Multiflow Feasibility Problem for $K_3 + K_3$

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Contents

H. Hirai, Bounded fractionality of the multiflow feasibility problem for demand graph $K_3 + K_3$ and related maximization problems, *RIMS-Preprint 1645*, 2008.

available at `http://www.kurims.kyoto-u.ac.jp/~hirai/index.html`

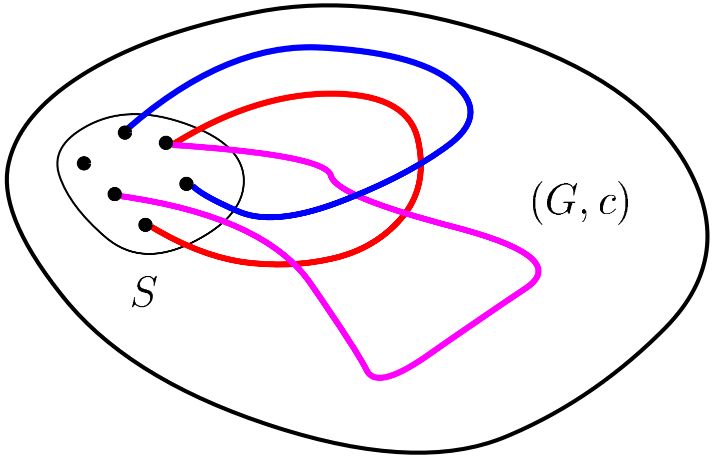
- Reviews on multiflow feasibility problems
- Karzanov's conjecture
- Main theorem
- Proof sketch

Multiflows (Multicommodity flows)

G: an undirected graph (*supply graph*)

$c : EG \to \mathbf{R}_+$: nonnegative edge capacity

$S \subseteq VG$: terminal set



A *multiflow* $f = (\mathcal{P}, \lambda) \overset{\text{def}}{\Longleftrightarrow}$

$\mathcal{P}$: set of S-paths

$\lambda : \mathcal{P} \to \mathbf{R}_+$: flow-value function satisfying capacity constraint

$$\sum \{\lambda(P) \mid P \in \mathcal{P} : e \in P\} \leq c(e) \quad (e \in EG).$$

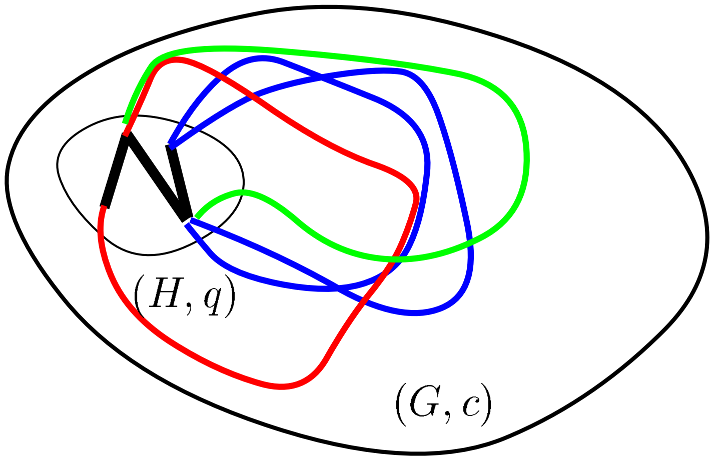
Multiflow feasibility problem

H: *demand graph* with $VH = S$
$q : EH \to \mathbf{R}_+$: demand function on edges EH.

Find a multiflow $f = (\mathcal{P}, \lambda)$ satisfying demand requirement

$$\sum\{\lambda(P) \mid P \in \mathcal{P} : P \text{ is } (s,t)\text{-path}\} = q(st) \quad (st \in EH),$$

or establish that no such a multiflow exists.



We are interested in behavior of multiflows for a fixed H and arbitrary G, c, q.

- $H = K_2$: single commodity flows

Theorem [Ford-Fulkerson 54]

c, q integral, feasible $\Rightarrow \exists$ *integral* solution.

