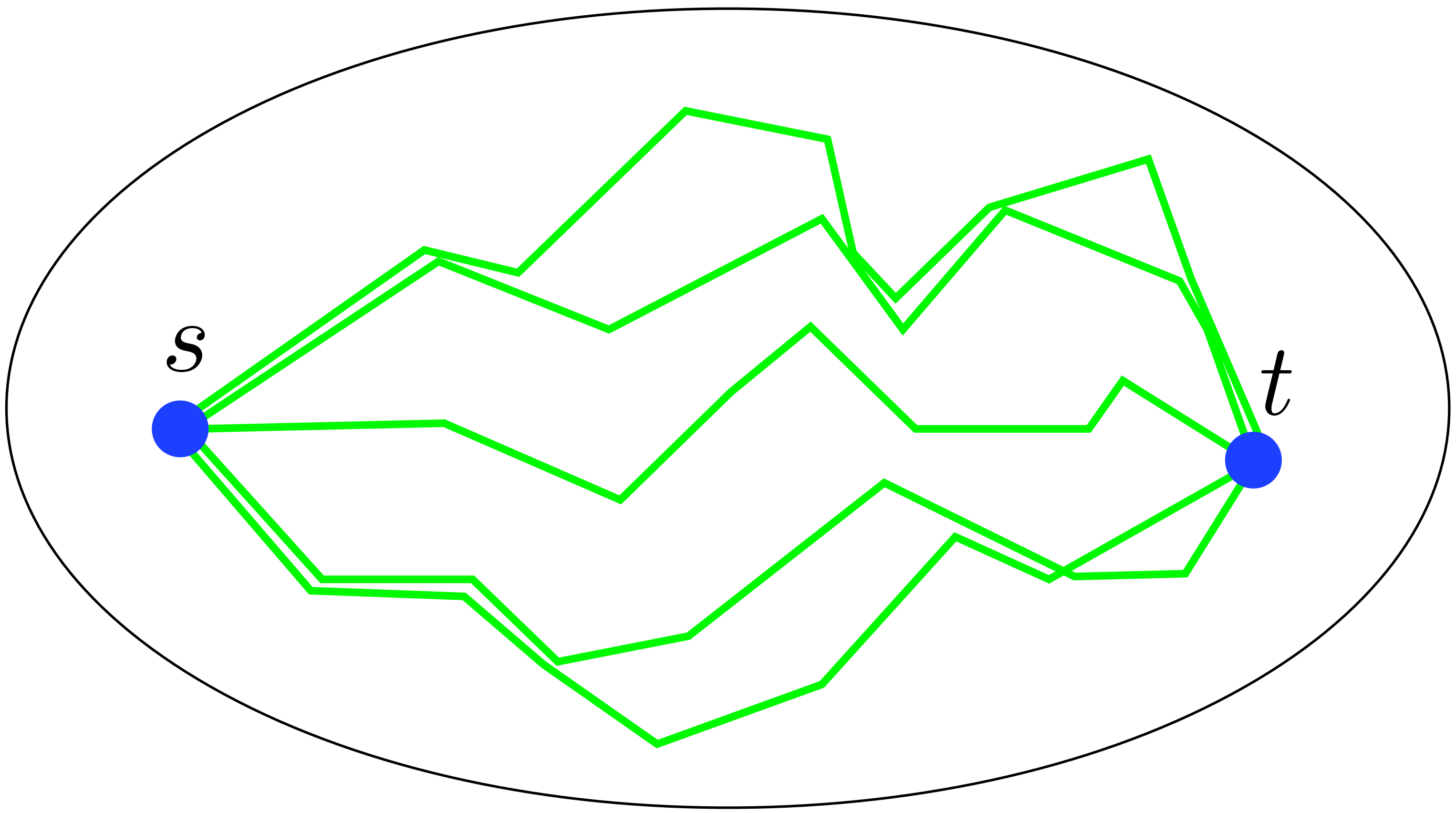


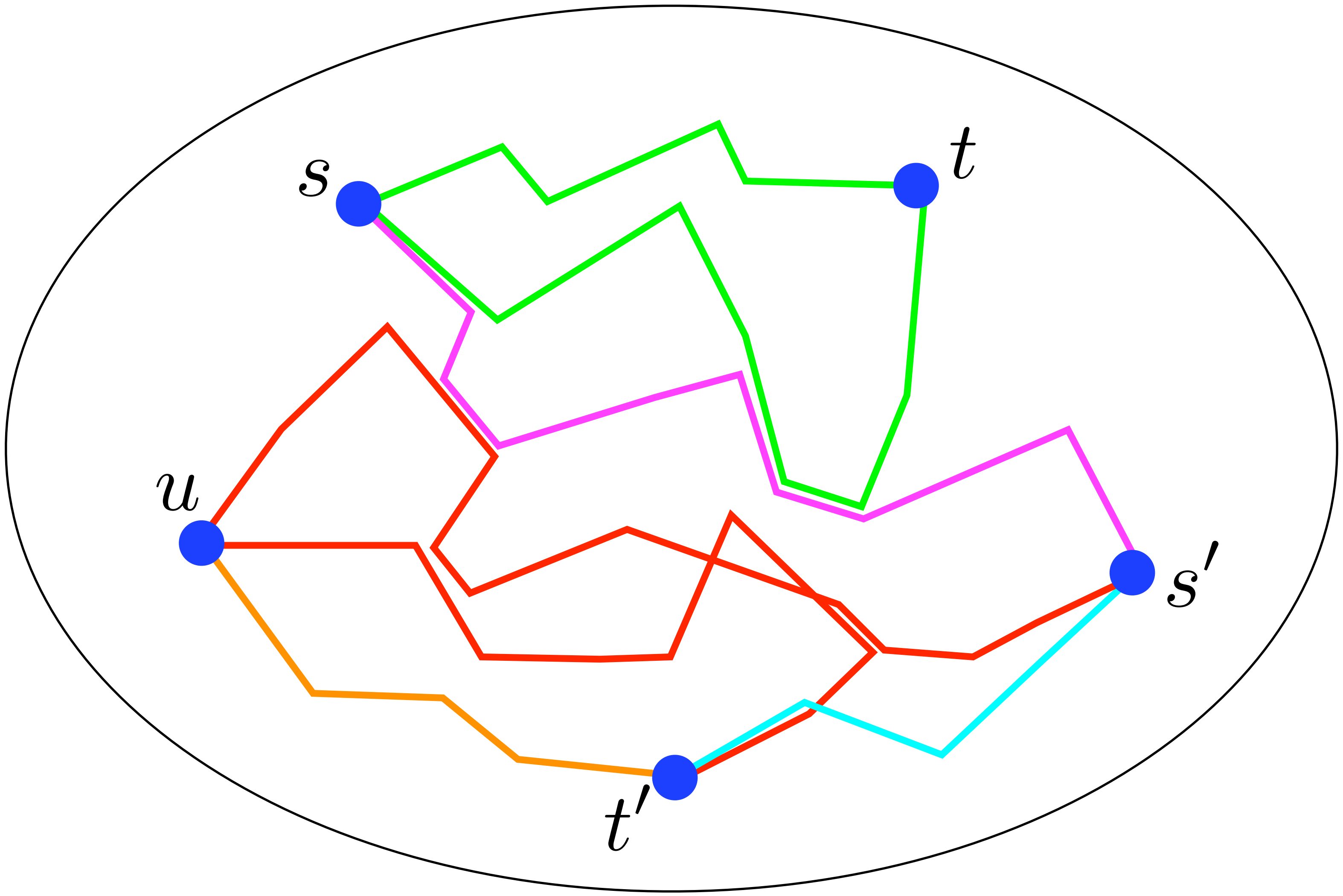
$$(V, E, c, \{s, t\})$$

Is there any analogue for multiflows ?

(Weighted) Maximum Multiflow Problem

$$\text{Max.} \qquad \sum \mu_{st}|f_{st}|$$

$$\text{s.t.} \qquad f = \{f_{st}\}$$



$$(V, E, c, \{s, t, u, s', t', \ldots\})$$

Combinatorial Min-max Theorems for special μ

- Hu 1963
- Lovasz, Cherkassky 1976
- Karzanov-Lomonosov 1978
- Karzanov 1989, 1998
- H. 2009 ~

A unified multiflow duality (H. 2009)

Max. $\sum \mu_{st}|f_{st}|$ = Min. ＊＊＊＊

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$$\begin{array}{lll} \text{Min.} & \sum c_{ij} d(p_i, p_j) & \\ \text{s.t.} & p_i \in X & (i : \text{node}) \\ & p_s \in X_s & (s : \text{terminal}) \end{array}$$

(X, d) : metric space (finite or infinite)

X_s : subspace indexed by s

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determined by μ

Max-flow Min-cut theorem

$$\text{Max. } |f_{st}| = \text{Min. } (s,t)\text{-cut}$$

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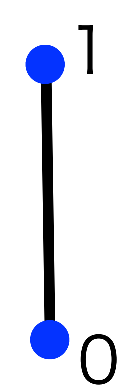
$$\begin{array}{ll} \text{Min.} & \sum c_{ij}|p_i - p_j| \\ \text{s.t.} & p_i \in \{0,1\} \quad (i : \text{node}) \\ & p_s = 0,\ p_t = 1 \end{array}$$

$$(X, d) \longleftarrow$$



Lovasz-Cherkassky Theorem

$$\text{Max.} \sum_{s,t \in S} |f_{st}| = \frac{1}{2} \sum_{u \in S} \text{Min.}\,(u, S \setminus u)\text{-cut}$$

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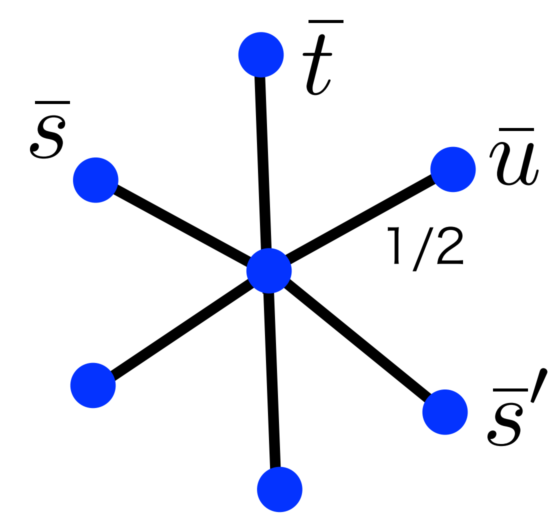
$$\begin{aligned} &\text{Max.} && \sum c_{ij} d(p_i, p_j) \\ &\text{s.t.} && p_i \in X \quad (i : \text{node}) \\ & && p_s = \bar{s} \quad (s : \text{terminal}) \end{aligned}$$

$(X, d) \longleftarrow$



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Hu's max-biflow min-cut theorem

$$\text{Max. } |f_{st}| + |f_{s't'}| = \text{Min.}\{(ss', tt')\text{-cut}, \ (st', ts')\text{-cut}\}$$

Hu's max-biflow min-cut theorem

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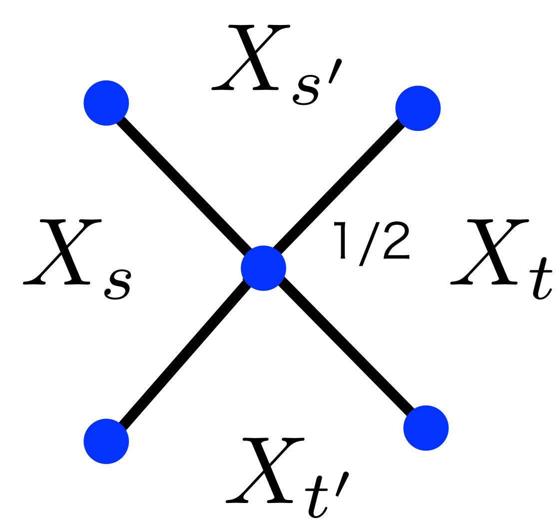
$$(X, d) \longleftarrow$$



In singleflow

Max. flow = Min. cut

submodular function

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submodular function

Is there a submodularity concept for multiflow min-max ?

