Some Combinatorial Optimization Problems Related to Metric Spaces of Nonpositive Curvature

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CAT(0) complexes appear in
some combinatorial optimization problems:

1. Multicommodity flow ~~ folder complex

2. Multifacility location ~~ orthoscheme complex

But I don't know the reason of these appearances…

N = (V, E, c, s,t): undirected network

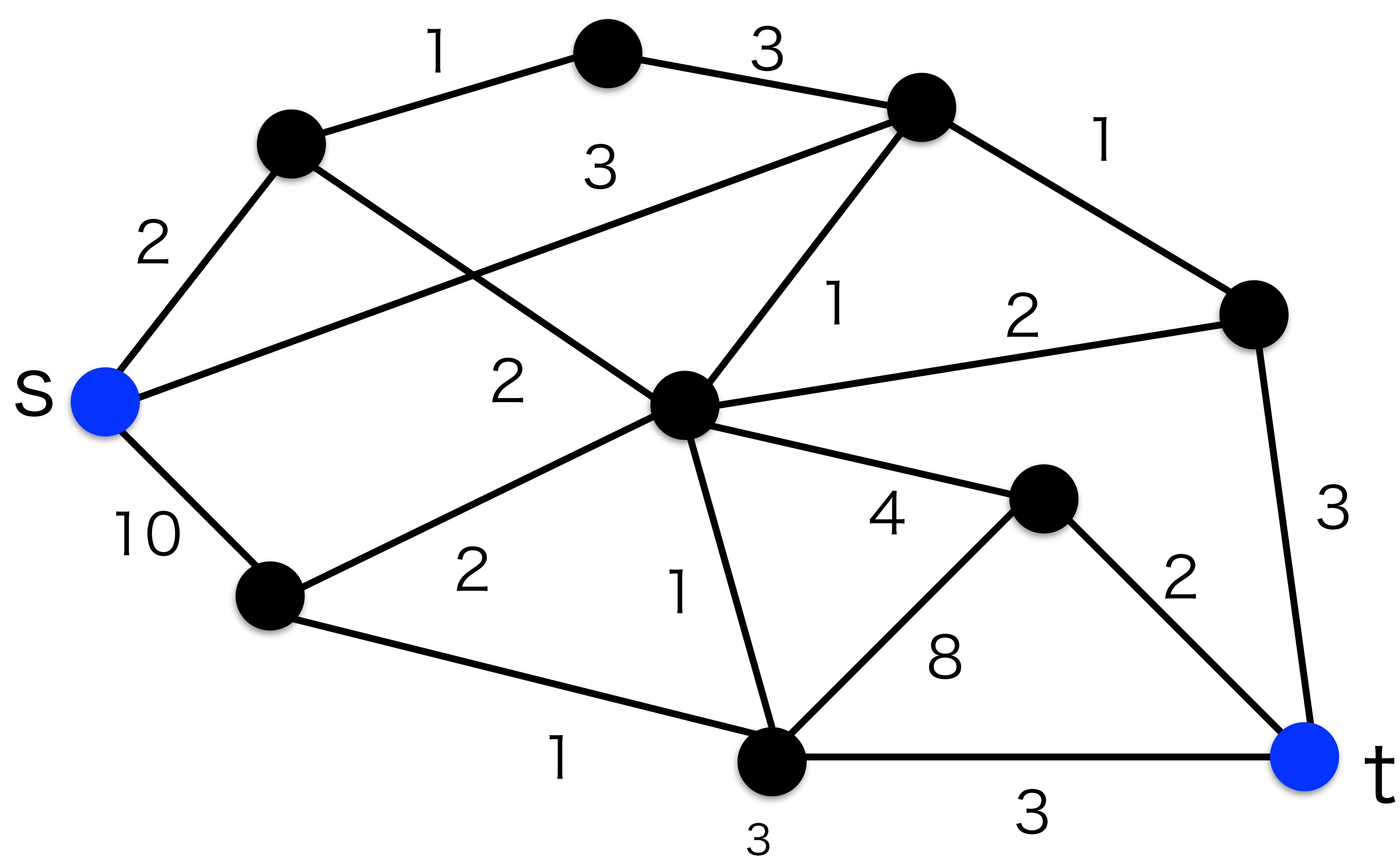
c: E → Z+: edge-capacity

s,t ∈ V: terminal pair

Def: (s,t)-flow

⇔ f: { (s,t)-paths } → R+,

Σ { f(P) | P: e in P } ≦ c(e)   (e in E)