- $H = K_2 + K_2$: 2-commodity flows

Theorem [Hu 63]

c, q integral, feasible $\Rightarrow \exists$ *half-integral* solution.

- $H = K_2 + K_2 + \cdots + K_2$: k-commodity flows

??? c, q integral, feasible $\Rightarrow \exists$ $1/p$-integral solution ($p \leq k$)???

(Jewell 67, Seymour 81)

Theorem [Lomonosov 85]

There is no integer $k > 0$ such that every feasible 3-commodity flow problem with integer capacity and demand has a $1/k$-integral solution.

Fractionality

frac(H) := the least positive integer k with the property:

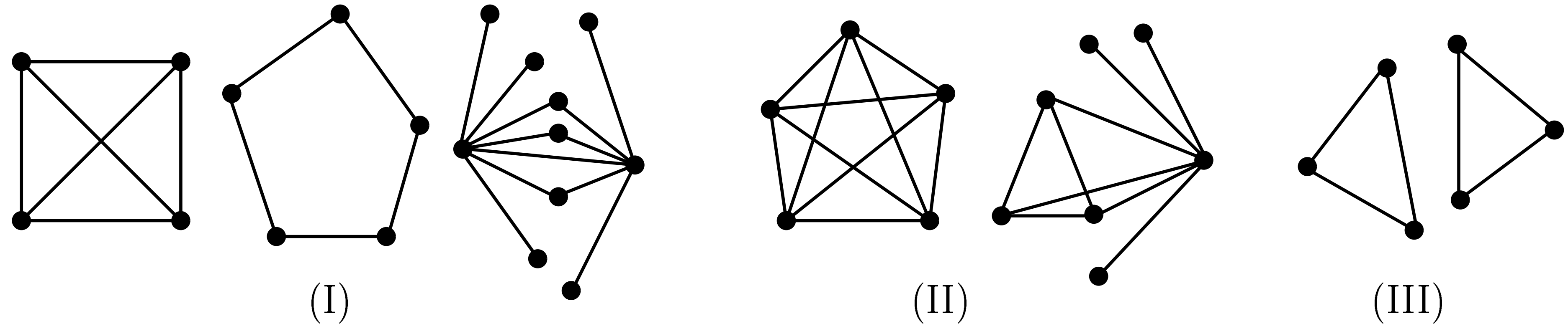
$\forall c, q$ integral, feasible $\Rightarrow \exists$ $1/k$-integral solution.

Problem [Karzanov 89,90]

Classify demand graphs H with frac$(H) < +\infty$.

Remark $H \supseteq K_2 + K_2 + K_2 \Rightarrow$ frac$(H) = +\infty$.

Demand graphs without $K_2 + K_2 + K_2$



(I) $H = K_4, C_5$, or star + star:

Theorem [Rothschild & Winston 66, Seymour 80, Lomonosov 76,85]
c, q Eulerian, feasible $\Rightarrow \exists$ integral solution ($\rightarrow$ frac(H) = 2).

by "c, q Eulerian" we mean $(G + H, c + q)$ is Eulerian.

(II) $H = K_5$ or star + K_3:

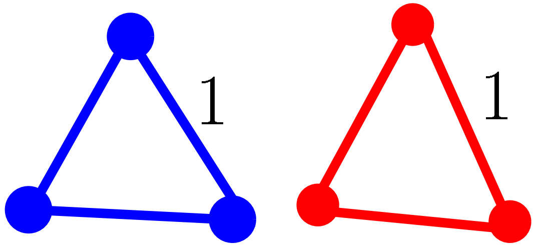
Theorem [Karzanov 87]
c, q Eulerian, feasible $\Rightarrow \exists$ integral solution ($\rightarrow$ frac(H) = 2).

