Discrete Convex Optimization for Left-Right Action (nc-rank & det)

Part II

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Contents

M. Hamada & H. Hirai: Computing the nc-rank via discrete convex optimization on CAT(0) spaces, *SIAM Journal on Applied Geometry and Algebra, 2021.*

H. Hirai & M. Ikeda: A cost-scaling algorithm for computing the degree of determinants, 2020, arXiv.

H. Hirai: Computing the degree of determinants via discrete convex optimization on Euclidean buildings, *SIAM Journal on Applied Geometry and Algebra,* 2019.

Edmonds Problem

Edmonds 1967

Can we compute the rank of

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

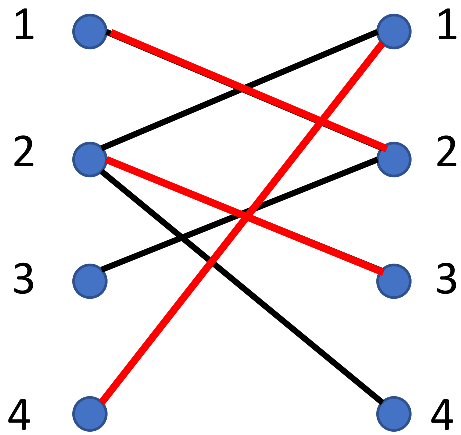
in polynomial time ?

x_i : variables, A_i: $n \times n$ matrices over field $\mathbb{K}$

A: matrix over $\mathbb{K}[x_1, x_2, \ldots, x_m] \subset \mathbb{K}(x_1, x_2, \ldots, x_m)$

- RP, but P ? (for large field)
- Related to fundamental problems in diverse areas
  ～ combinatorial optimization, rigidity theory, TCS,…

Motivation in combinatorial optimization: Algebraic interpretation of bipartite matching



$G = (U, V; E)$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} & x_{12} & & \\ x_{21} & & x_{23} & x_{24} \\ & x_{32} & & \\ & & & x_{41} \end{pmatrix} \end{array}$$

$$= \sum_{ij \in E} E_{ij} x_{ij}$$

$$E_{ij} = \begin{array}{cc} & \begin{array}{cc} & j \end{array} \\ i & \begin{bmatrix} \cdots & 1 \end{bmatrix} \end{array}$$

Lem. # maximum matching of G $= \operatorname{rank} A$

$$\because \det A = \sum_{M} \pm \prod_{ij \in M} x_{ij}$$

- min-max formula (Hall, König-Egerváry)
- polynomial time algorithm

$$T_{ij} = \begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \end{array}$$

Non-bipartite matching ---- $A = \sum_{ij \in E} T_{ij} x_{ij}$

Linear matroid intersection ---- $A = \sum_{k=1}^{m} a_k b_k^T \, x_k$

Linear matroid matching ----- $A = \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T) \, x_k$

$\exists$ min-max theorem + polynomial time algorithm (Edmonds, Lovász)

Non-commutative Edmonds Problem

Ivanyos-Qiao-Subrahmanyam 2017

Can we compute the rank ***(ncrank)*** of

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

in polynomial time ?

x_i : noncommutative variables, A_i: matrices over field $\mathbb{K}$

A: matrix over free ring $\mathbb{K}\langle x_1, x_2, \dots, x_m \rangle$

$\cap$

free skew field $\mathbb{K}(\langle x_1, x_2, \dots, x_m \rangle)$

- $\mathrm{rank}\, A \leq \mathrm{ncrank}\, A$

FR-formula → min-max relations in combinatorial optimization

Thm (Fortin-Reutenauer 2004)

$$\mathrm{ncrank}\ \textstyle\sum_k A_k x_k = 2n - \mathrm{Max.}\ r + s$$

$$\text{s.t.}\quad PA_kQ = \begin{array}{c|cc|}\cline{2-3} & * & * \\ & * & * \\ \cline{2-2} r & \multicolumn{1}{|c|}{0} & * \\ \cline{2-3} & s & \end{array} \quad (\forall k)$$