$N = (V, E, c, s, t)$: undirected network

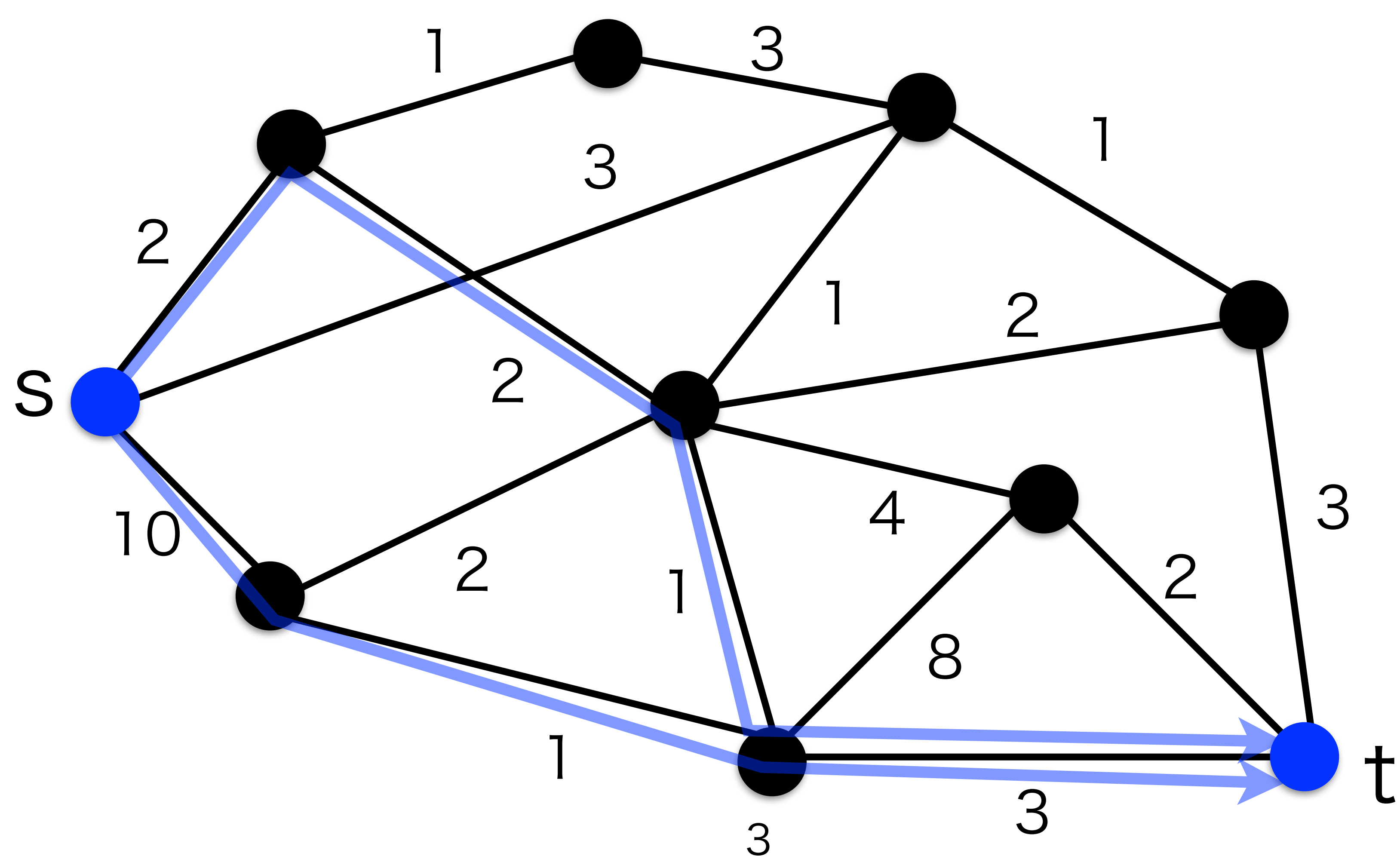
$c: E \to \mathbb{Z}_+$: edge-capacity

$s, t \in V$: terminal pair

Def: (s,t)-flow

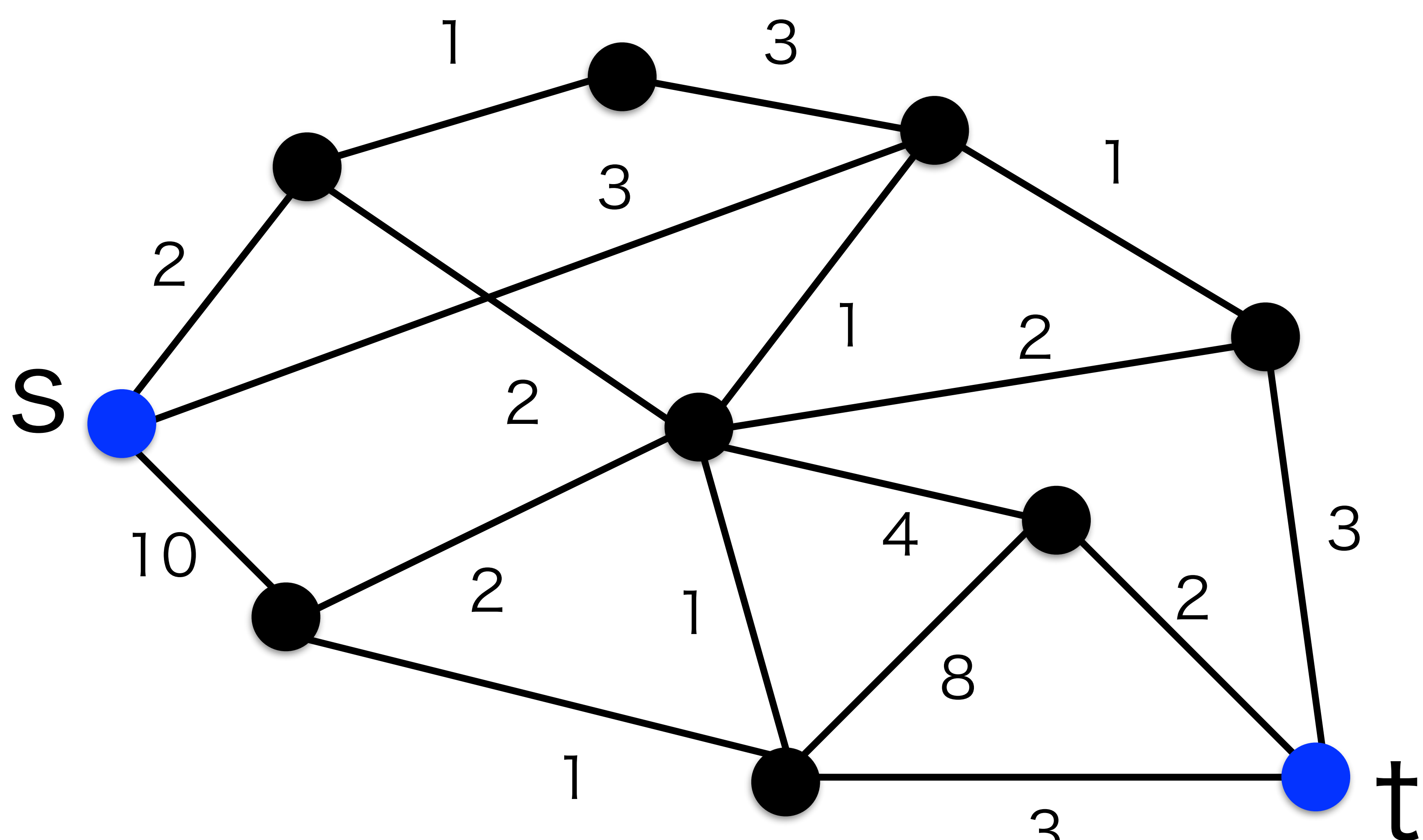
$\Leftrightarrow f: \{ (s,t)\text{-paths} \} \to \mathbb{R}_+$,