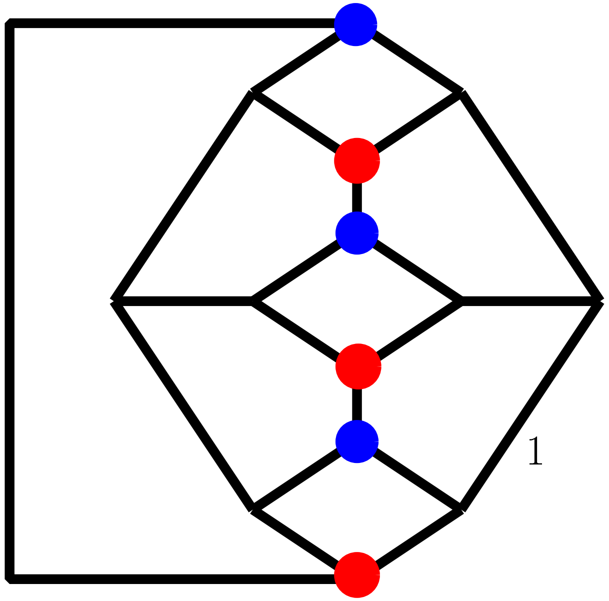
(III) $H = K_3 + K_3$ ....?

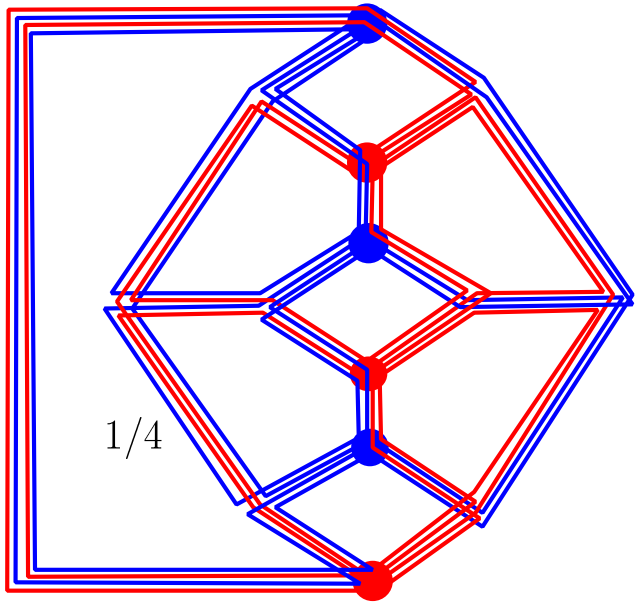
(III) $H = K_3 + K_3$

An example from A. Schrijver: *Combinatorial Optimaization*, p.1275.

$H =$







$\rightarrow$ frac$(K_3 + K_3) \geq 4$.

Conjecture [Karzanov 90, ICM, Kyoto]

1. frac$(K_3 + K_3) < +\infty$.
2. c, q Eulerian, feasible $\Rightarrow \exists$ half-integral solution ($\rightarrow$ frac$(K_3 + K_3) = 4$)

Cf. Problems 51, 52 in Schrijver's book

Main Theorem [H. 08]

$H = K_3 + K_3$, c, q Eulerian, feasible

$\Rightarrow \exists$ 1/12-integral solution.

$\rightarrow$ *the complete classification of*

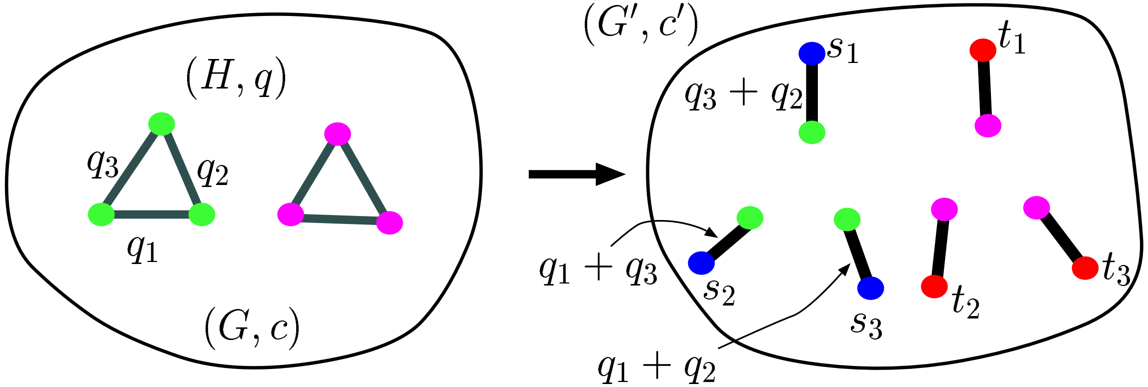
demand graphs having finite fractionality

Corollary frac$(H) < +\infty \Leftrightarrow H \not\supseteq K_2 + K_2 + K_2$.