$$P, Q \in \mathrm{GL}_n(\mathbb{K})$$

In the case of bipartite matching $A = \sum_{ij \in E} E_{ij} x_{ij}$,
P, Q can be chosen as permutation matrices

$$\begin{aligned} \mathrm{ncrank\ A} &= \mathrm{Min}\ 2n - |S| \ \text{ s.t. } S\text{: stable set of } G \\ &= \mathrm{Max}\ |M| \text{ s.t. } M\text{: matching in } G \\ &= \mathrm{rank}\ A \end{aligned}$$

→ $\mathrm{rank}\ A = \mathrm{ncrank}\ A$ (Edmods-Rado property)

Non-bipartite matching ---- $A = \sum_{ij \in E} T_{ij} x_{ij}$ → $\mathrm{rank}\, A < \mathrm{ncrank}\, A$

Linear matroid intersection ---- $A = \sum_{k=1}^{m} a_k b_k^T x_k$ → $\mathrm{rank}\, A = \mathrm{ncrank}\, A$

Linear matroid matching ---- $A = \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T)\, x_k$ → $\mathrm{rank}\, A < \mathrm{ncrank}\, A$

2x2-partitioned matrix

$$A = \begin{pmatrix} A_{11}x_{11} & A_{12}x_{12} & \cdots & A_{1n}x_{1n} \\ A_{21}x_{21} & A_{22}x_{22} & \cdots & A_{2n}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}x_{n1} & A_{n2}x_{n2} & \cdots & A_{nn}x_{nn} \end{pmatrix}$$ where $A_{ij} \in \mathbb{K}^{2\times 2}$

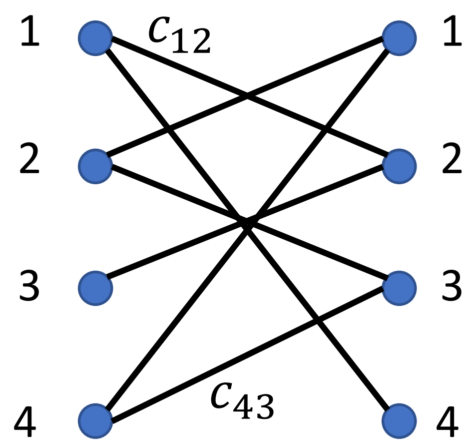
$\mathrm{rank}\, A = \mathrm{ncrank}\, A$ (from Iwata-Murota 95)

“Matching” concept & combinatorial “blow-up free” algorithm
based on Wong sequence (H-Iwamasa 2020)

Weighted Edmonds problem: Motivation

$\sim$ capture "weighted" combinatorial optimization problems from (non-commutative) linear algebra

Ex: Weighted bipartite matching

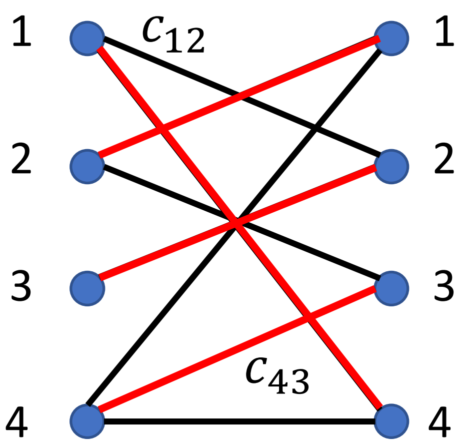


$c_{ij} \in \mathbb{Z}_+$: edge-weight

Max. $c(M) := \sum_{ij \in M} c_{ij}$

s.t. M: perfect matching

Algebraic Interpretation of Weighted Matching



$$A(t) = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} & x_{12}t^{c_{12}} & & x_{14}t^{c_{14}} \\ x_{21}t^{c_{21}} & & x_{23}t^{c_{23}} & \\ & x_{32}t^{c_{32}} & & \\ x_{41}t^{c_{41}} & & x_{43}t^{c_{43}} & x_{44}t^{c_{44}} \end{pmatrix} \end{array}$$

$$= \sum_{ij \in E} E_{ij} x_{ij} t^{c_{ij}}$$

Lem. Max weight of perfect matching $= \deg_t \det A(t)$

$$\because \deg \det A = \deg \sum_{M} \pm t^{c(M)} \prod_{ij \in M} x_{ij} = \max_{M} c(M)$$