$\Sigma \{ f(P) \mid P: e \text{ in } P \} \leqq c(e) \quad (e \text{ in } E)$



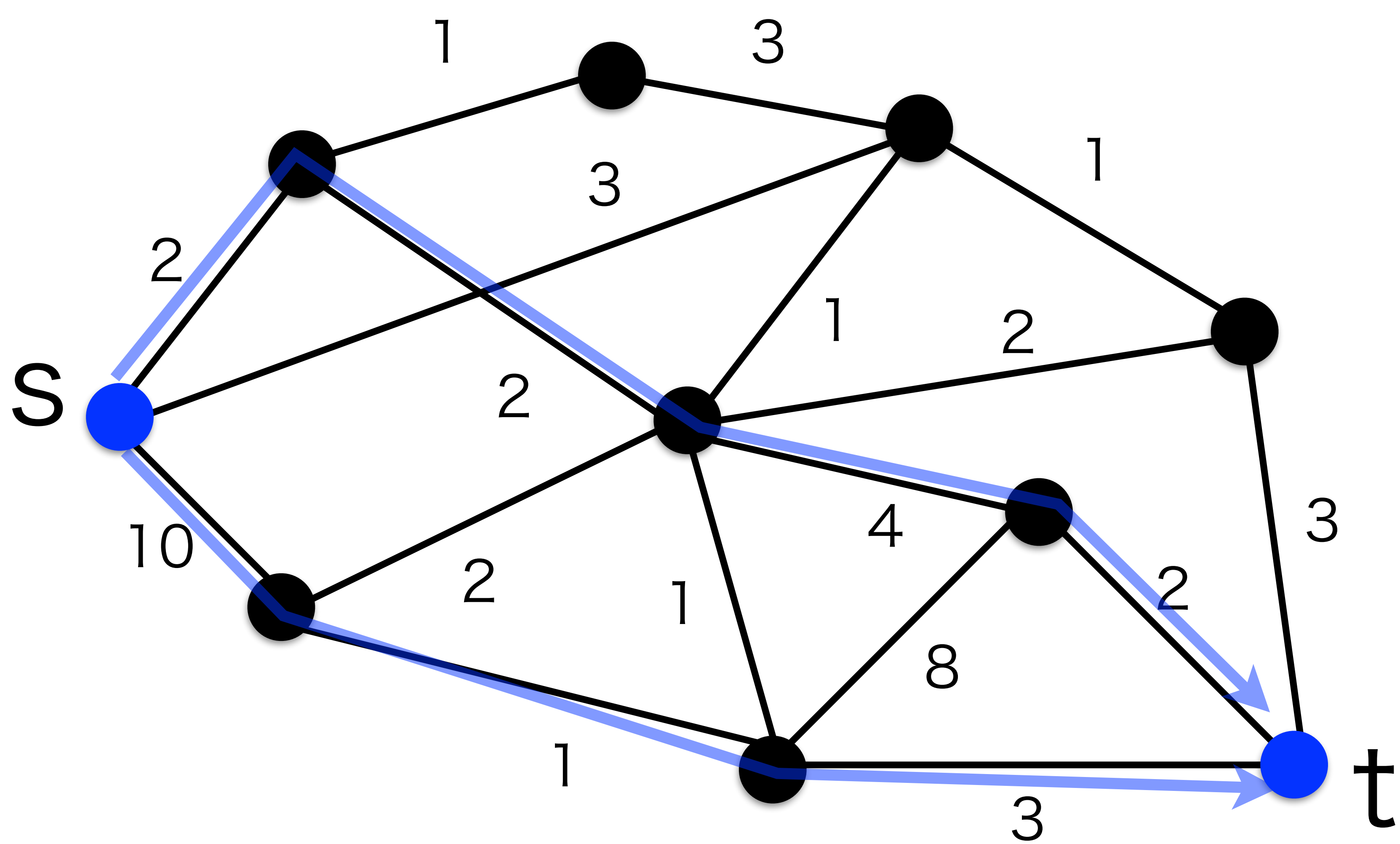
Maximum flow problem

Maximize $\Sigma f(P)$ over (s,t)-flows f



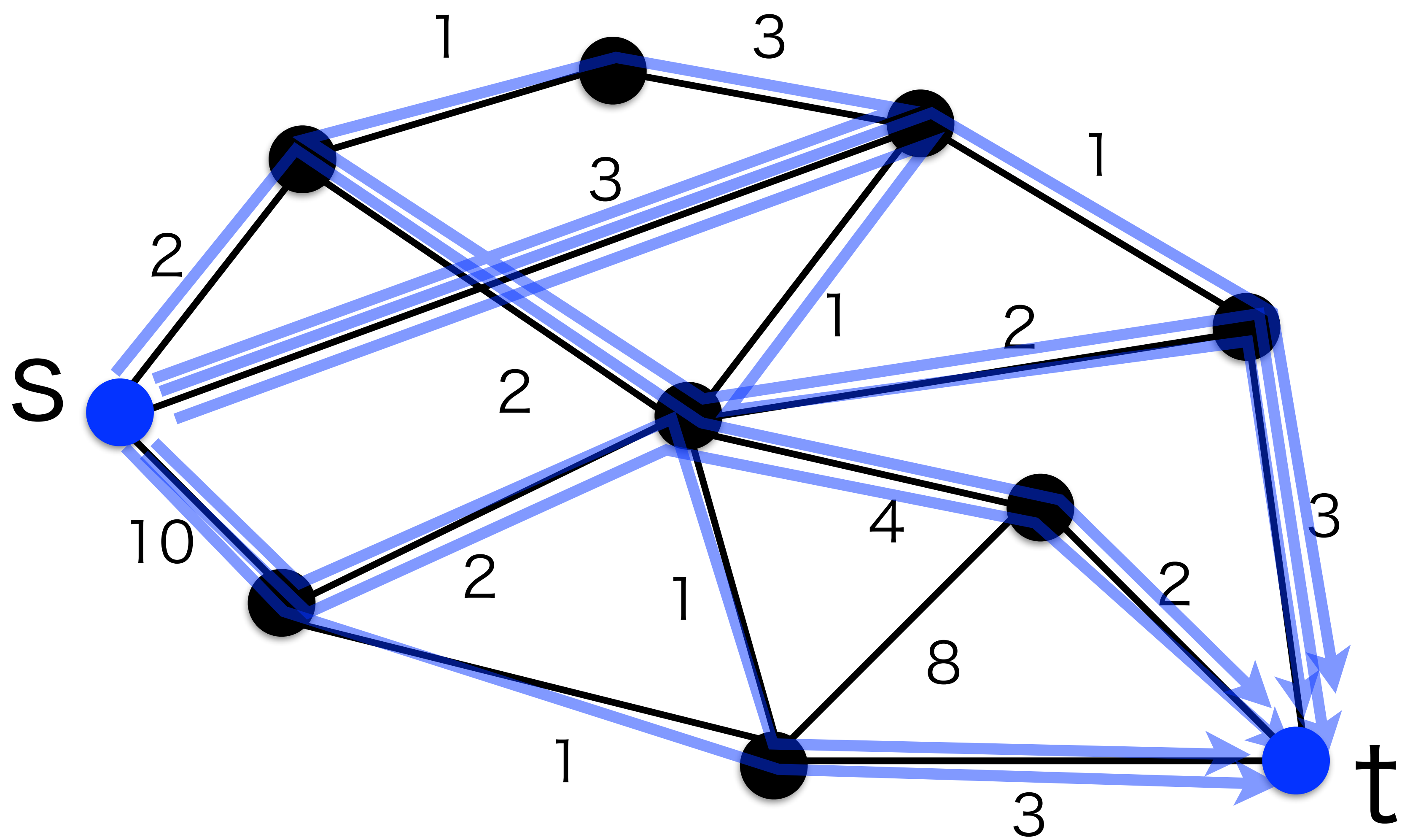
Maximum flow problem

Maximize $\Sigma f(P)$ over (s,t)-flows f



Maximum flow problem

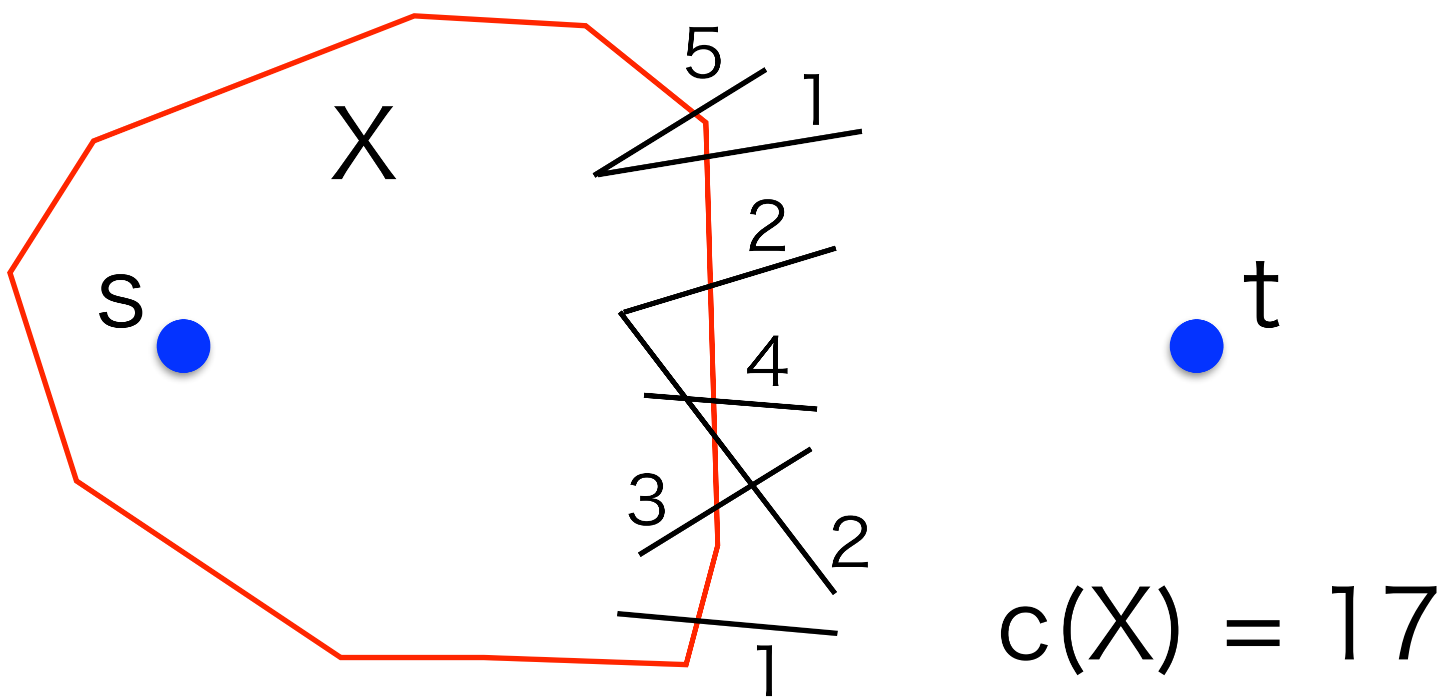
Maximize Σ f(P) over (s,t)-flows f



Def: (s,t)-cut

<=> $X \subseteq V$: s in X , t not in X

Cut capacity $c(X) := \Sigma\{ c(xy) \mid x \text{ in } X, y \text{ not in } X \}$