Corollary frac$(K_3 + K_3) \in \{4, 8, 12, 24\}$.

Proof Sketch (very rough)

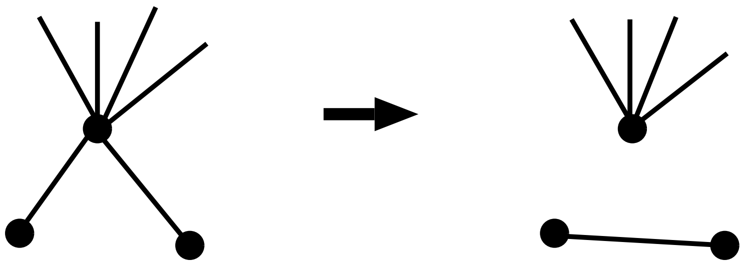
1. reduction to $K_{3,3}$-metric weighted multiflow maximization



2. combinatorial duality relation (of max-flow min-cut type)

3. fractional splitting-off and potential update

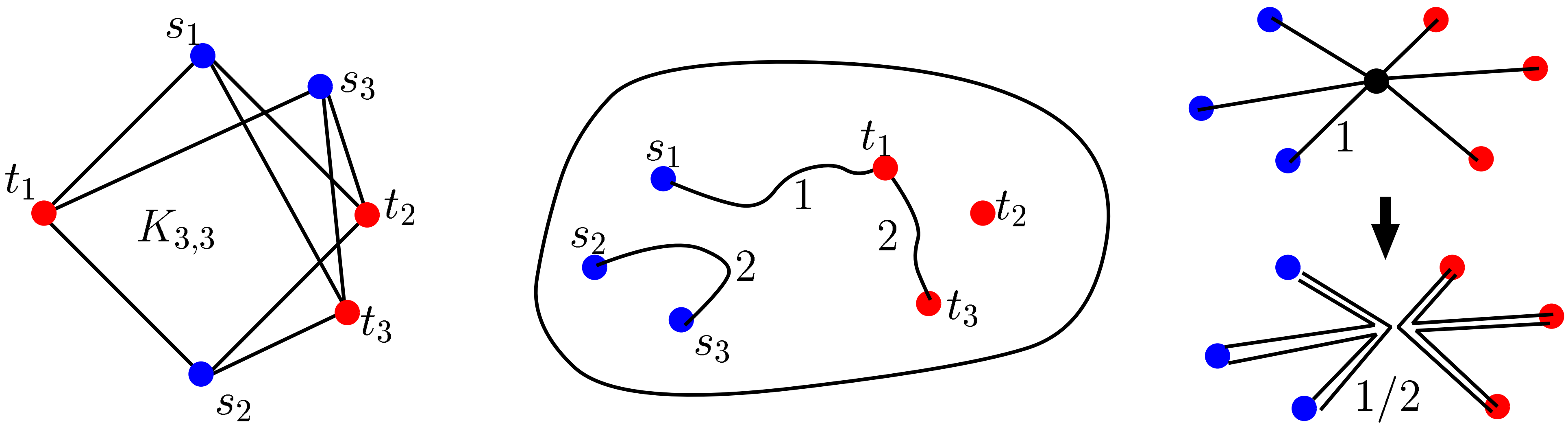
Recall: splitting-off



$K_{3,3}$-metric weighted maximum multiflow problem:

$S \subseteq VG$: 6-element terminal set with $S = VK_{3,3}$.

Maximize $\sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P)\lambda(P)$ s.t. $f = (\mathcal{P}, \lambda)$ for $(G, c; S)$,



Remark no integral optimum even if (G, c) inner Eulerian.