Weighted non-bipartite matching ---- $A = \sum_{ij \in E} T_{ij} x_{ij} t^{c_{ij}}$

Weighted linear matroid intersection ---- $A = \sum_{k=1}^{m} a_k b_k^T \, x_k t^{c_k}$

Weighted linear matroid matching ----- $A = \sum_{k=1}^{m} (a_k b_k^T - b_k a_k^T) \, x_k t^{c_k}$

A weighted version of Edmonds Problem

Can we compute deg det of

$$A(t) = A_1(t)x_1 + \cdots + A_m(t)x_m$$

in polynomial time ?

$x_1, x_2, \ldots, x_m$: variables

$A_1(t), \ldots, A_m(t)$: matrices over $\mathbb{K}[t]$

A: matrix over $\mathbb{K}(x_1, x_2, \ldots, x_m)(t)$

Important special case:

$$A(t) = A_1 x_1 t^{c_1} + \cdots + A_m x_m t^{c_m}$$

$A_1, \ldots, A_m$: matrices over $\mathbb{K}$, $c_1, \ldots, c_m$: (nonnegative) integers

Goal: develop a non-commutative version

Contribution of [H.19] & [H-Ikeda 20]

- Formulate weighted non-commutative Edmonds problem

  by using *Dieudonné determinant* Det
- Establish a formula for deg Det ($\sim$ an analogue of FR formula for nc-rank)

  $\sim$ *L-convex function on Euclidean building*
- Algorithm: minimization of L-convex function

  $\sim O(nd)$ nc-rank computation, d: max degree
- Special case of deg det = deg Det (strong Edmonds-Rado property):

  $\sim$ weighted linear matroid intersection

- Polytime algorithm for deg Det of $A_1 x_1 t^{c_1} + \cdots + A_m x_m t^{c_m}$

  → Polytime algorithm for deg det = deg Det of 2x2-partitioned matrix
- Polyhedral interpretation of deg Det by nc-version of Newton polytope

How to see $A(t) = A_1(t)x_1 + \cdots + A_m(t)x_m$

- Matrix over (skew) polynomial ring $\mathbb{K}(\langle x \rangle)[\mathrm{t}]$

$$\text{Ore ring ---- } \forall p, q, \exists \underset{\neq 0}{u, v}: pu = qv$$

- Matrix over the rational function skew field $\mathbb{K}(\langle x \rangle)(\mathrm{t})$

$$p/q = p'/q' \Leftrightarrow \exists \underset{\neq 0}{u, v},\ pu = p'v, qu = q'v$$

- Degree: $\deg p/q \coloneqq \deg p - \deg q$

How to define "determinant" of matrices over skew field

A: nonsingular over $\mathbb{F}$ (Our case: $\mathbb{F} = \mathbb{K}(\langle x \rangle)(\mathrm{t})$)

Bruhat decomposition: LU-decomposition of matrices over skew field

$$A = \underbrace{L}_{\text{uni-lower-triangular}} \ \underbrace{D}_{\text{diagonal}} \ \underbrace{P}_{\text{permutation}} \ \underbrace{U}_{\text{uni-upper-triangular}}$$

unique

Dieudonné determinant $\in$ Abelianization $\mathbb{F}^*/[\mathbb{F}^*, \mathbb{F}^*]$ of $\mathbb{F}^*$

$$\mathrm{Det}\, A := \mathrm{sgn}(P) D_{11} D_{22} \dots D_{nn} \ \bmod \underbrace{[\mathbb{F}^*, \mathbb{F}^*]}_{\text{commutator group}}$$