Minimum cut problem

Minimize c(X) over all (s,t)-cuts X

Max-Flow Min-Cut Theorem
(Ford-Fulkerson 56)

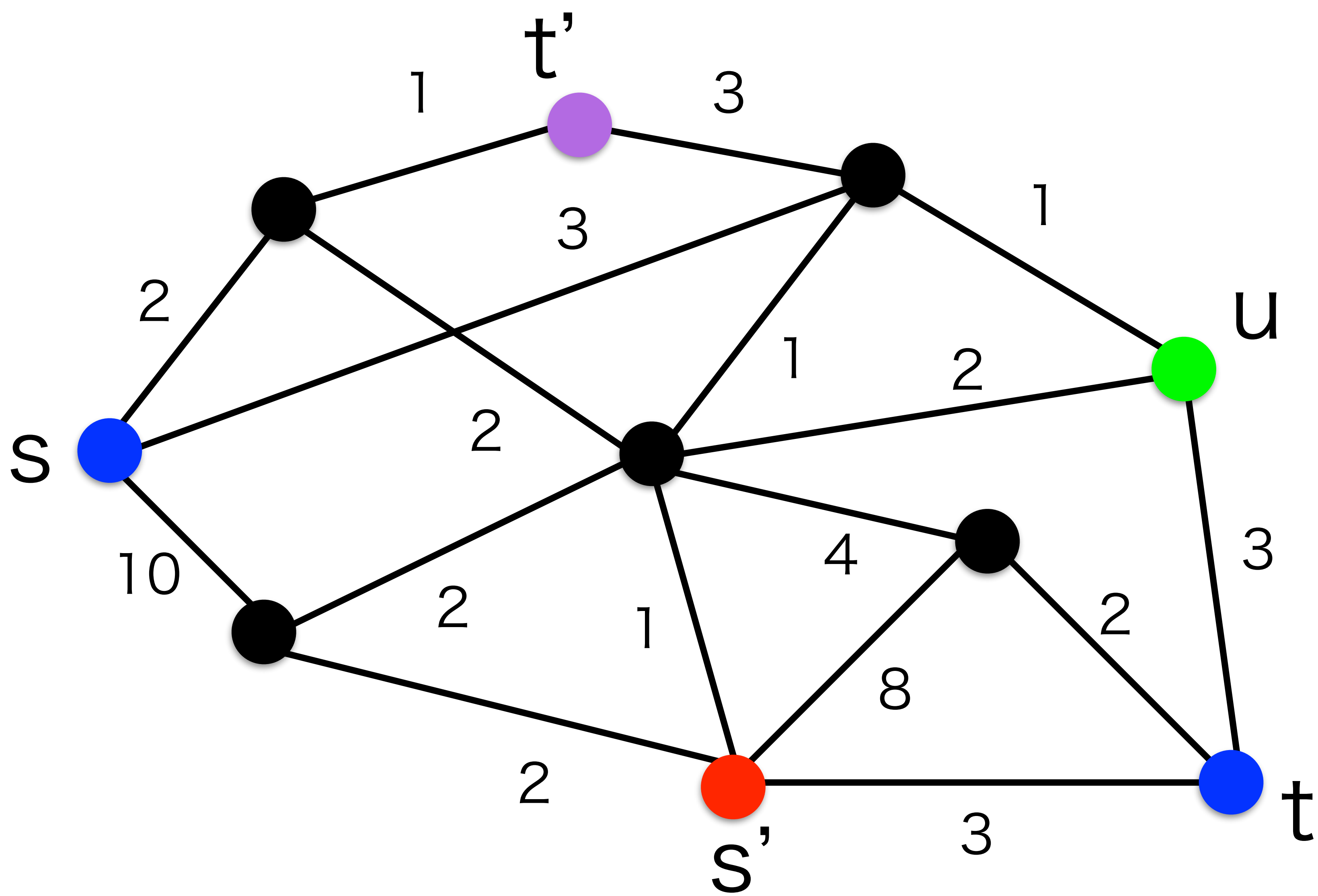
$$\text{Max } \sum f(P) = \text{Min } c(X)$$

f: (s,t)-flow X: (s,t)-cut

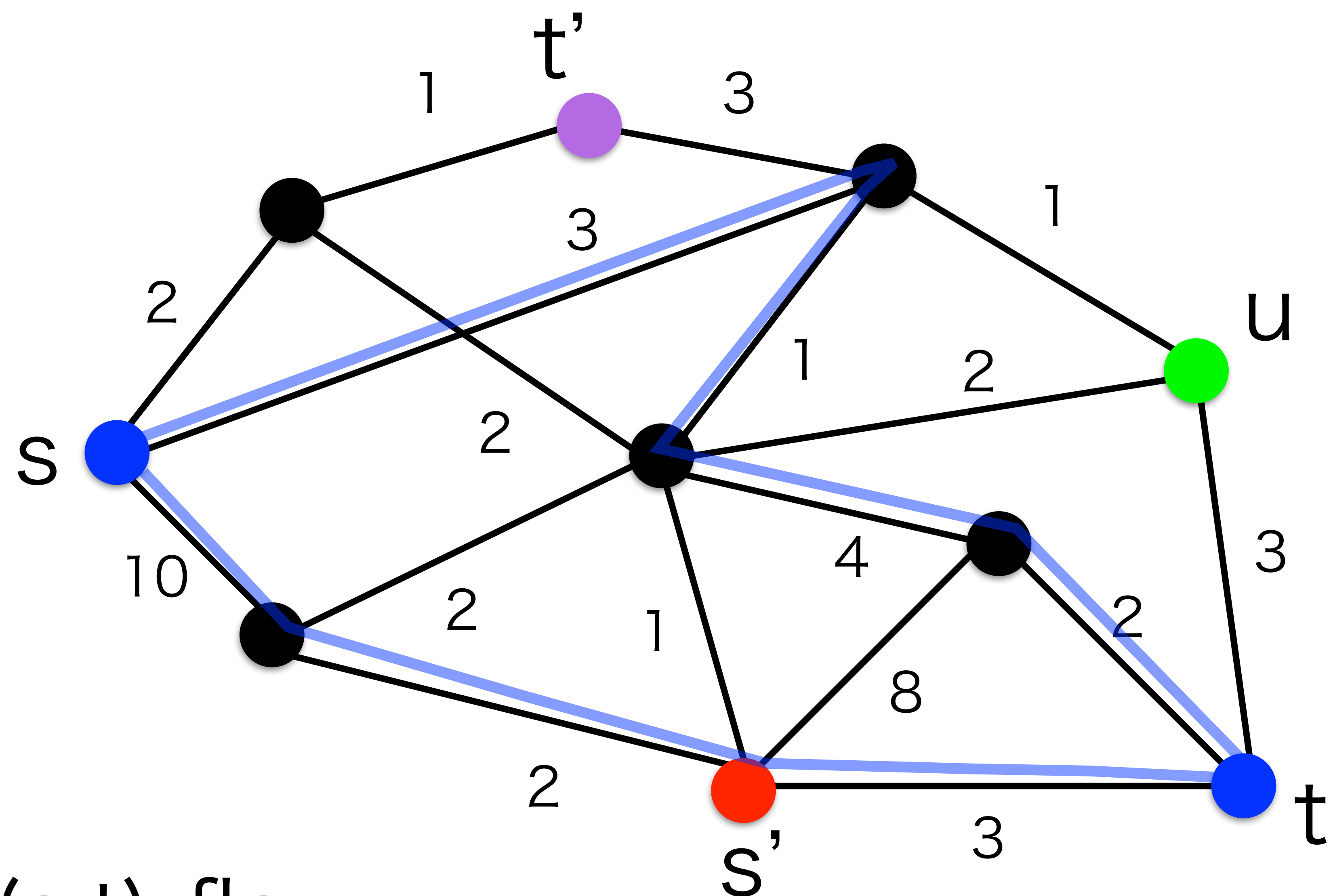
∃ integer-valued max-flow (integrality)