Theorem [H. 08]

$\exists$ 1/12-integral optimum in every inner Eulerian $K_{3,3}$-max problem.

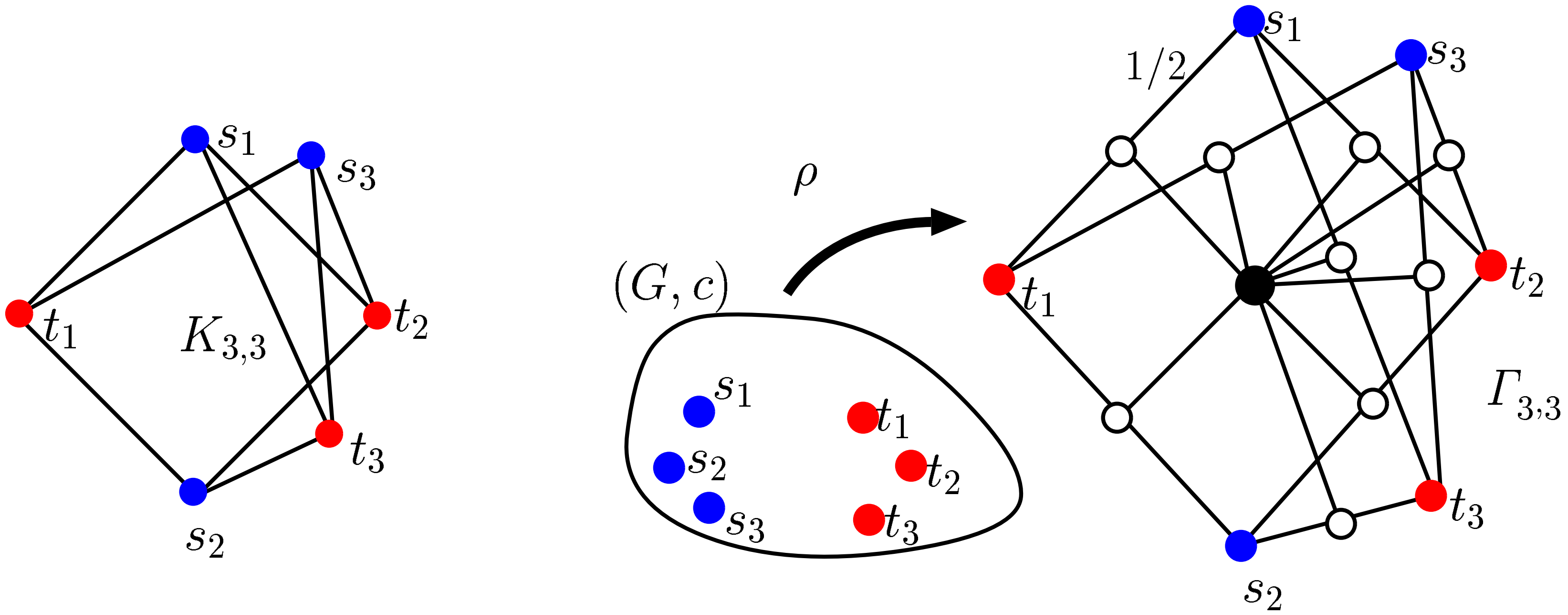
$\Rightarrow$ $\exists$ 1/12-integral solution in Eulerian $K_3 + K_3$-feasibiltiy problem

Combinatorial duality relation [Karzanov 89, 98]

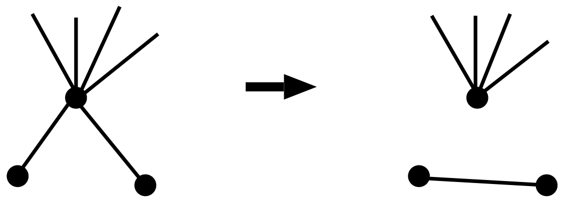
$$\text{Max.} \quad \sum_{P \in \mathcal{P}} d_{K_{3,3}}(s_P, t_P)\lambda(P) \quad \text{s.t.} \quad f = (\mathcal{P}, \lambda) \text{ for } (G, c; S)$$

$$= \text{Min.} \quad \sum_{xy \in EG} c(xy) d_{\Gamma_{3,3,\frac{1}{2}}}(\rho(x), \rho(y))$$

$$\text{s. t.} \quad \rho : VG \to V\Gamma_{3,3}, \ \rho|_S = id, \qquad \leftarrow \textit{potential}$$