Minimum 0-extension Problem

[facility location form]

Γ : graph, d_Γ : graph metric

$y_1, y_2, \ldots, y_k$: facilities on Γ

Minimum 0-extension Problem

[facility location form]

Γ : graph, d_Γ : graph metric

$y_1, y_2, \ldots, y_k$: facilities on Γ

Locate new facilities $p_1, p_2, \ldots, p_n$ on Γ s.t.

mutual communication cost

$$\sum_{i,l} b_{il} d_\Gamma(p_i, y_l) + \sum_{i,j} c_{ij} d_\Gamma(p_i, p_j)$$

is minimum

Fix Γ, suppose $V_\Gamma = \{y_1, y_2, \ldots, y_k\}$

0-EXT$[\Gamma]$

Input: n, b, c

Min. $$\sum b_{il} d_\Gamma(p_i, y_l) + \sum c_{ij} d_\Gamma(p_i, p_j)$$

s.t. $$(p_1, p_2, \ldots, p_n) \in V_\Gamma \times V_\Gamma \times \cdots \times V_\Gamma$$

Fix Γ, suppose $V_\Gamma = \{y_1, y_2, \ldots, y_k\}$

$$\mathbf{0\text{-}EXT}[\Gamma]$$

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Rem. $\mathbf{0\text{-}EXT}[K_2]$ = Minimum cut

Rem. Multiflow-dual is (a variation of) $\mathbf{0\text{-}EXT}$

$\mathbf{0\text{-}EXT}[K_n]$:

Minimum-cut ~ P (n = 2)

Multiway-cut ~ NP-hard (n > 2)

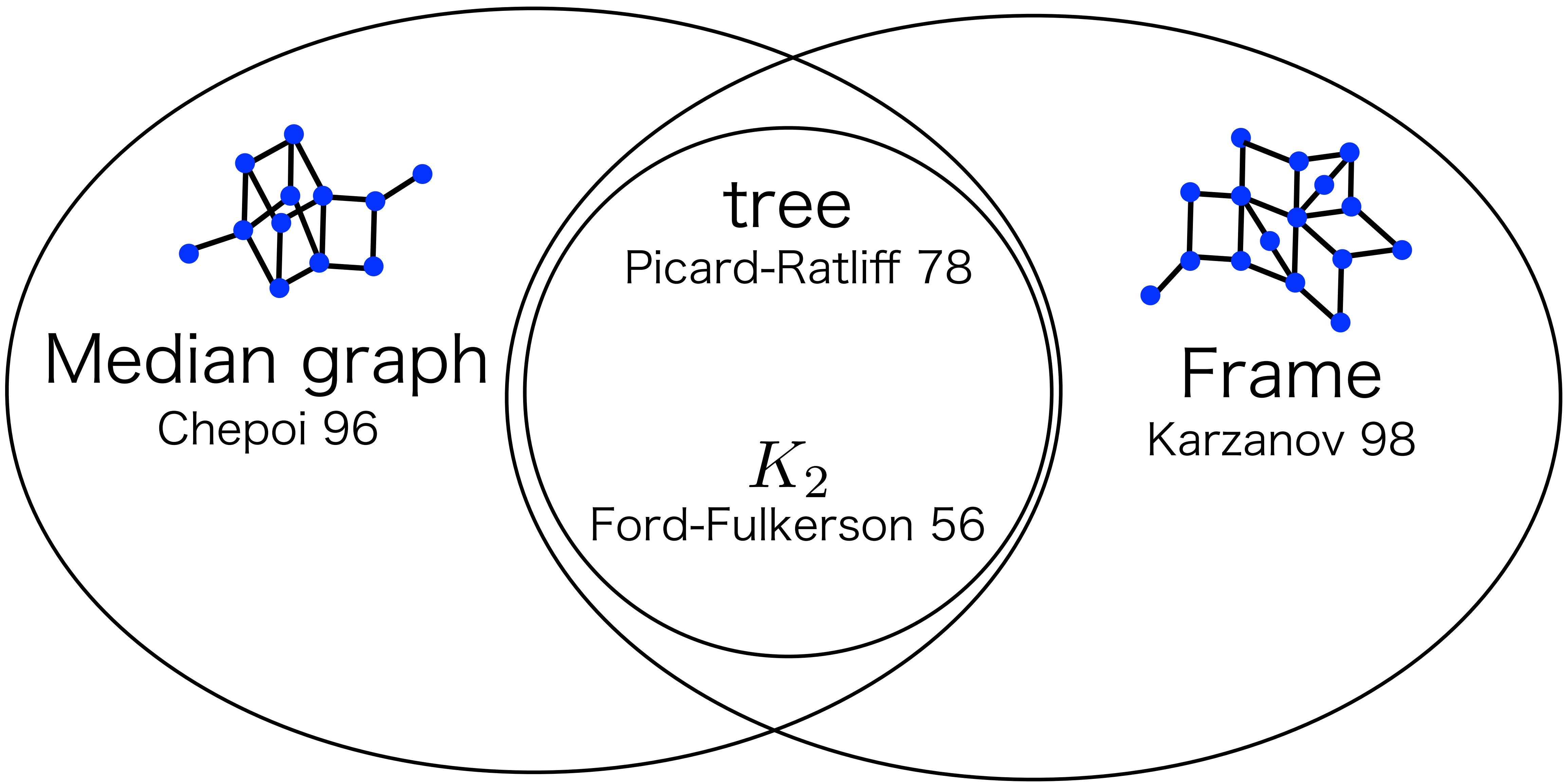
$\mathbf{0\text{-}EXT}[K_n]$: Minimum-cut ~ P (n = 2)

Multiway-cut ~ NP-hard (n > 2)

Karzanov 1998, 2004:

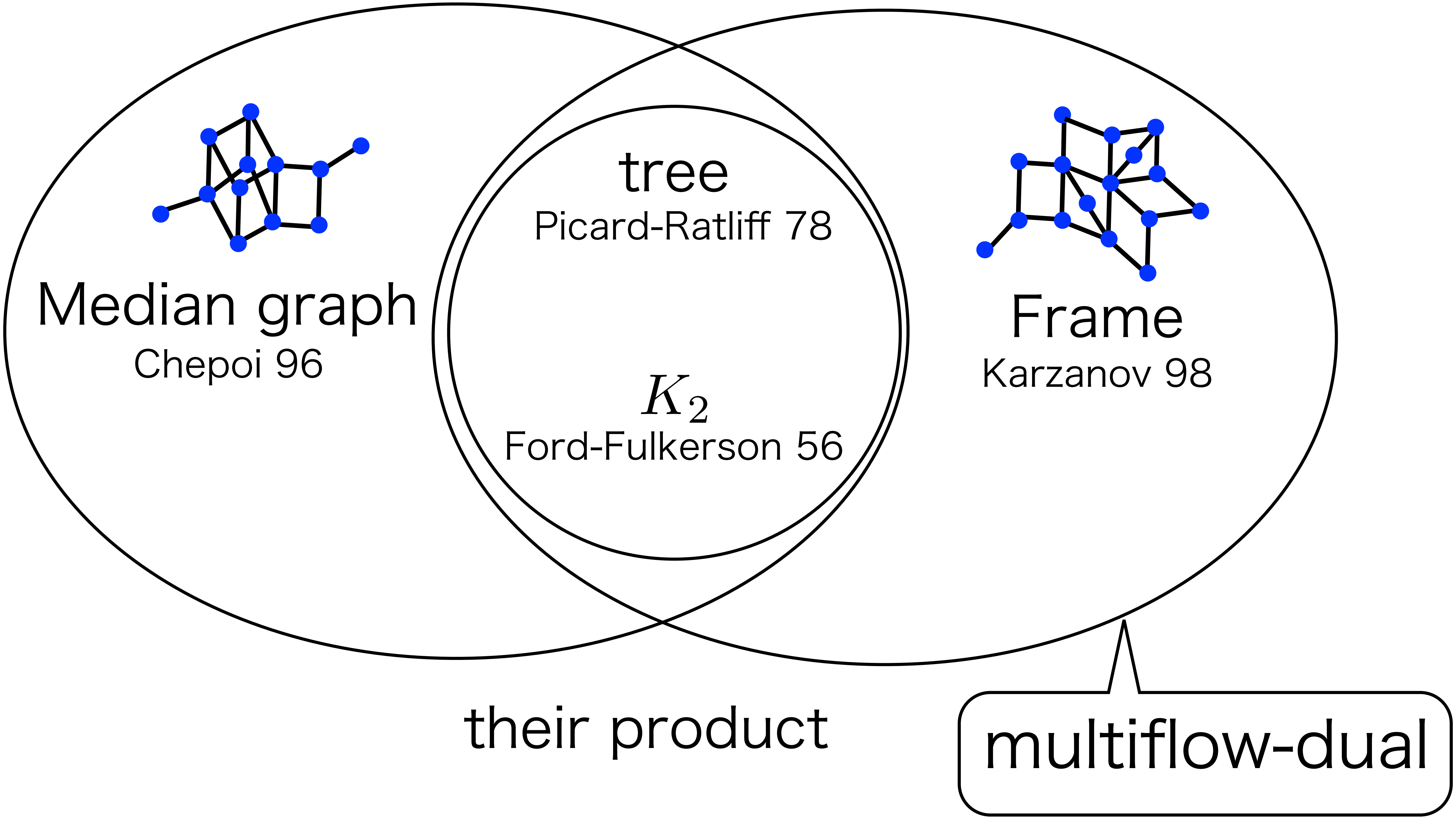
What is Γ for which $\mathbf{0\text{-}EXT}[\Gamma]$ is tractable ?