∃ Polynomial time algorithm for max-flow/min-cut

Multicommodity flow

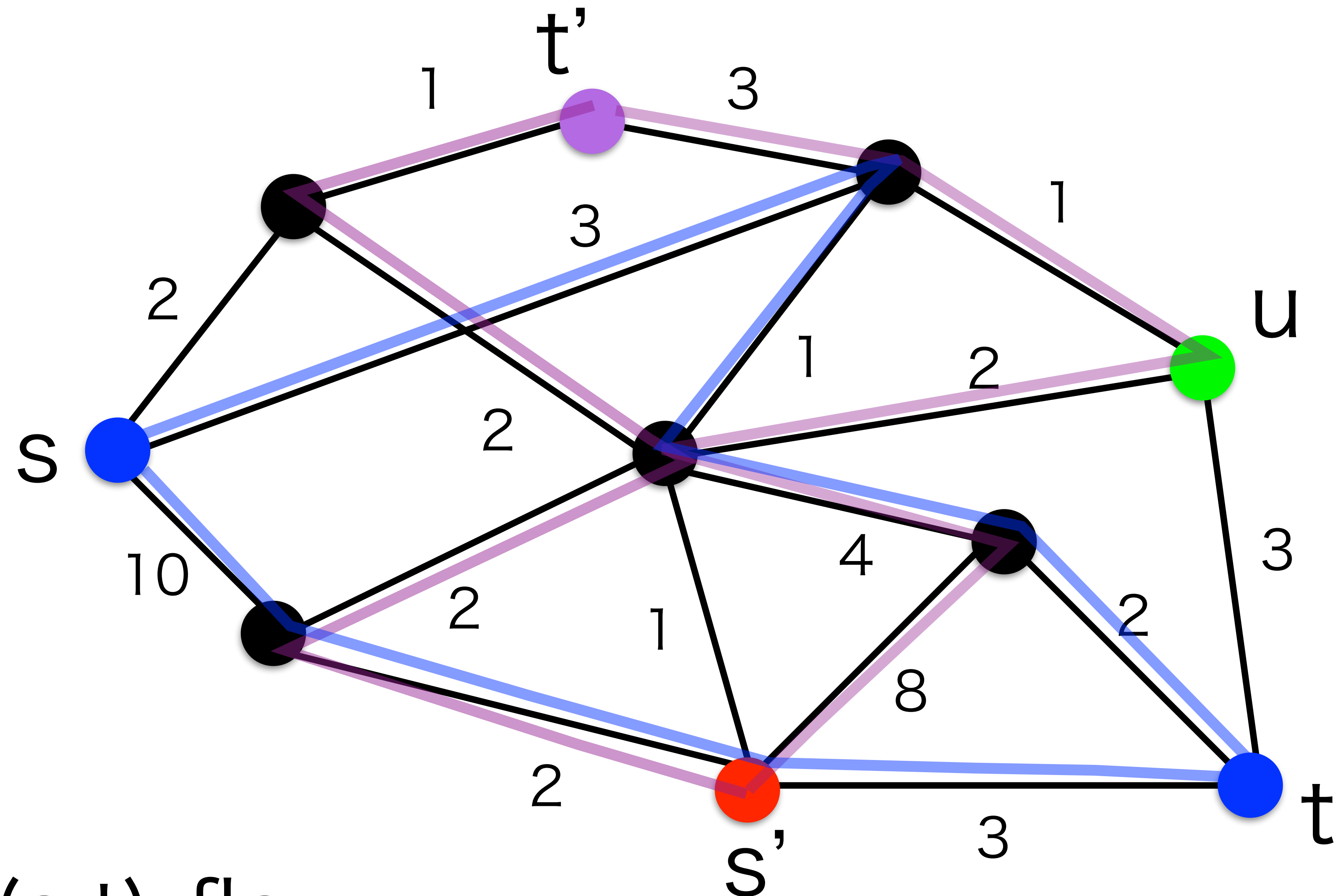


Multicommodity flow



(s,t)-flow

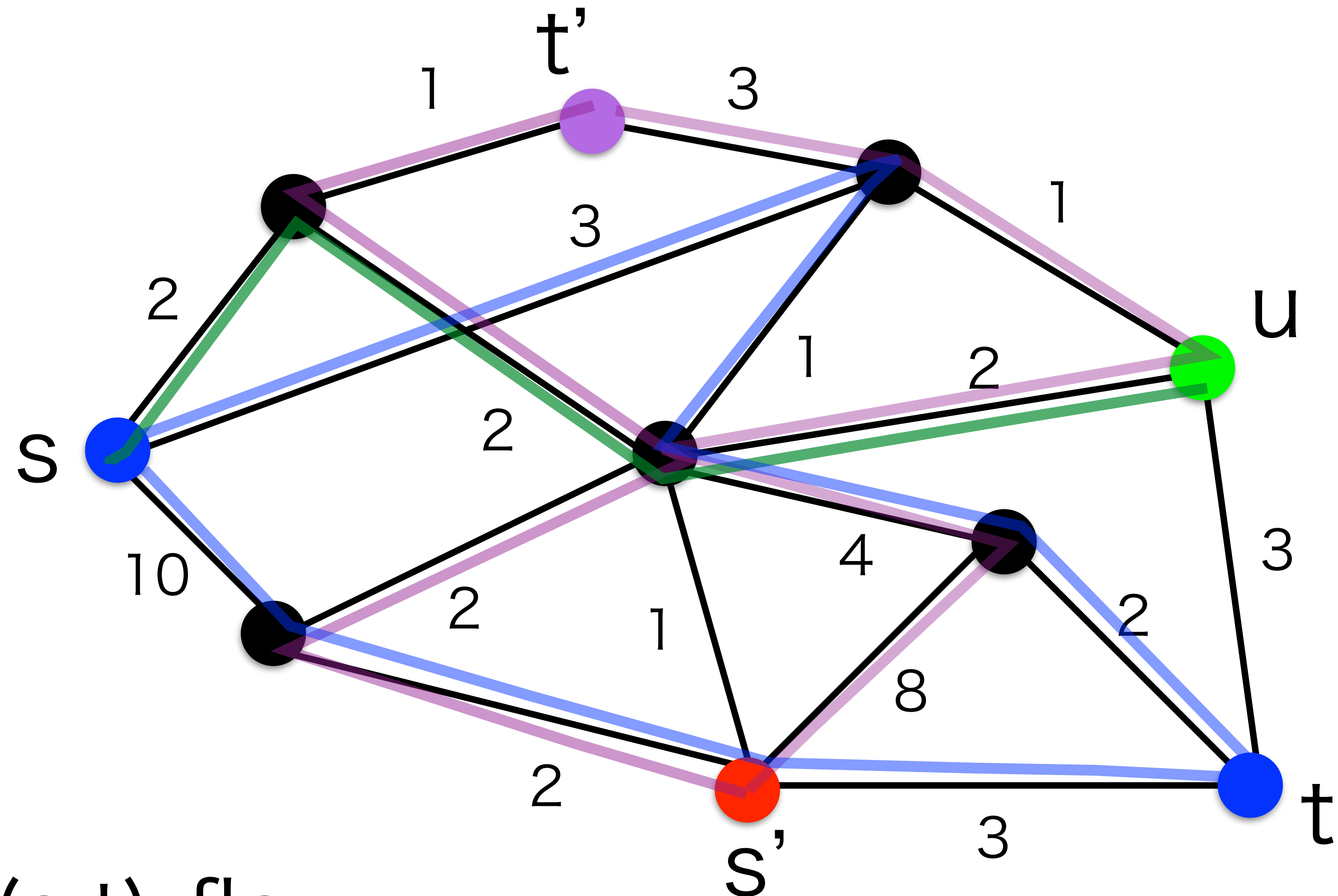
Multicommodity flow



(s,t)-flow

(s’,t’)-flow

Multicommodity flow



(s,t)-flow

(s',t')-flow

(s,u)-flow

. . .