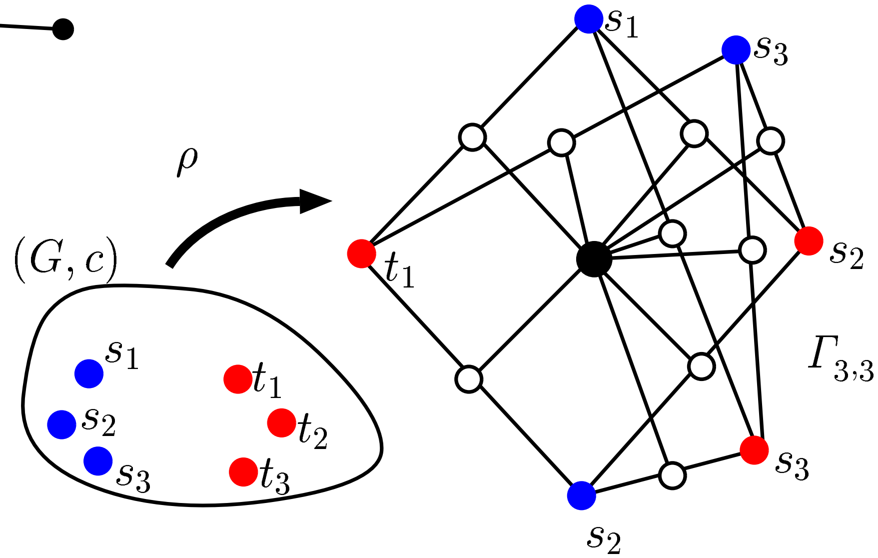


Splitting-off



ρ: optimal potential

$S_\rho = \{x \in VG \mid \rho(x) = \bullet$ or $\bullet\}$
$M_\rho = \{x \in VG \mid \rho(x) = \circ\}$
$C_\rho = \{x \in VG \mid \rho(x) = \bullet\}$



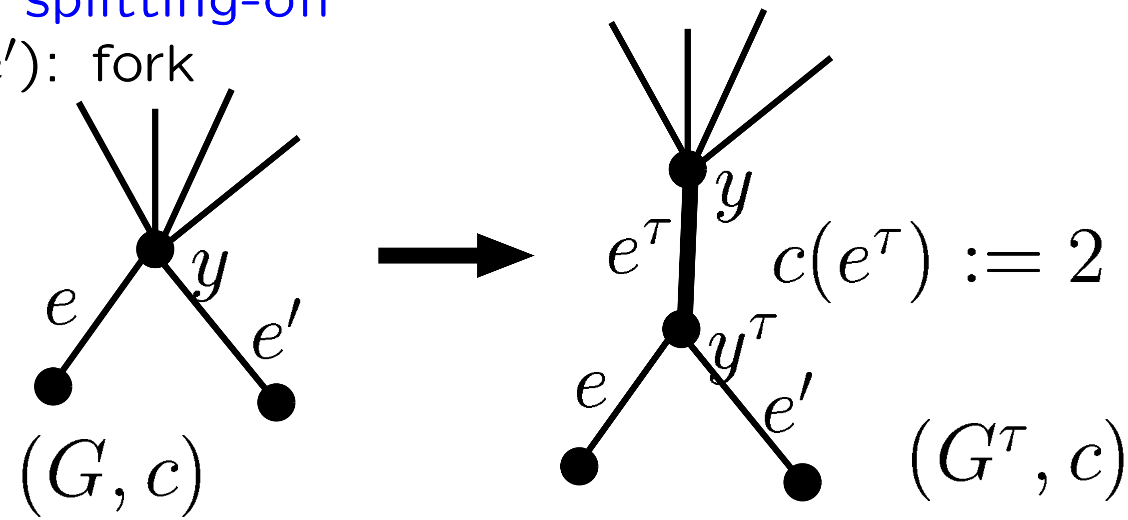
Proposition (G, c) inner Eulerian, y inner node in S_ρ
$\Rightarrow$ y is splittable. (the proof based on [Karzanov 87])

Corollary $M_\rho \cup C_\rho = \emptyset \Rightarrow \exists$ integral optimum.