Known tractable graphs



their product

Known tractable graphs



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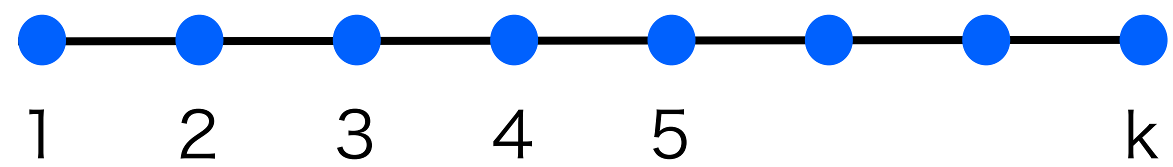
Connection to DCA

Γ =



$$\text{Min.} \quad \sum b_{il}|p_i - l| + \sum c_{ij}|p_i - p_j|$$

$$\text{s.t.} \quad p \in \{1, 2, 3, \ldots, k\}^n \subseteq \mathbf{Z}^n$$

Connection to DCA

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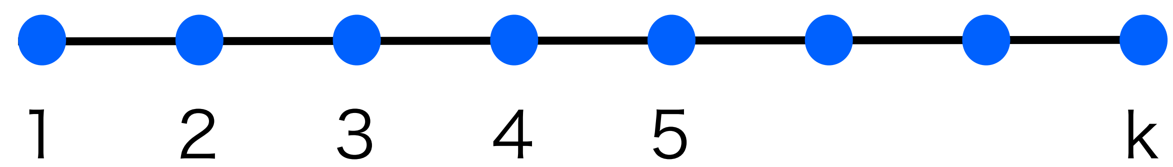

$\mathrm{L}^\natural$-convex function

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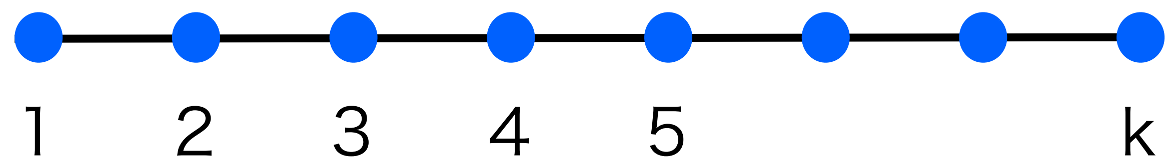

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L-convexity for more general tractable Γ ?

Our result

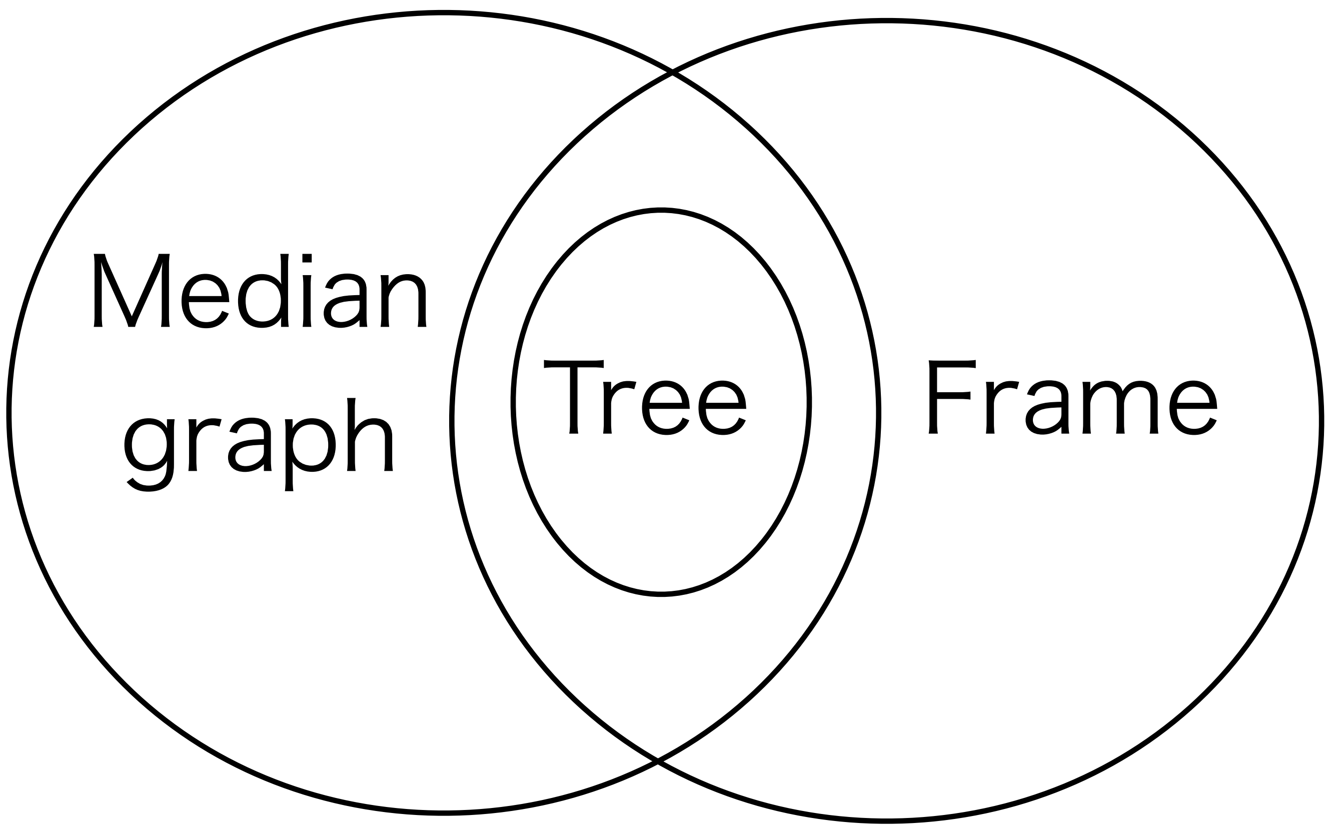
A theory of L-convex functions
on certain graph structures
captures
combinatorial multiflow-duals
&
tractable min. 0-extensions

Consequence

Thm (H. 2012, SODA'13)

If Γ is orientable modular, $\mathbf{0\text{-}EXT}[\Gamma]$ is in P

Orientable modular



Consequence

Thm (H. 2012, SODA'13)

If Γ is orientable modular, $\mathbf{0\text{-}EXT}[\Gamma]$ is in P

Thm (Karzanov 1998)

Otherwise, $\mathbf{0\text{-}EXT}[\Gamma]$ is NP-hard

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II. Framework

- Submodular functions on modular semilattice
- L-convex functions on modular complex

≒ orientable modular graph

Tools

Discrete metric geometry

Bandelt, Chepoi, Dress, Isbell, Van de Vel...