- Many practical applications: trafic, internet, communication networks, ⋯
- One of prominent research areas in TCS & combinatorial optimization
- Many problem formulations
- Various extensions of MFMC to multiflows

N = (V, E, c, S): undirected network

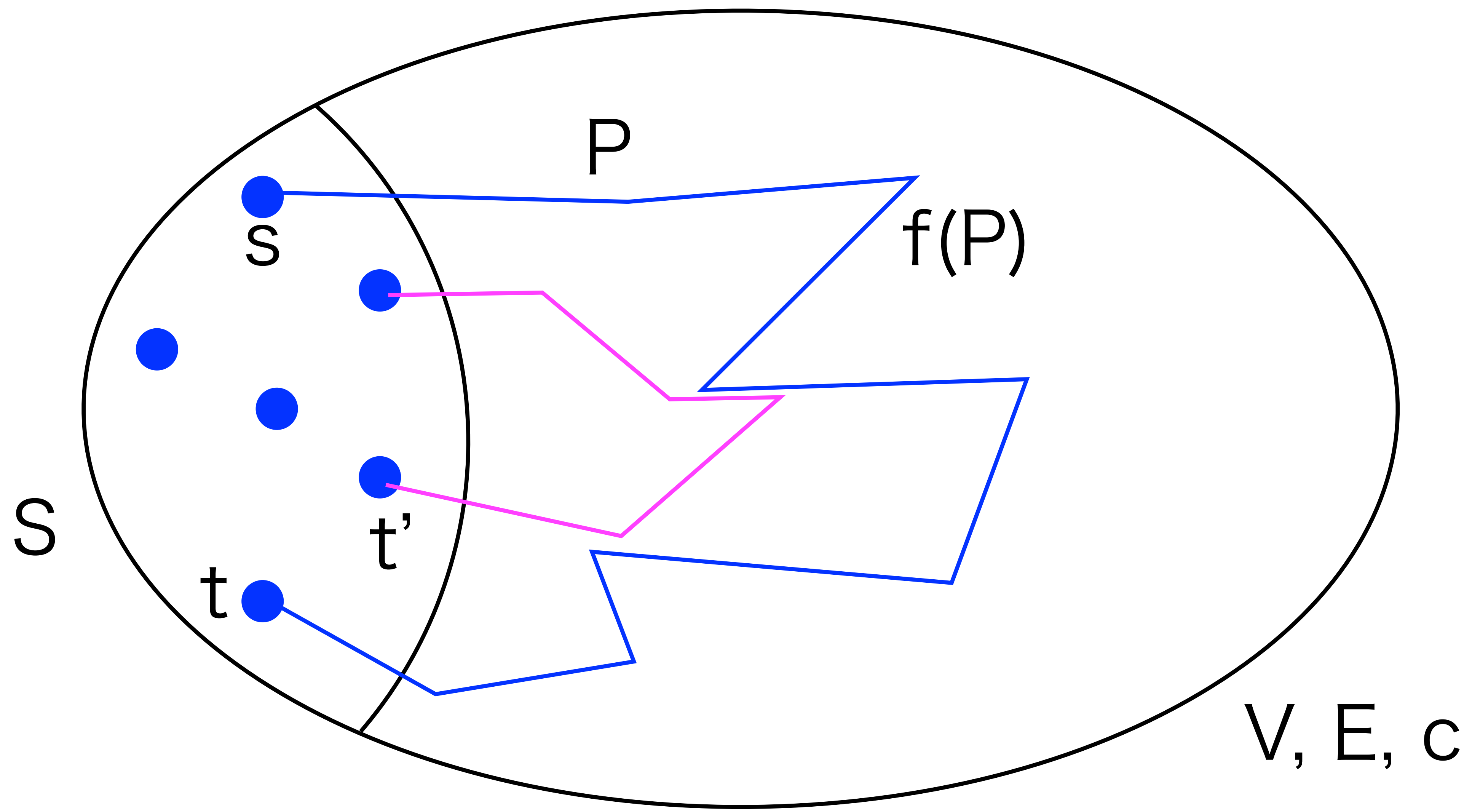
$c: E \to Z+$

$S \subseteq V$: terminal set

Def: multiflow

$\Leftrightarrow$ f: { S-paths } $\to$ R+,

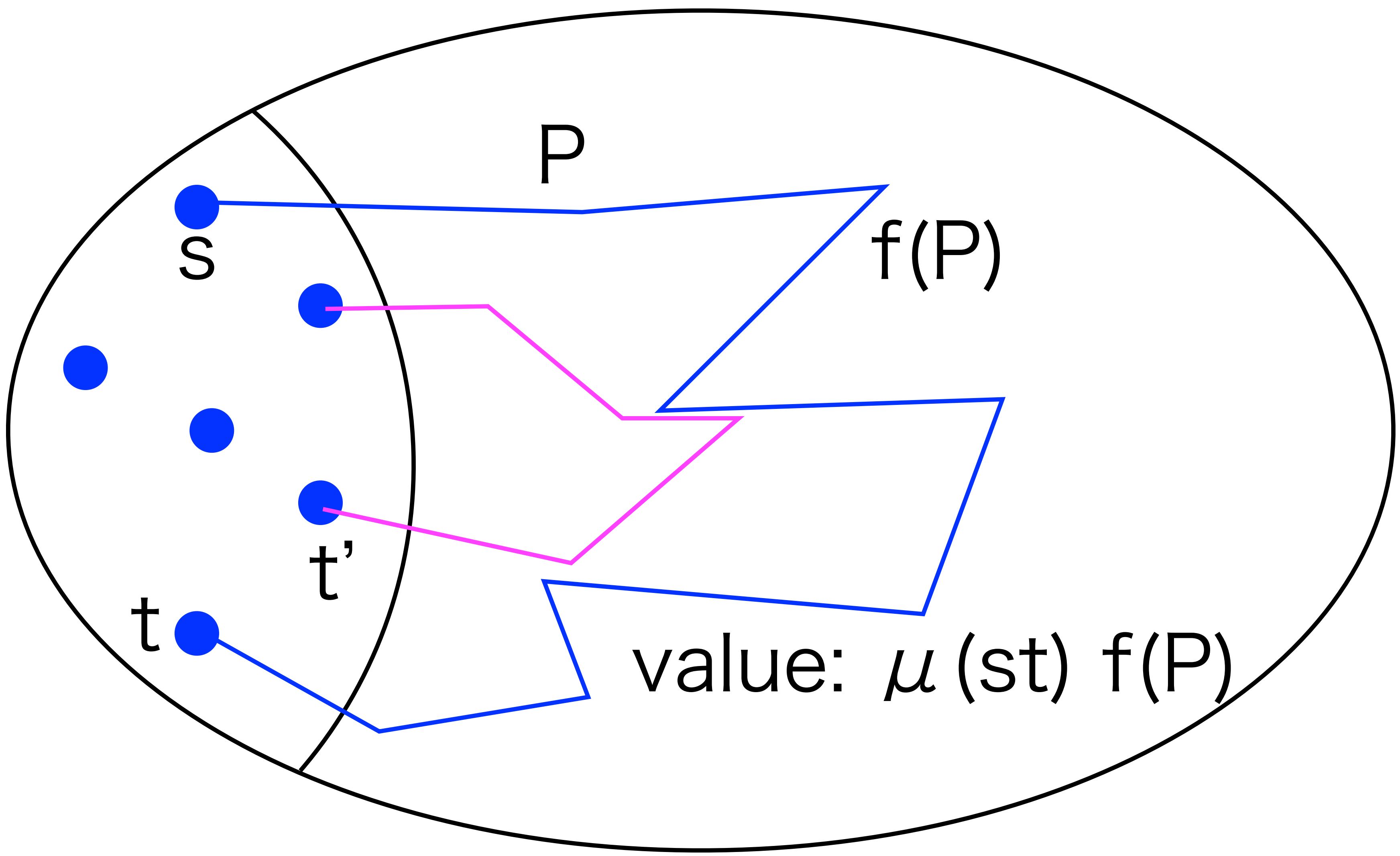
Σ { f(P) | P: e in P } $\leqq$ c(e)   (e in E)