Main step: (G, c): inner Eulerian, ρ: optimal potential
$(G, c; \rho) \to \cdots \to (G', c'; \rho')$ until $M_{\rho'} \cup C_{\rho'} = \emptyset$ keeping (G', kc') inner Eulerian

Fractional splitting-off

$\tau = (e, y, e')$: fork



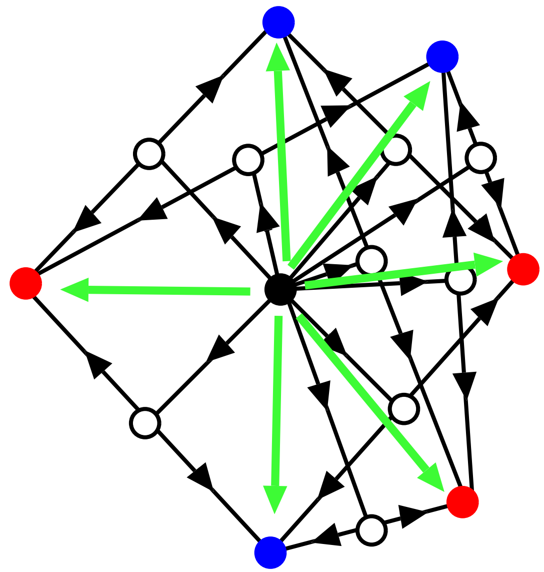
Splitting capacity:

$$\begin{aligned}\alpha(\tau) \;&:=\; \max\{0 \le \alpha \le 2 \mid \mathrm{opt}(G, c) = \mathrm{opt}(G^\tau, c - \alpha\chi_{e^\tau})\} \\ &=\; \min\left\{ \frac{\langle c, d_{\Gamma_{3,3}} \circ \rho' \rangle - \mathrm{opt}(G, c)}{d_{\Gamma_{3,3}}(\rho'(y), \rho'(y^\tau))} \;\middle|\; \rho' : \text{potential with } \rho'(y) \neq \rho'(y^\tau) \right\},\end{aligned}$$

where $\langle c, d_{\Gamma_{3,3}} \circ \rho' \rangle := \sum c(xy) d_{\Gamma_{3,3}}(\rho'(x), \rho'(y))$.

Remark τ is splittable if and only if $\alpha(\tau) = 2$.

ρ: optimal potential $\Rightarrow$ we can take ρ' combinatorially close to ρ.



Splitting-off with Potential UPdate (SPUP)

Lemma ρ: optimal potential, (G, c): Eulerian at $M_\rho \cup C_\rho$, $\forall \tau$: fork at C_ρ, $\exists \rho'$ s.t.

$\rho'(x) \neq \rho(x)$ implies $\rho(x) \longrightarrow \rho'(x)$ or $\rho(x) \longrightarrow \rho'(x)$ and