Valued CSP & fractional polymorphism

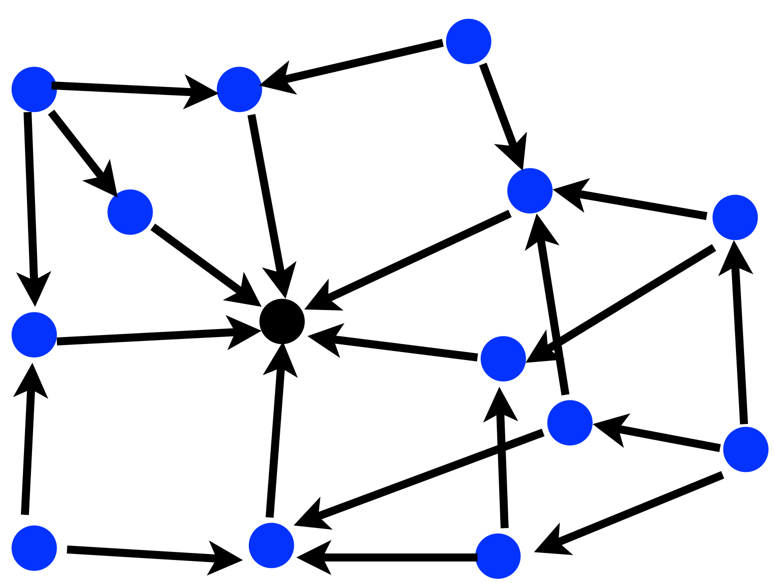
Thapper-Zivny FOCS'12

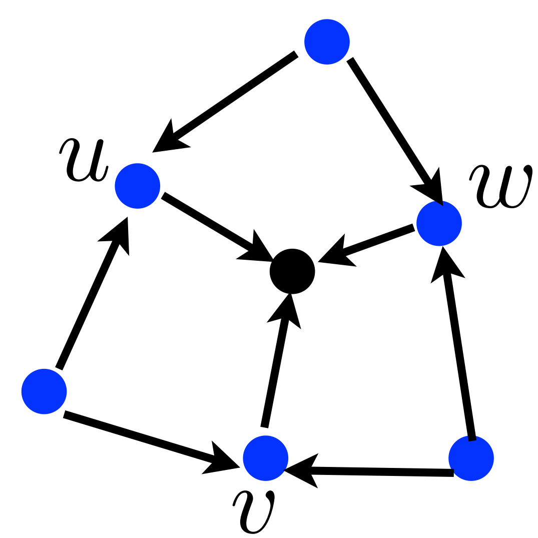
Modular semilattice

Bandelt-Van de Vel-Verheul 1993

A semilattice $\mathcal{L}$ is modular if

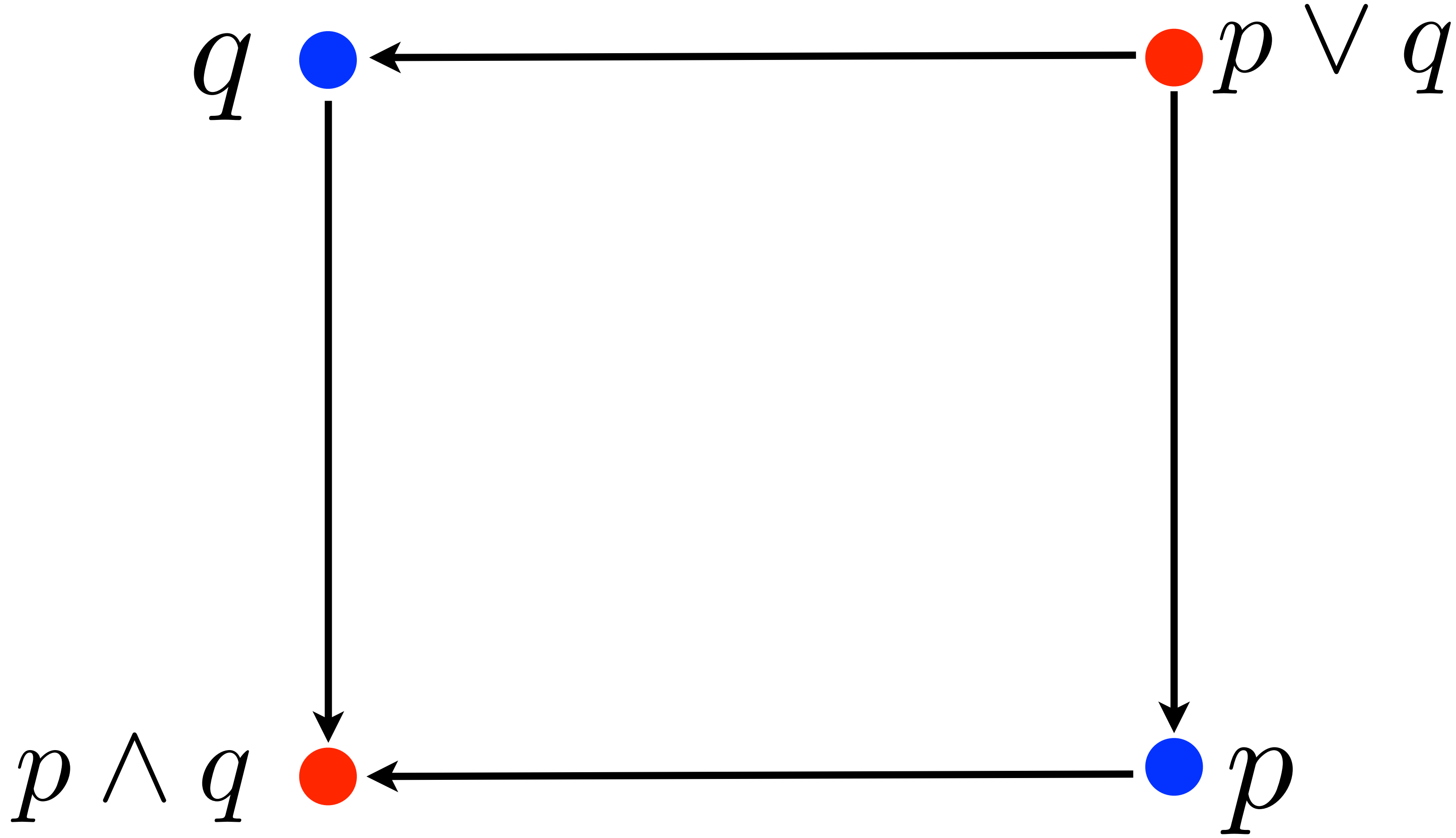
- every lower ideal is modular
- if $u \vee v, v \vee w, w \vee u$ exist,
  then $u \vee v \vee w$ exists





not modular !

Submodularity on $\mathcal{L}$



$$f(p) + f(q) \geq f(p \wedge q) + f(p \vee q)$$

But $p \vee q$ may not exist ...

We can define fractional join $\sum_{u \in \bigvee(p,q)} \nu_u u$

Def. $f : \mathcal{L} \to \mathbf{R}$ is submodular if

$$f(p) + f(q) \geq f(p \wedge q) + \sum \nu_u f(u)$$

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metric interval of p, q in $\mathcal{L}$

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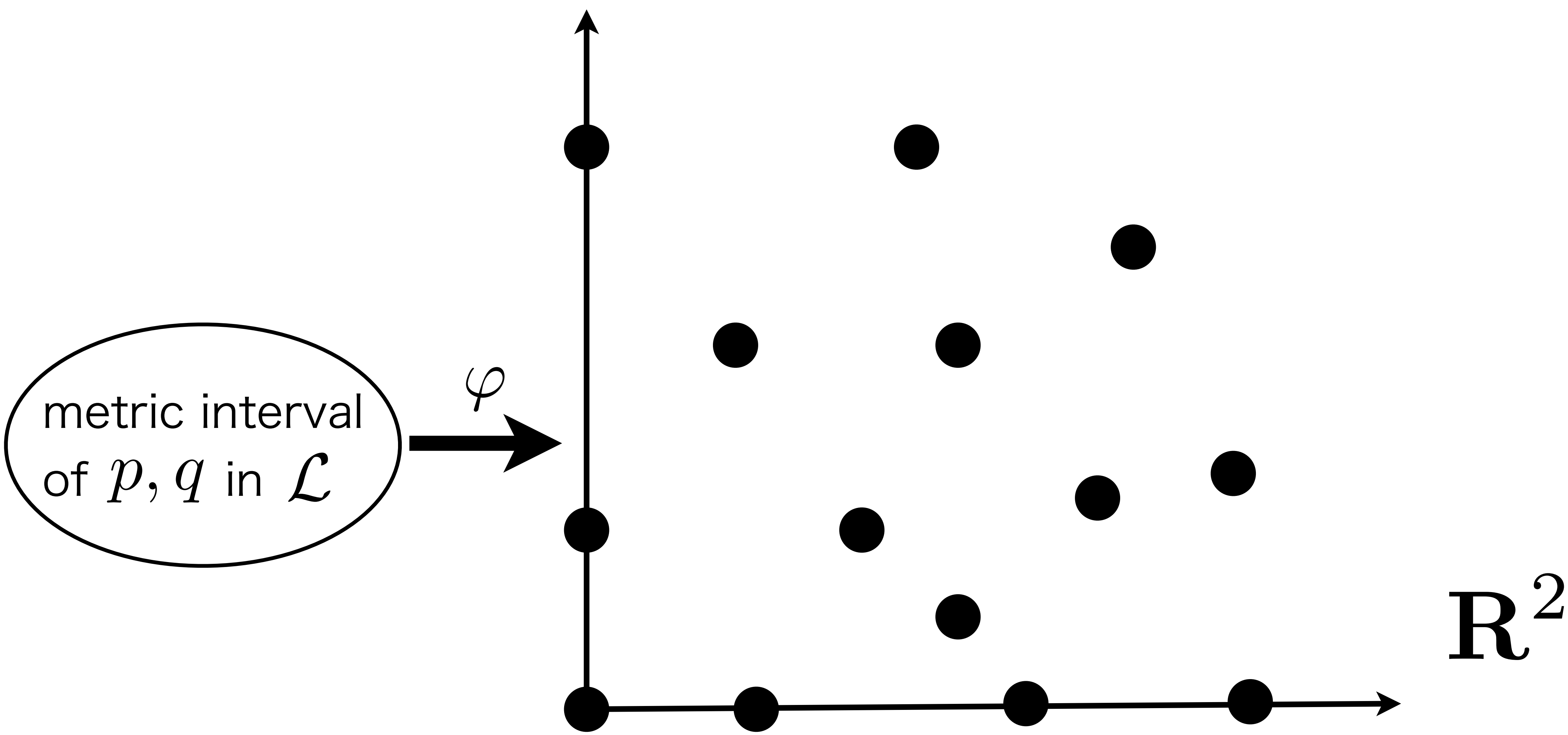

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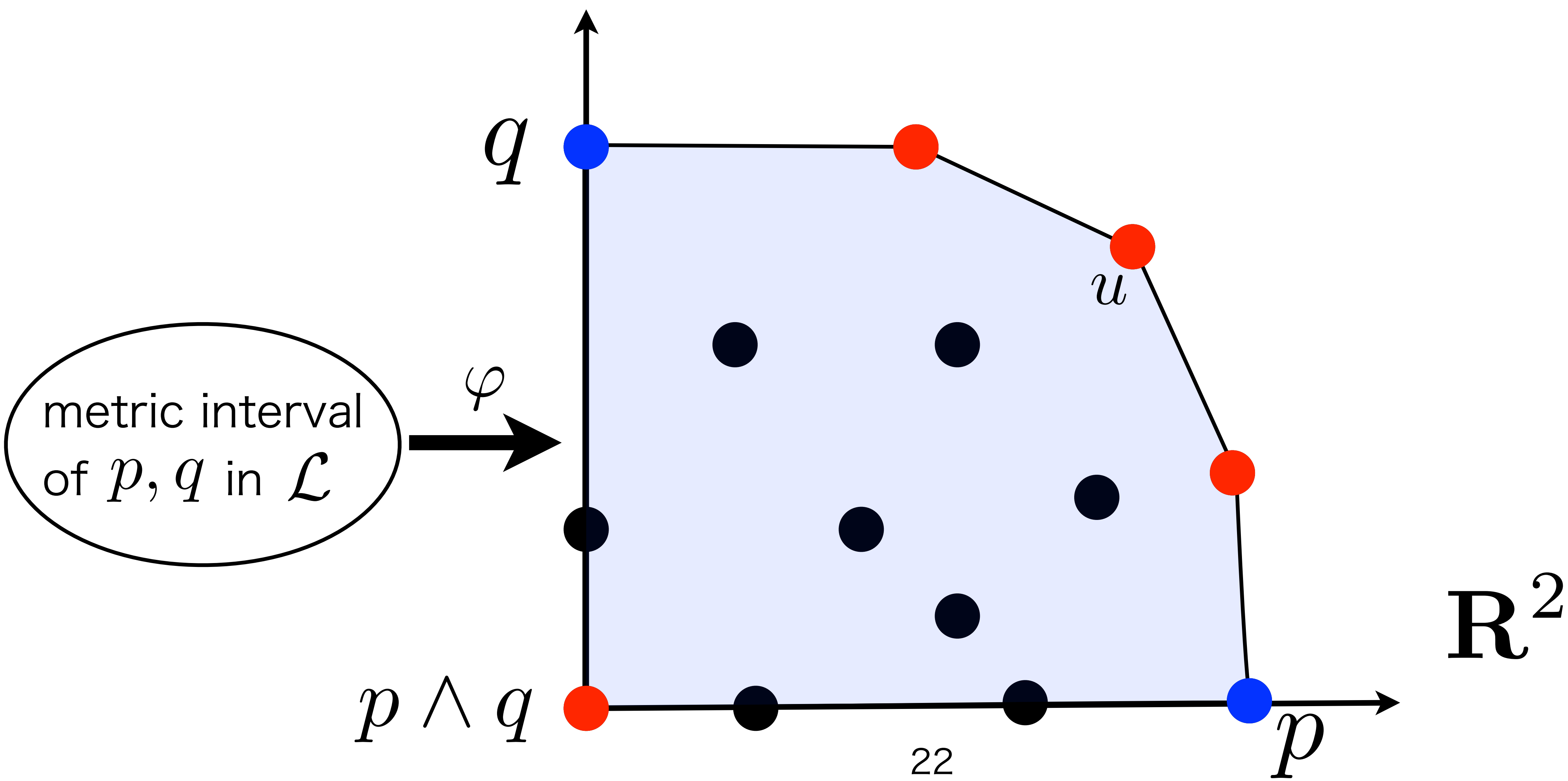