$\mathrm{Det}\, A := 0$ for singular A

Lem. $\mathrm{Det}\, AB = \mathrm{Det}\, A\ \mathrm{Det}\, B$

See e.g., P. Cohn: *Further Algebra and Applications*, Springer, 2003

Weighted Non-commutative Edmonds Problem

Can we compute $\deg \mathrm{Det}$ of

$$A(t) = A_1(t)x_1 + \cdots + A_m(t)x_m$$

in polynomial time ?

$x_1, x_2, \ldots, x_m$: variables $x_i x_j \neq x_j x_i$

$A_1(t), \ldots, A_m(t)$: matrices over $\mathbb{K}[t]$

A: matrix over $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)(t)$

Rem. $\deg \mathrm{Det}$ is well-defined since $\deg$ is zero on commutators

$$\deg\,(pqp^{-1}q^{-1}) = 0$$

Formula for deg Det

$$A(t) = A_1(t)x_1 + \cdots + A_m(t)x_m$$

This problem is viewed as discrete convex optimization over Euclidean building

Thm. [H. 19]

$$\deg \mathrm{Det}\, A = \quad \mathrm{Min.} -\deg \det P \; - \; \deg \det \mathrm{Q}$$

$$\text{s.t.}\;\; \deg\, (PA_kQ)_{ij} \leq 0 \;\; (\forall k, \forall ij)$$

$$P, Q \in \mathrm{GL}_n(\mathbb{K}(t))$$

is definable on any field with valuation, such as $\mathbb{Q}$

(Proof of $\leq$; Murota 95 for deg det)

$$\deg\, (PA_kQ)_{ij} \leq 0 \;\; (\forall k, \forall ij) \Leftrightarrow \deg\, (PAQ)_{ij} \leq 0 \;\; (\forall ij)$$

$$\Rightarrow \deg \mathrm{Det}\, PAQ \leq 0 \;\Rightarrow \deg\, (\mathrm{Det}\, P\, \mathrm{Det}\, A\, \mathrm{Det}\, Q) \leq 0$$

$$\Rightarrow \underbrace{\deg \mathrm{Det}\, P}_{\deg \det P} + \deg \mathrm{Det}\, A + \underbrace{\deg \mathrm{Det}\, Q}_{\deg \det Q} \leq 0$$

Auxiliary lemma

$\mathbb{F}$: skew field, $\mathbb{F}(t)$: rational function skew field (our case: $\mathbb{F} = \mathbb{K}(\langle x \rangle)$)

$B = B(t) \in \mathbb{F}(t)^{n\times n}$

over $\mathbb{F}$

Lem: Suppose $\deg B_{ij} \leq 0 \leftrightarrow B(t) = B^{(0)} + B^{(-1)}t^{-1} + B^{(-2)}t^{-2} + \cdots$

$\deg \mathrm{Det}\, B = 0$ if and only if $B^{(0)}$ is nonsingular over $\mathbb{F}$

Proof of Thm

P, Q: feasible ($\deg(PAQ)_{ij} \leq 0$)

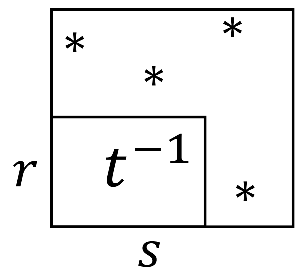
$\deg \mathrm{Det}\, PAQ = 0$

If $\mathrm{ncrank}(PAQ)^{(0)} = n$ → P, Q: optimal; $\deg \mathrm{Det}\,\mathrm{A} = -\deg\det P - \deg\det Q$