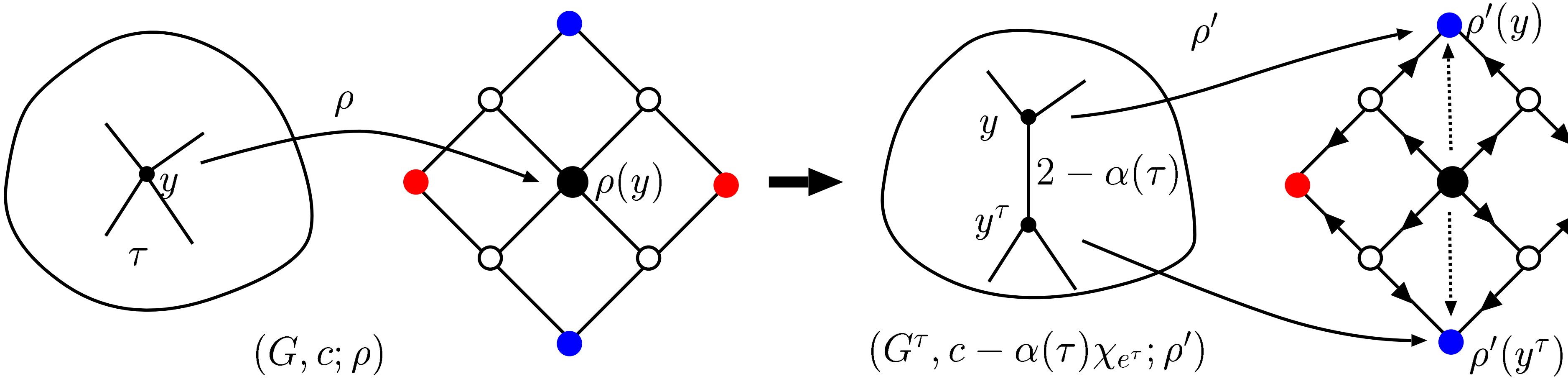
$$\alpha(\tau) = \frac{\langle c, d_{\Gamma_{3,3}} \circ \rho' \rangle - \langle c, d_{\Gamma_{3,3}} \circ \rho \rangle}{d_{\Gamma_{3,3}}(\rho'(y), \rho'(y^\tau))} \in \left\{0, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{6}{4}, 2\right\}.$$

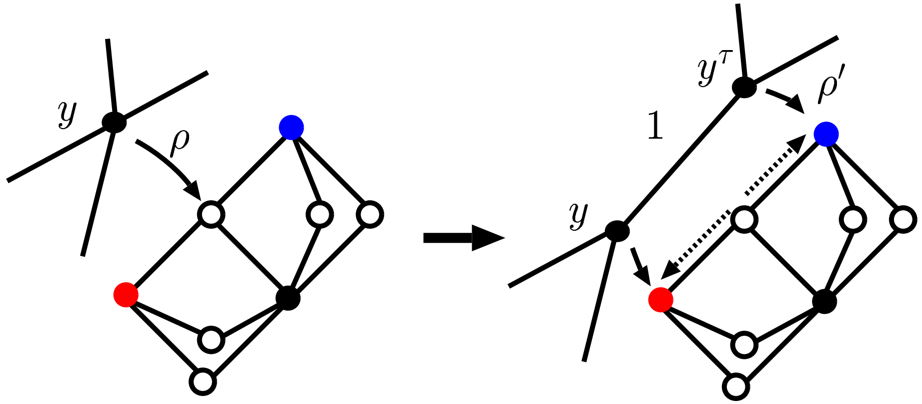
$$S_\rho = \{x \mid \rho(x) = \bullet \text{ or } \bullet\}$$
$$M_\rho = \{x \mid \rho(x) = \circ\}$$
$$C_\rho = \{x \mid \rho(x) = \bullet\}$$

S_ρ is splittable if inner Eulerian.

In addition, if $C_\rho = \emptyset$ and c is integral, then so does for $\exists \tau$ at M_ρ.

SPUP: $(G, c; \rho) \leftarrow (G^\tau, c - \alpha(\tau)\chi_{e^\tau}; \rho')$





Proposition

We can repeat SPUPs until $M_\rho \cup C_\rho = \emptyset$ keeping $(G, 12c)$ inner Eulerian.

Concluding remarks

- We do not know whether $1/12$ is tight.
- Recently we proved a generalized conjecture for $k = 12$:

  For a terminal weight $\mu : \binom{S}{2} \to \mathbf{R}_+$,

  $\dim T_\mu \leq 2$ *if and only if there exists* $k > 0$ *such that every Eulerian* μ*-max problem has a* $1/k$*-integral optimum,*

  where $T_\mu :=$ Minimal $\{p \in \mathbf{R}_+^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}$

- Half-integral $\Gamma_{3,3}$-metric packing [H. 07, *Combinatorica*, to appear]