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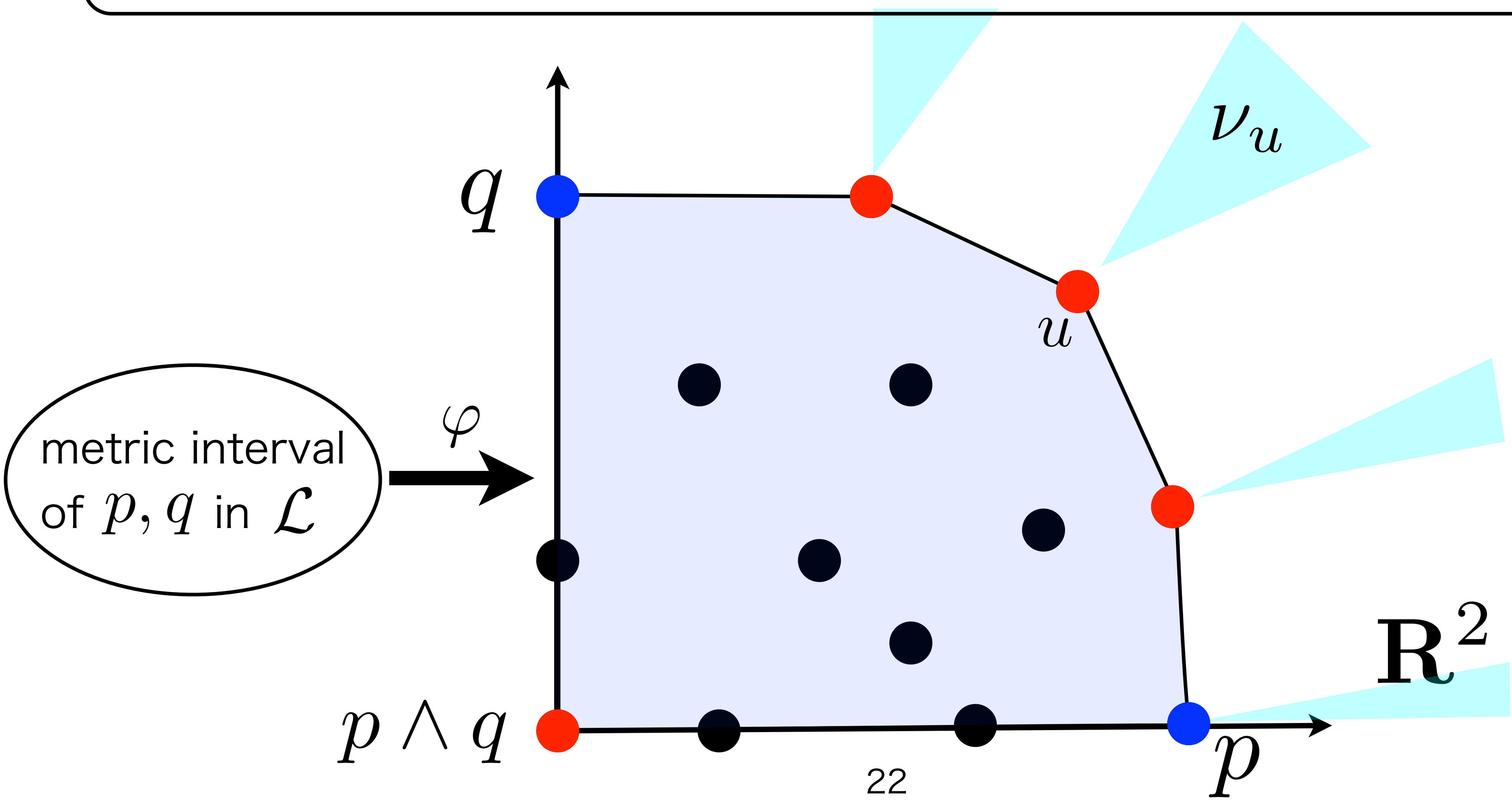

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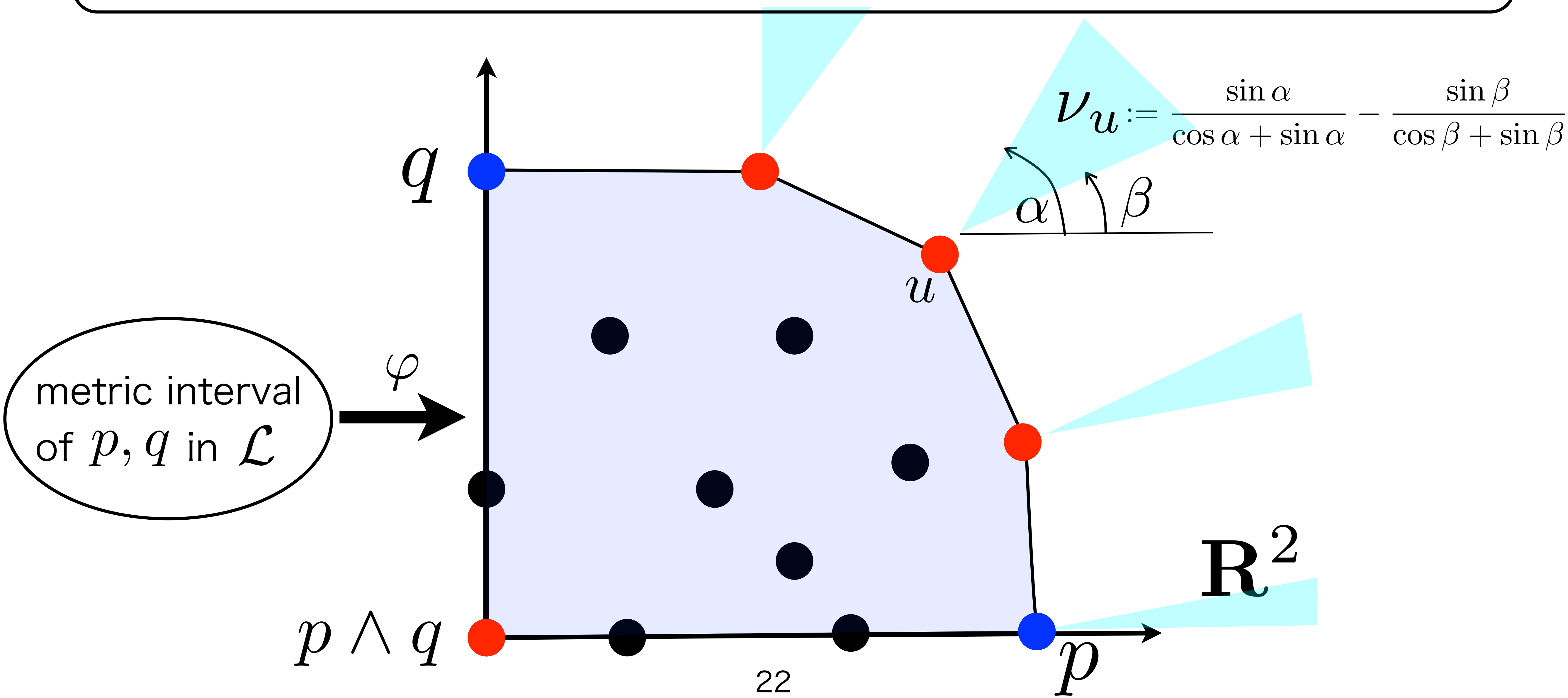

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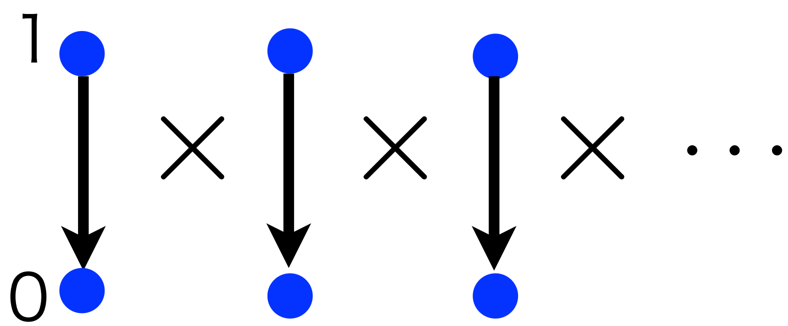
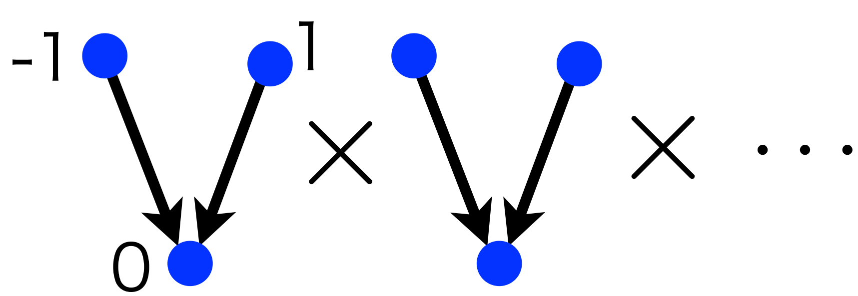
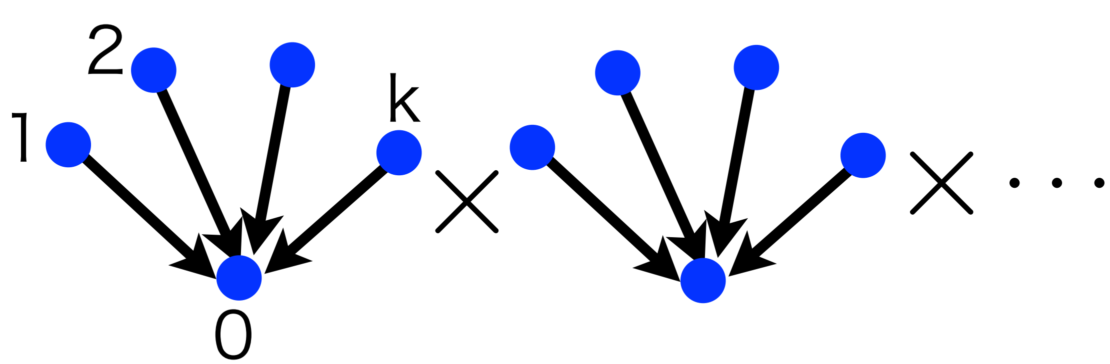
$\mathcal{L} \longrightarrow$ submodular func. on $\mathcal{L}$



submodular set func.