Otherwise, $\exists S, T \in \mathrm{GL}_n(\mathbb{K})$ s.t. $S(PAQ)^0 T = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{} \\ \cline{2-3} & \multicolumn{2}{c|}{* \quad * \quad *} \\ \cline{2-3} r & 0 & * \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{s} & \multicolumn{1}{c}{} \end{array}$ $\quad r + s > n$

Rem. If $\mathrm{rank}\,(PAQ)^{(0)} = n$ → P, Q: optimal; $\deg\det \mathrm{A} = -\deg\det P - \deg\det Q$

$\rightarrow S(PAQ)T =$



$$\deg \begin{pmatrix} I & O \\ O & tI_r \end{pmatrix} SPAQT \begin{pmatrix} I & O \\ O & t^{-1}I_{n-s} \end{pmatrix} \leq 0$$

Update $P, Q \rightarrow P', Q' := \begin{pmatrix} I & O \\ O & tI_r \end{pmatrix} SP, QT \begin{pmatrix} I & O \\ O & t^{-1}I_{n-s} \end{pmatrix}$ feasible,

$$-\deg \det P' - \deg \det Q' = -\deg \det P - \deg \det Q + (n - r - s)$$

Repeat this procedure:

If terminates then $\deg \mathrm{Det}\, A = -\deg \det P - \deg \det Q$

Otherwise, $\deg \mathrm{Det}\, A \leq \mathrm{RHS} = -\infty$

Deg-Det algorithm

Minimizing an L-convex func on Euclidean building

Input: $A(t) = x_1 A_1(t) + x_2 A_2(t) + \cdots + x_m A_m(t)$

0: Choose $P, Q \in \mathrm{GL}_n(\mathbb{K}(t))$ s.t. $\deg\,(PAQ)_{ij} \leq 0$

1: $PAQ = (PAQ)^{(0)} + t^{-1}M$

2: Choose $S, T \in \mathrm{GL}_n(\mathbb{K})$ s.t. $S(PAQ)^0 T =$
$$\begin{array}{cc} & \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \\ r \text{ (rows of } 0) & s \text{ (columns of } 0) \end{array}$$

with maximum $r + s$

Optimization over the link at $\langle P \rangle, \langle Q \rangle$ (= spherical building) = nc-rank computation

$(PAQ)^0$: nc-nonsingular

3: If $r + s = n \Rightarrow P, Q$:optimal; $\deg \mathrm{Det}\, A = -\deg \det P - \deg \det Q$

4: If $r + s > n \Rightarrow$

$$P, Q \leftarrow \begin{pmatrix} I & O \\ O & tI_r \end{pmatrix} SP,\ QT \begin{pmatrix} I & O \\ O & t^{-1}I_{n-s} \end{pmatrix}$$; go to 1

$\langle P \rangle, \langle Q \rangle$ moves on the 1-skeleton of the Euclidean building

Building theoretic view

$$\text{Min.}\ \ -\deg\det P\ -\ \deg\det Q$$

$$\text{s.t.}\ \ \deg\,(PA_kQ)_{ij} \le 0\ \ (\forall k, \forall ij)$$

$$P, Q \in GL_n(\mathbb{K}(t))$$

$\mathbb{K}(t)^- := \{\, p/q \in \mathbb{K}(t) \mid \deg p/q \le 0\}$ (= valuation ring)

$GL_n(\mathbb{K}(t)^-)$: group of invertible $n \times n$ matrices over $\mathbb{K}(t)^-$

Obs. $P, Q \to UP, QV \quad (U, V \in GL_n(\mathbb{K}(t)^-)\,)$

does not change feasibility & objective value

→ The problem is over $\mathcal{L} \times \mathcal{L}$, where $\mathcal{L} := GL_n(\mathbb{K}(t))/GL_n(\mathbb{K}(t)^-)$