Our multiflow problem

μ: { terminal pairs } → Z+

Maximize $\sum_{s,t,\ (s,t)\text{-path } P} \mu(st)\, f(P)$ over all multiflows f



MMP[μ]:

Maximize $\Sigma \ \mu(st) \ f(P)$ over all multiflows f

MMP[μ]:

Maximize Σ μ(st) f(P) over all multiflows f

Thm [H. 09-14, build on works of Karzanov, Chepoi,⋯]

MFMC-type formula holds in MMP[μ]

$\Leftrightarrow$

μ "embeds" into a folder complex

(= CAT(0)B2-complex)

“MFMC-type formula holds in MMP[μ]”

means

∃ k, 1/k-integer max-flow
for every instance of MMP[μ]

$$\text{Max } \Sigma \ \mu(st)\, f(P) \ = \text{Min} \ {*}{*}{*}{*}$$

Optimization over
“combinatorial objects”

$$\mu \quad \begin{array}{c|cc} & s & t \\ \hline s & & 1 \\ t & 1 & \end{array}$$

MFMC

k=1

$$\begin{array}{c|cccc} & s & t & s' & t' \\ \hline s & & 1 & & \\ t & 1 & & & \\ s' & & & & 1 \\ t' & & & 1 & \end{array}$$

2-commodity

k=2 Hu 63

$$\begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{bmatrix}$$

3-commodity

k=∞

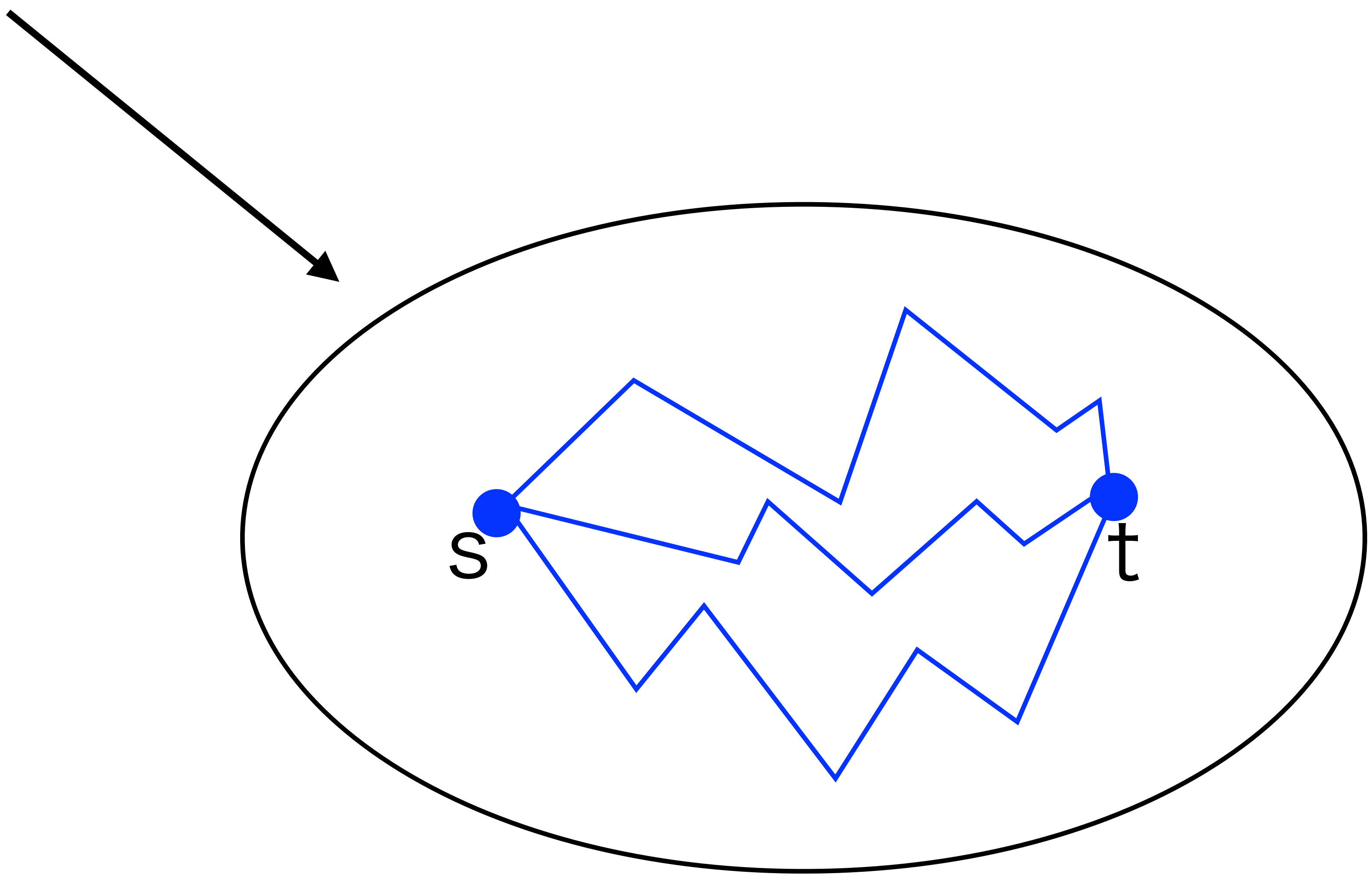
$$\begin{array}{c|ccccc} & s & t & u & & \\ \hline s & & 1 & 1 & 1 & 1 \\ t & 1 & & 1 & 1 & 1 \\ u & 1 & 1 & & 1 & 1 \\ & 1 & 1 & 1 & & 1 \\ & 1 & 1 & 1 & 1 & \end{array}$$

free multiflow

Lovasz 76,

Cherkassky 77

k=2



$$\mu \quad \begin{array}{c|cc} & s & t \\ \hline s & & 1 \\ t & 1 & \end{array}$$

MFMC

k=1

$$\begin{array}{c|cccc} & s & t & s' & t' \\ \hline s & & 1 & & \\ t & 1 & & & \\ s' & & & & 1 \\ t' & & & 1 & \end{array}$$