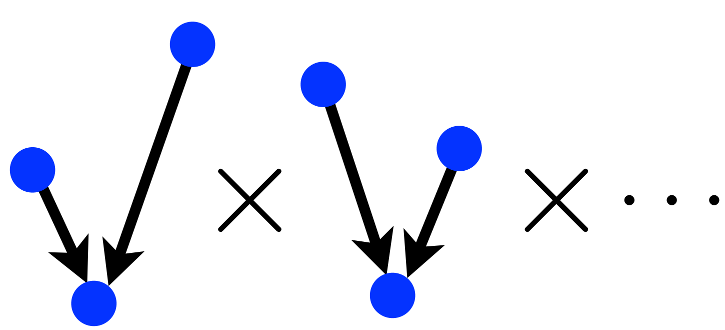


bisubmodular func.



k-submodular func.

Huber-Kolmogorov 12



skew-bisubmodular func.

Huber-Krokhin-Powell 13

Fujishige-Tanigawa-Yoshida 13

SFM (oracle model)

Input: oracle of f

$$\begin{array}{ll} \text{Min.} & f(p) \\ \text{s.t.} & p \in \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n \end{array}$$

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??? P or not P ???

SFM (VCSP model)

fixed constant : K

Input: all values of all $f_{i_1,i_2,\ldots,i_K}$

$$\text{Min.} \quad \sum_{i_1,i_2,\ldots,i_K} f_{i_1,i_2,\ldots,i_K}(p_{i_1},p_{i_2},\ldots,p_{i_K})$$

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P

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$$\text{s.t.} \quad p \in \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$$

P

fractional polymorphism criterion

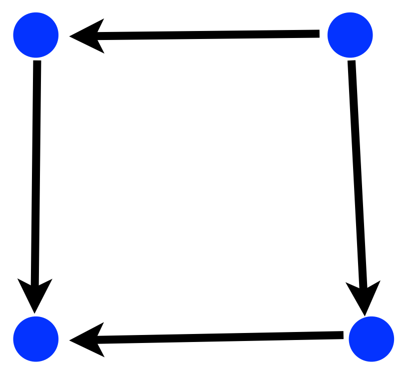
Thapper-Zivny FOCS'12

Γ is orientable modular if

$$\forall y_1, y_2, y_3, \exists w, \ \ d_\Gamma(y_i, y_j) = d_\Gamma(y_i, w) + d_\Gamma(w, y_j)$$

&

$\exists o$: orientation s.t.



Ex. Hasse diagram of modular (semi)lattice

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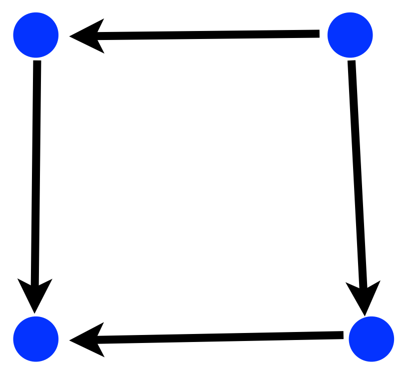
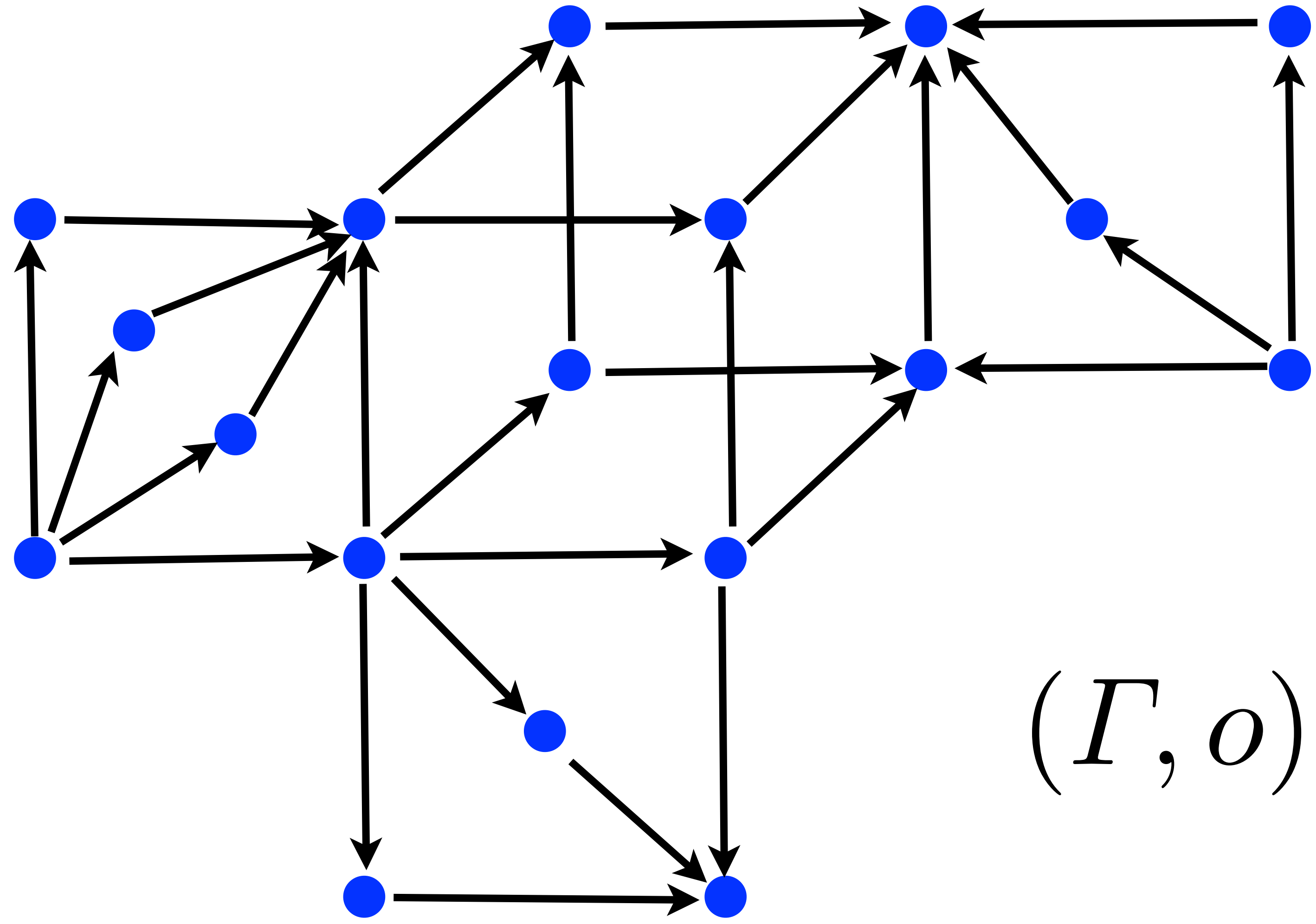
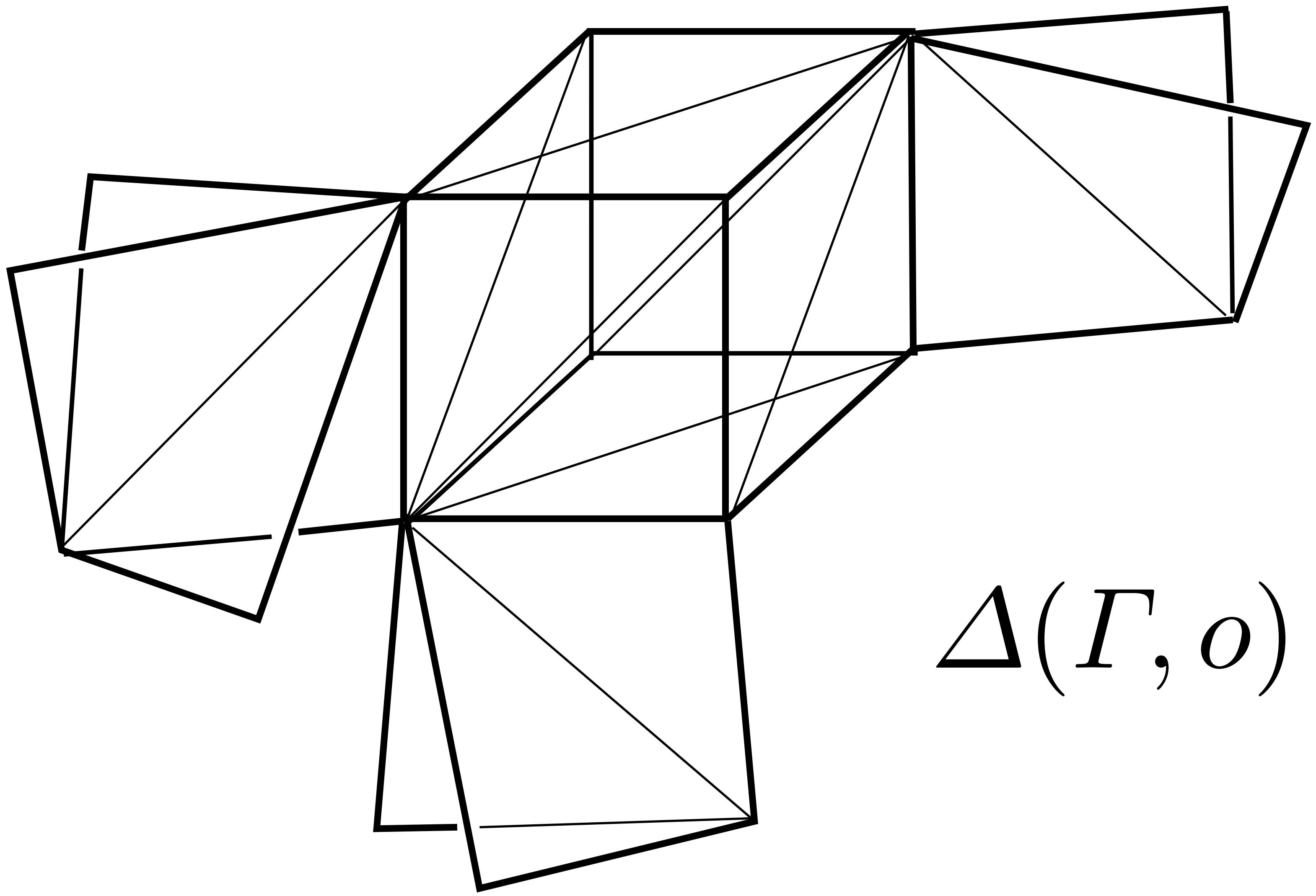