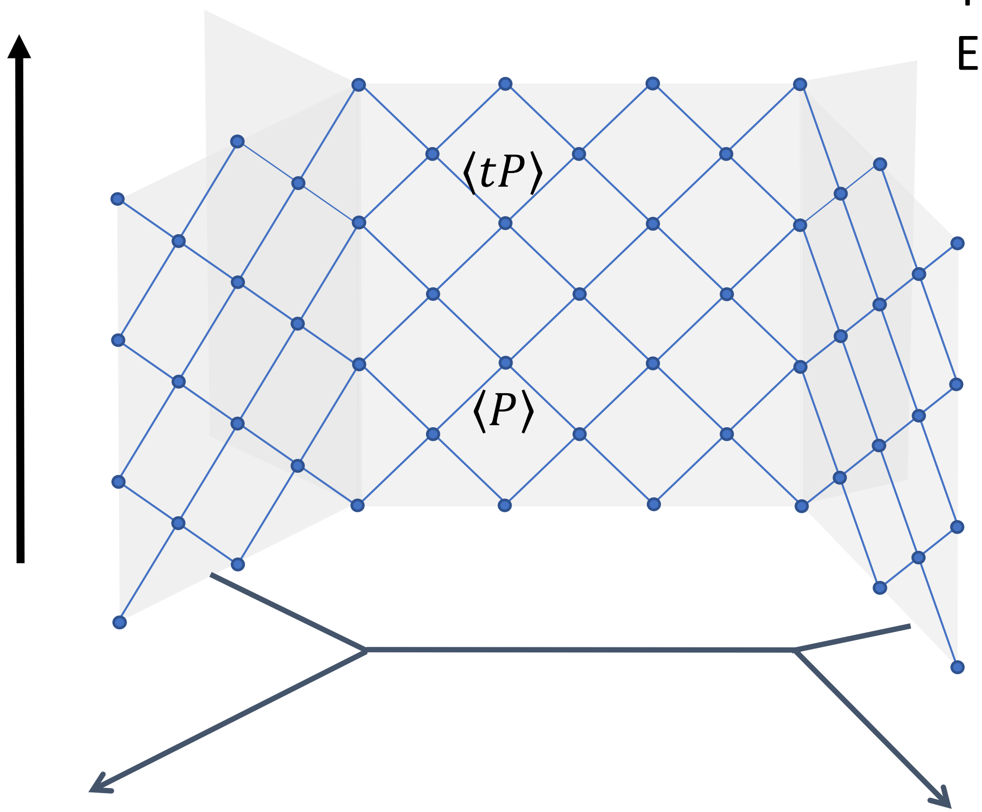
$$= \{\langle P\rangle \mid P \in GL_n(\mathbb{K}(t))\}$$

Def. $\langle P\rangle :=$ $\mathbb{K}(t)^-$-module generated by columns of $P \in GL_n(\mathbb{K}(t))$

Lem. $\mathcal{L} = \{\langle P \rangle \mid P \in GL_n(\mathbb{K}(t))\}$ is a modular lattice,

where $\preccurlyeq := \subseteq$, $\wedge = \cap$, $\vee = +$, and $t\mathcal{L} = \mathcal{L}$

This structure is equivalent to Euclidean building of $GL_n(\mathbb{K}(t))$



$\leftarrow$ Part of Euclidean building for $GL_2(\mathbb{K}(t)))$

Define L-convex function: $f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$, $f(tx) = f(x)$

$\rightarrow$ The deg Det problem is L-convex function minimization on $\mathcal{L} \times \mathcal{L}^*$

Remarks

- Idea comes from *combinatorial relaxation method* (Murota 90) for deg det

- Bipartite matching $A = \sum E_{ij} x_{ij} t^{c_{ij}}$ → Hungarian method

- Linear matroid intersection $A = \sum_k a_k b_k^T x_k t^{c_k}$

  → Frank's weight splitting algorithm (Furue-H 20)

- Computation of deg Det for skew polynomial matrix (Oki 19)

- Further discrete convex analysis & cost-scaling imply

  a polytime algorithm for $\deg \mathrm{Det}\, A = \sum_k A_k x_k t^{c_k}$ (H-Ikeda 20)