Ex. Hasse diagram of modular (semi)lattice

(Γ, o) : modular complex



(Γ, o)



$\Delta(\Gamma, o)$

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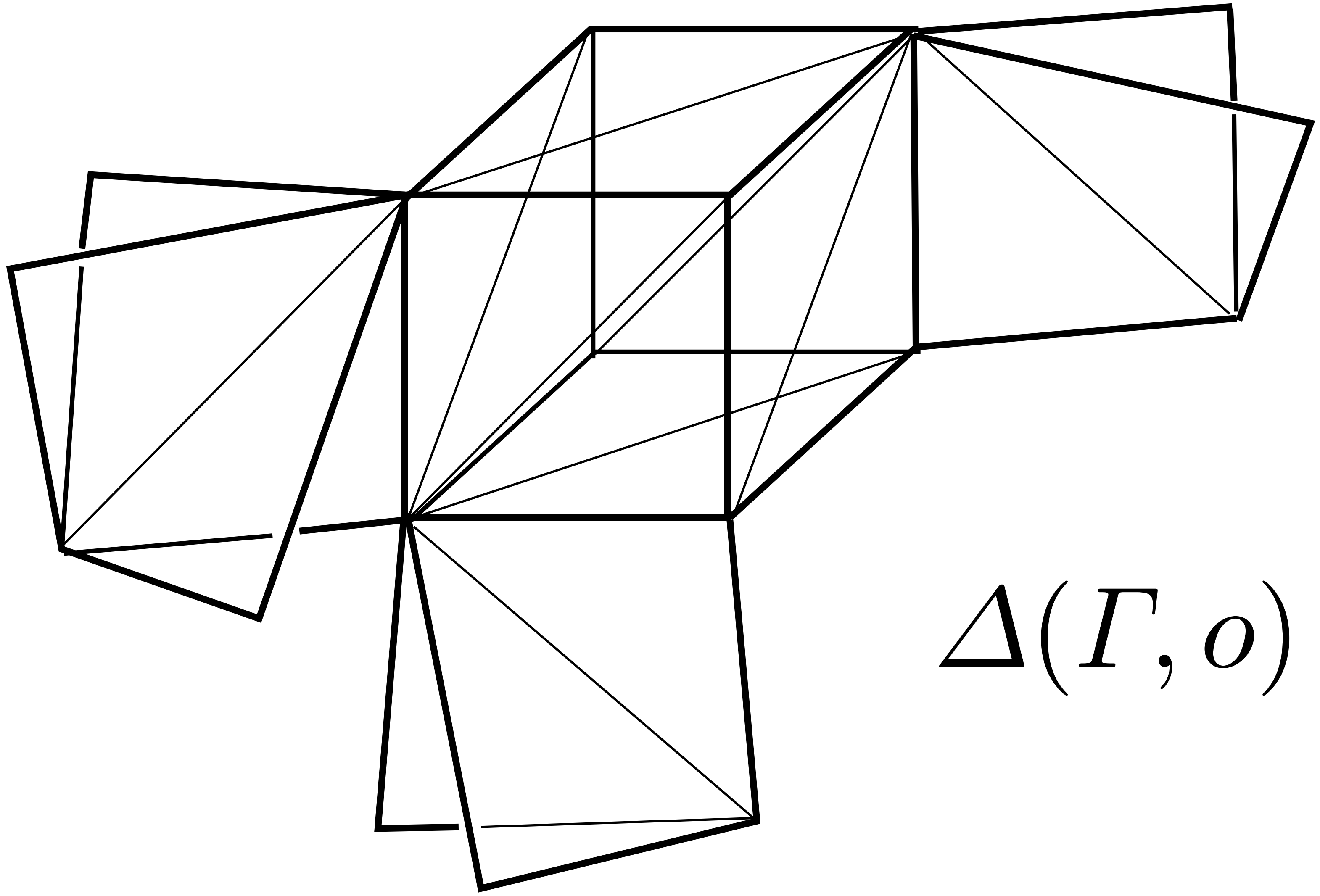


$\Delta(\Gamma, o)$

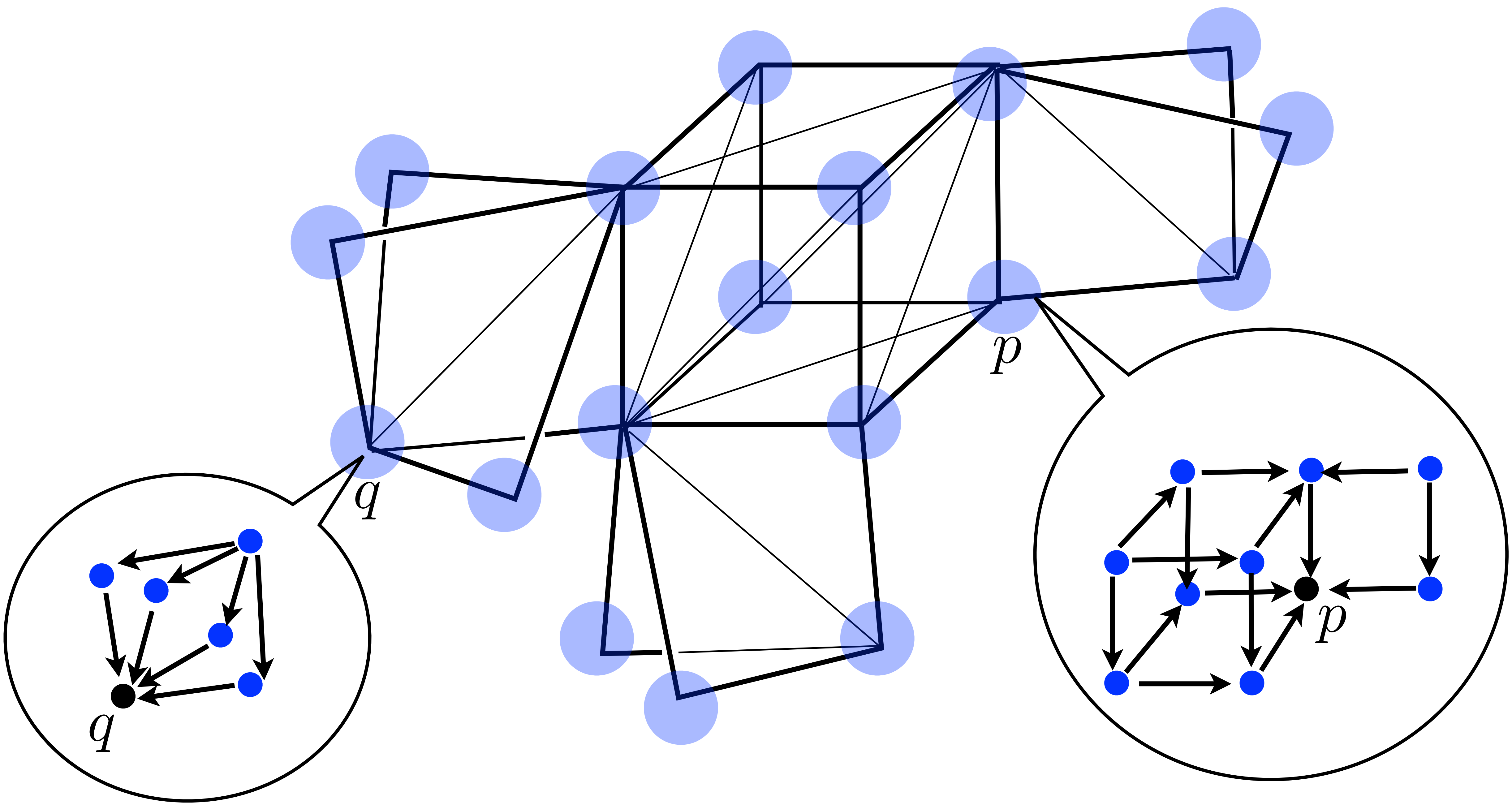
Lovasz extension

$$g : V_\Gamma \to \mathbf{R} \quad \Rightarrow \quad \bar{g} : \Delta(\Gamma, o) \to \mathbf{R}$$

Def. g is L-convex if

$\bar{g}$ is submodular on

each neighborhood semilattice



Minimization of g over Γ

is

Minimization of $\bar{g}$ over Δ

- Local optimality check = SFM
- Descent algorithm

Minimization of g over Γ

is

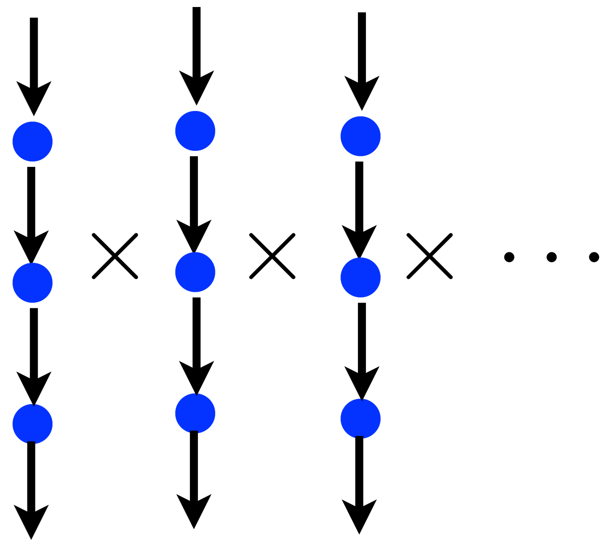
Minimization of $\bar{g}$ over Δ

- Local optimality check = SFM
- Descent algorithm

Local opt = Global opt

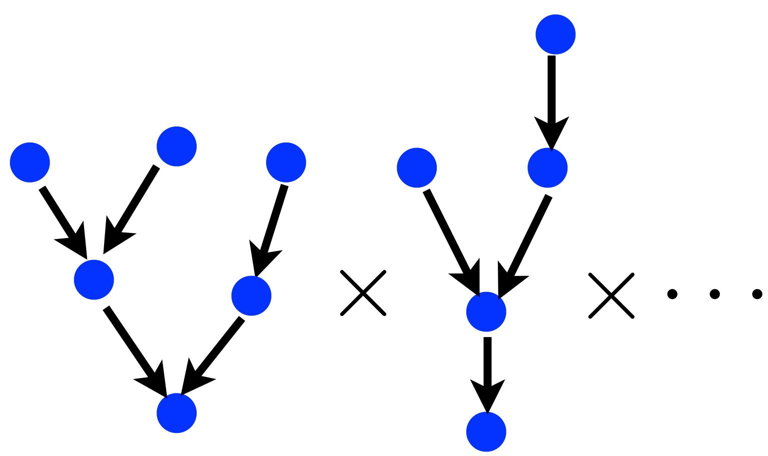
$(\Gamma, o) \longrightarrow$ L-convex func. on (Γ, o)

$\mathbf{Z}^n$



$\mathrm{L}^\natural$-convex func.