Polyhedral interpretation of deg Det [H-Ikeda 20]

Obs. $\deg \det \sum_{k=1}^{m} A_k x_k t^{c_k} = \max\{c^T u \mid u \in \mathrm{Newton}\ A\}$

where $\mathrm{Newton}\ A :=$ *Newton polytope* of $\det A$

Def. Newton polytope of polynomial $p(x_1, x_2, \dots, x_m)$

$$:= \mathrm{Conv}\,\{\, \alpha \in \mathbb{Z}^m \mid x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m} \text{ is a monomial of } p \,\} \subseteq \mathbb{R}^m$$

$$p(x_1, x_2, \dots, x_m) = \textstyle\sum_\alpha b_\alpha x^\alpha \implies p(x_1 t^{c_1}, x_2 t^{c_2}, \dots, x_m t^{c_m}) = \sum_\alpha b_\alpha x^\alpha t^{c^T \alpha}$$

Def. $\mathrm{NcNewton}\ A := \bigcup_{d \geq 1} \mathrm{proj}\ \mathrm{Newton}\ A^{\{d\}}$

d-Blow-up $A^{\{d\}} := \sum_{k=1}^{m} A_k \otimes X_k$, $X_k = (x_{k,ij})$: $d \times d$ variable matrix

proj: $(u_{k,ij})_{k,ij} \to \left(\frac{1}{d} \sum_{i,j} u_{k,ij}\right)_k \in \mathbb{R}^m$

Prop. $\deg \mathrm{Det} \sum_{k=1}^{m} A_k x_k t^{c_k} = \max\{c^T u \mid u \in \mathrm{NcNewton}\ A\}$

→ $\mathrm{NcNewton}\ A$ is an ***integral*** polytope $\supseteq \mathrm{Newton}\ A$

Proof sketch:

If d: large

Derksen-Makam, IQS

Fact: $\mathrm{ncrank}\, A = \frac{1}{d}\mathrm{ncrank}\, A^{\{d\}} = \frac{1}{d}\mathrm{rank}\, A^{\{d\}}$

If d: large

→ $\deg \mathrm{Det}\, A(t) = \frac{1}{d}\deg \mathrm{Det}\, A(t)^{\{d\}} = \frac{1}{d}\deg \det A(t)^{\{d\}}$

$P, Q \to P \otimes I_d, Q \otimes I_d$

$= \max\{c^T u \mid u \in \mathrm{proj\ Newton}\, A^{\{d\}}\}$

Def. Strong Edmonds-Rado property ⇔ $\mathrm{Newton}\, A = \mathrm{NcNewton}\, A$

If $A = \sum_k a_k b_k^T x_k$ (matroid intersection)

→ Matroid intersection polytope $=$ $\mathrm{Newton}\, A = \mathrm{NcNewton}\, A$

If $A = \sum_{ij} T_{ij}\, x_{ij}$ (nonbipartite matching)

→ $2 \cdot$ Matching polytope $=$ $\mathrm{Newton}\, A \subset \mathrm{NcNewton}\, A$

?? $2 \cdot$ Fractional matching polytope ??

Problem: Further study on $\mathrm{NcNewton}\, A$ (char. of vertices/facets, LP-description)

2. Reduction of nc-singularity over $\mathbb{Q}$

to that over $GF(p)$ by p-adic valuation

Def. (p-adic valuation) p: prime, $v_p\colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$

$$v_p(u) \coloneqq k \ \text{ if } \ u = p^k \frac{a}{b} \in \mathbb{Q},\ \ a, b \in \mathbb{Z} \text{ prime to } p$$

$$v_p(0) \coloneqq \infty$$

p-adic expansion of $u \in \mathbb{Q}$:

$$u = a_k p^k + a_{k+1} p^{k+1} + \cdots$$

where $k = v_p(u)$, $a_i \in \{0,1,2,\ldots,p-1\} = GF(p)$

Gauss extension: $v_p\colon \mathbb{Q}(x_1, x_2, \ldots, x_m) \to \mathbb{Z} \cup \{\infty\}$

$$v_p(f) \coloneqq \min_i v_p(c_i) \quad (f = \textstyle\sum_i c_i x^{\alpha_i})$$

$$v_p(f/g) \coloneqq v_p(f) - v_p(g)$$

$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$, where A_k over $\mathbb{Z} \subseteq \mathbb{Q}$

with $|A_{k,ij}| < D$

Instead of nc-rank / FR formula, we solve

$$v_p \mathrm{Det}' A := \quad \text{Max.} \quad -v_p \det P - v_p \det Q$$
$$\text{s.t.} \quad v_p(PA_kQ)_{ij} \geq 0 \quad (\forall ij, k)$$
$$P, Q \in GL_n(\mathbb{Q})$$

If d: large

Lem. $v_p \mathrm{Det}' A = \frac{1}{d} v_p \mathrm{Det}' A^{\{d\}} = \frac{1}{d} v_p \det A^{\{d\}}$

Cor. A: nc-nonsingular $\Leftrightarrow$ $v_p \mathrm{Det}' A < O(n \log nD) < \infty$

$\because$ | coefficient of $\det A^{\{d\}}| \leq (nD)^{O(nd)}$

$\frac{1}{d} v_p \det A^{\{d\}} = O(n \log_p nD)$ $\leftarrow v_p(z) \leq \log_p |z|$ for $0 \neq z \in \mathbb{Z}$

$$z = \underbrace{1011\,\overbrace{000}^{v_2(z)}}_{bitsize}$$

Problem: How can we extend v_p to $\mathbb{Q}(\langle x \rangle)$ so that $v_p \mathrm{Det}' A = v_p \mathrm{Det}\, A$?

Initially, $(P, Q) := (I, I)$; obj $= 0$

$v_p(PA_kQ) \geq 0$

over $\mathrm{GF}(p)$

p-adic expansion

$$\rightarrow PA_kQ = (PA_kQ)^{(0)} + (PA_kQ)^{(1)}p + (PA_kQ)^{(2)}p^2 + \cdots$$

- If $\operatorname{ncrank} \sum_k (PA_kQ)^{(0)} x_k = n$ over $\mathrm{GF}(p)$

  $\rightarrow \operatorname{ncrank} \sum_k PA_kQ\, x_k = \operatorname{ncrank} \sum_k A_k\, x_k = n$ over $\mathbb{Q}$

- Otherwise $\exists S, T$ s.t. $S(PA_kQ)^{(0)}T = \begin{array}{c|c} & \begin{pmatrix} * & * \\ \underbrace{0}_{s} & * \end{pmatrix} \\ r & \end{array}$ mod p $(\forall k)$

  $r + s > n$

  Update $P, Q \rightarrow P', Q' := \begin{pmatrix} I_{n-r} & \\ & p^{-1}I_r \end{pmatrix} SP, QT \begin{pmatrix} I_s & \\ & pI_{n-s} \end{pmatrix}$

  $$-v_p \det P' - v_p \det Q' = -v_p \det P - v_p \det Q + (r + s - n)$$

Final remarks

- I gave up to find a shrunk subspace (Hall blocker) by this approach.

  Difficulty: p-adic expansion of a rational x is cycling,

  where the length of the cycle may be exponential of bit-size(x).

- $v\,\mathrm{Det}'A$ problem is useful and has many analogies of the FR formula for nc-rank.

  Question: Find a natural reasoning of $v\,\mathrm{Det}'A$ in left-right action :

$$SL_n(\mathbb{K}) \times SL_n(\mathbb{K}) \ni (S,T) \mapsto (SA_kT)_{k=1,2,\ldots,m}$$

  in a field $\mathbb{K}$ with discrete valuation v.

END

Additional references

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