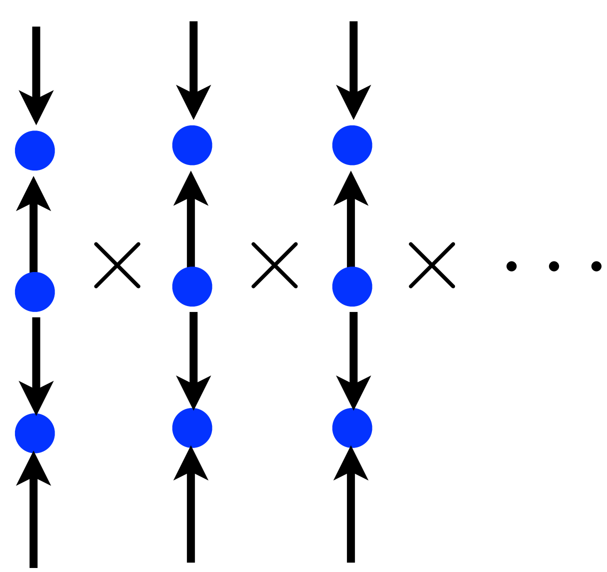
Favati-Tardella 1990, Murota 1996,
Fujishige-Murota 2000



tree-submodular func.

Kolmogorov 2012

$\mathbf{Z}^n$



UJ-convex func.

Fujishige 2013

d_Γ is L-convex on $(\Gamma \times \Gamma, o \times o)$

➡

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d_Γ is L-convex on $(\Gamma \times \Gamma, o \times o)$

➡ $\sum b_{il} d_\Gamma(y_l, \cdot) + \sum c_{ij} d_\Gamma(\cdot, \cdot)$

is L-convex on

$(\Gamma \times \Gamma \times \Gamma \cdots, o \times o \times o \cdots)$

➡

d_Γ is L-convex on $(\Gamma \times \Gamma, o \times o)$

arity 2

$$\Rightarrow \quad \sum b_{il} d_\Gamma(y_l, \cdot) + \sum c_{ij} d_\Gamma(\cdot, \cdot)$$

is L-convex on

$$(\Gamma \times \Gamma \times \Gamma \cdots, o \times o \times o \cdots)$$

$\Rightarrow$

d_Γ is L-convex on $(\Gamma \times \Gamma, o \times o)$

arity 2

➡ $\sum b_{il} d_\Gamma(y_l, \cdot) + \sum c_{ij} d_\Gamma(\cdot, \cdot)$

is L-convex on

$(\Gamma \times \Gamma \times \Gamma \cdots, o \times o \times o \cdots)$

➡ $\mathbf{0\text{-}EXT}[\Gamma]$ in P

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Combinatorial Multiflow Min-max

Max. flow = Min. ＊＊＊＊

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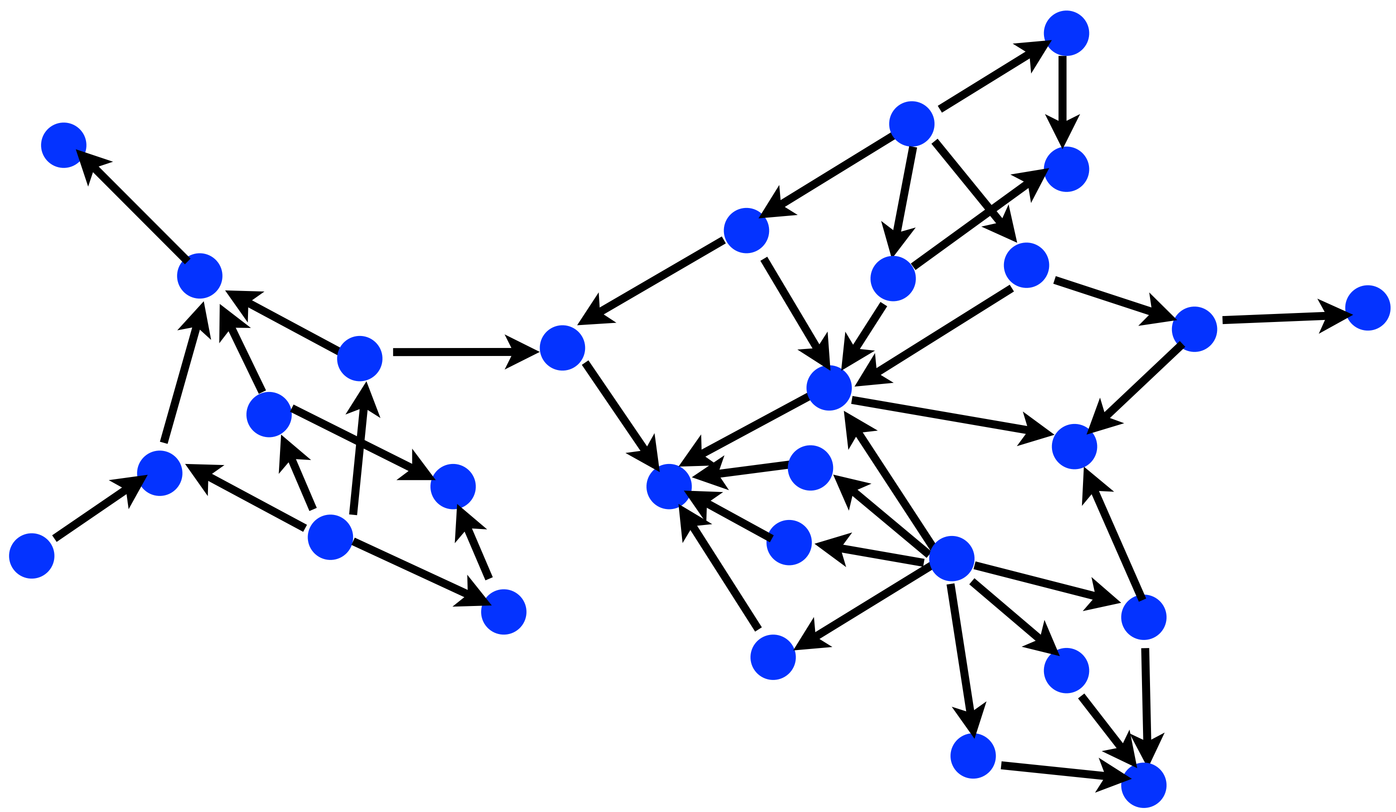
L-convex function on



2-dim. modular complex

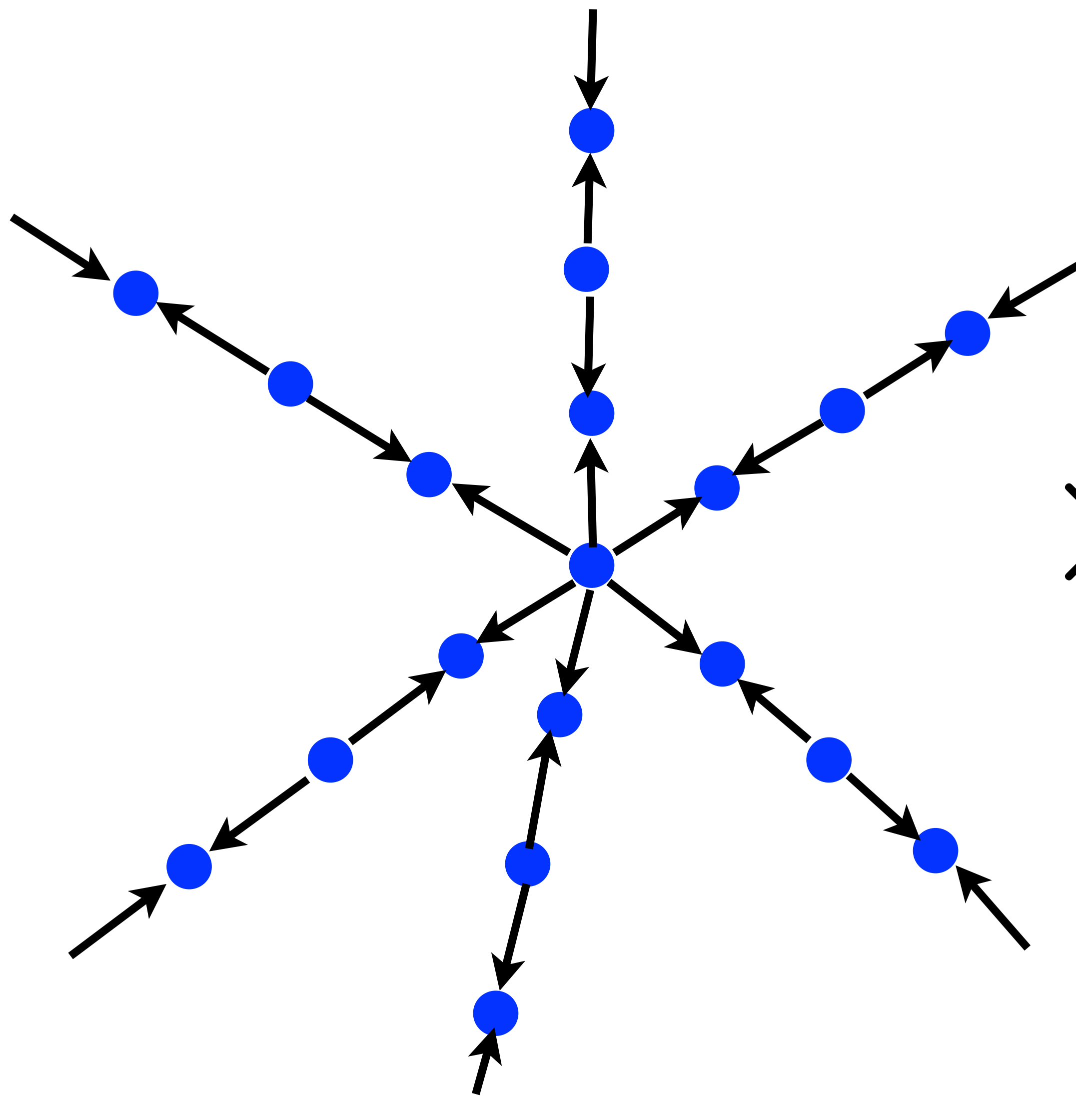
≒ 2-dim. tight span

Mincost free-multiflow

Karzanov 94

$$\text{Max.} \sum_{s,t \in S} |f_{st}| - \text{cost}\,(f) = \text{Min.} \ast\ast\ast\ast$$

L-convex function on



$\times \cdots$

Summary

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- Many issues …

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Thank you for your attention !