2-commodity

k=2 Hu 63

$$\begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{bmatrix}$$

3-commodity

k=∞

$$\begin{array}{c|ccccc} & s & t & u & & \\ \hline s & & 1 & 1 & 1 & 1 \\ t & 1 & & 1 & 1 & 1 \\ u & 1 & 1 & & 1 & 1 \\ & 1 & 1 & 1 & & 1 \\ & 1 & 1 & 1 & 1 & \end{array}$$

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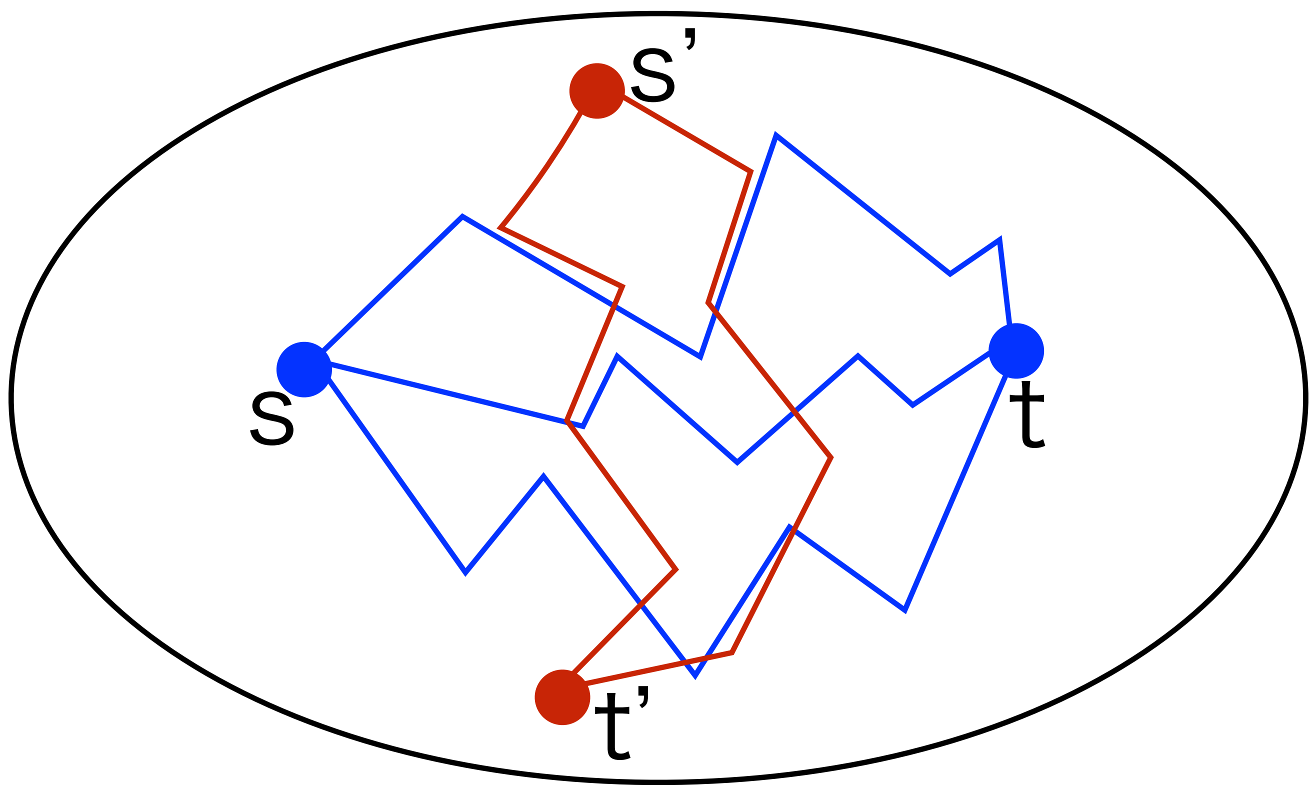


μ

	s	t
s		1
t	1	

MFMC

k=1

	s	t	s'	t'
s		1		
t	1			
s'				1
t'			1	

2-commodity

k=2 Hu 63

	1				
1					
			1		
		1			
					1
				1	

3-commodity

k=∞

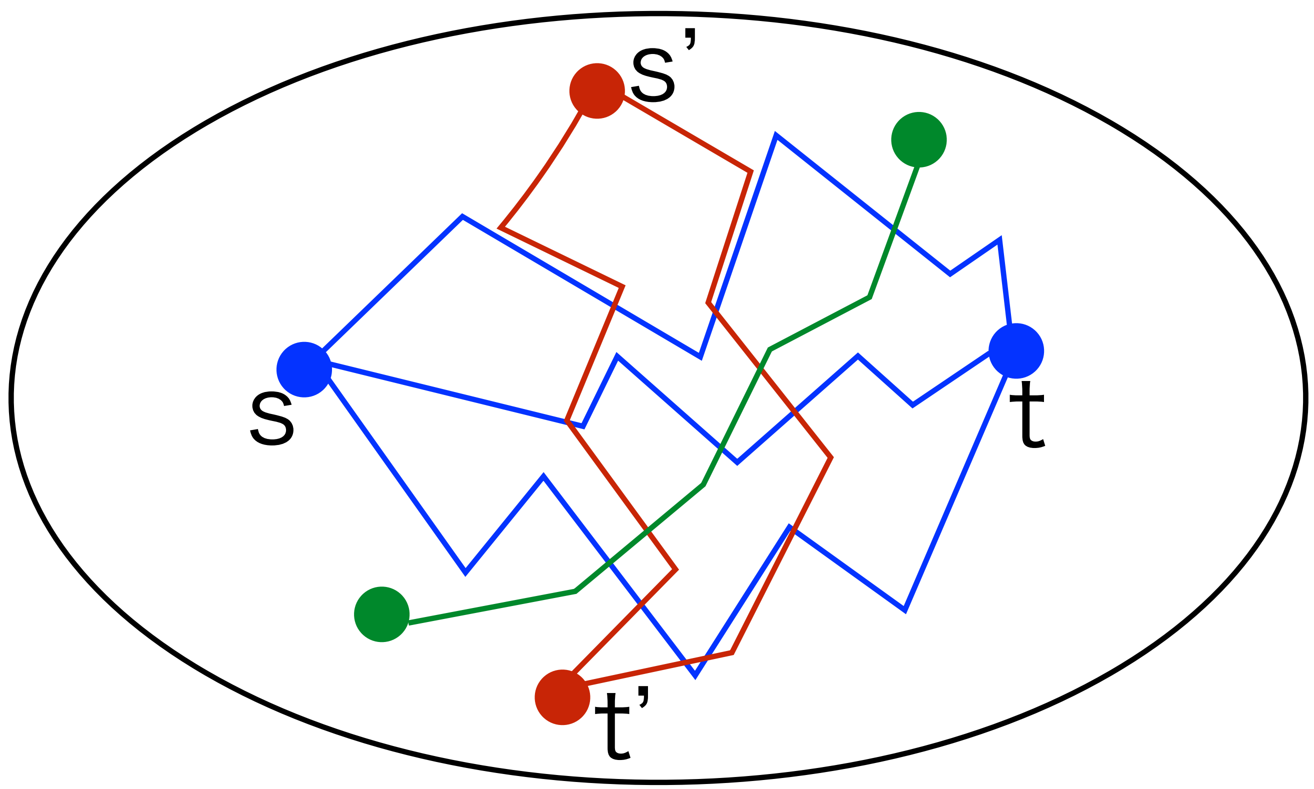
	s	t	u		
s		1	1	1	1
t	1		1	1	1
u	1	1		1	1
	1	1	1		1
	1	1	1	1	

free multiflow

Lovasz 76,

Cherkassky 77

k=2



μ

	s	t
s		1
t	1	

MFMC

k=1

	s	t	s’	t’
s		1		
t	1			
s’				1
t’			1	

2-commodity

k=2 Hu 63

$$\begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{bmatrix}$$

3-commodity

k=∞

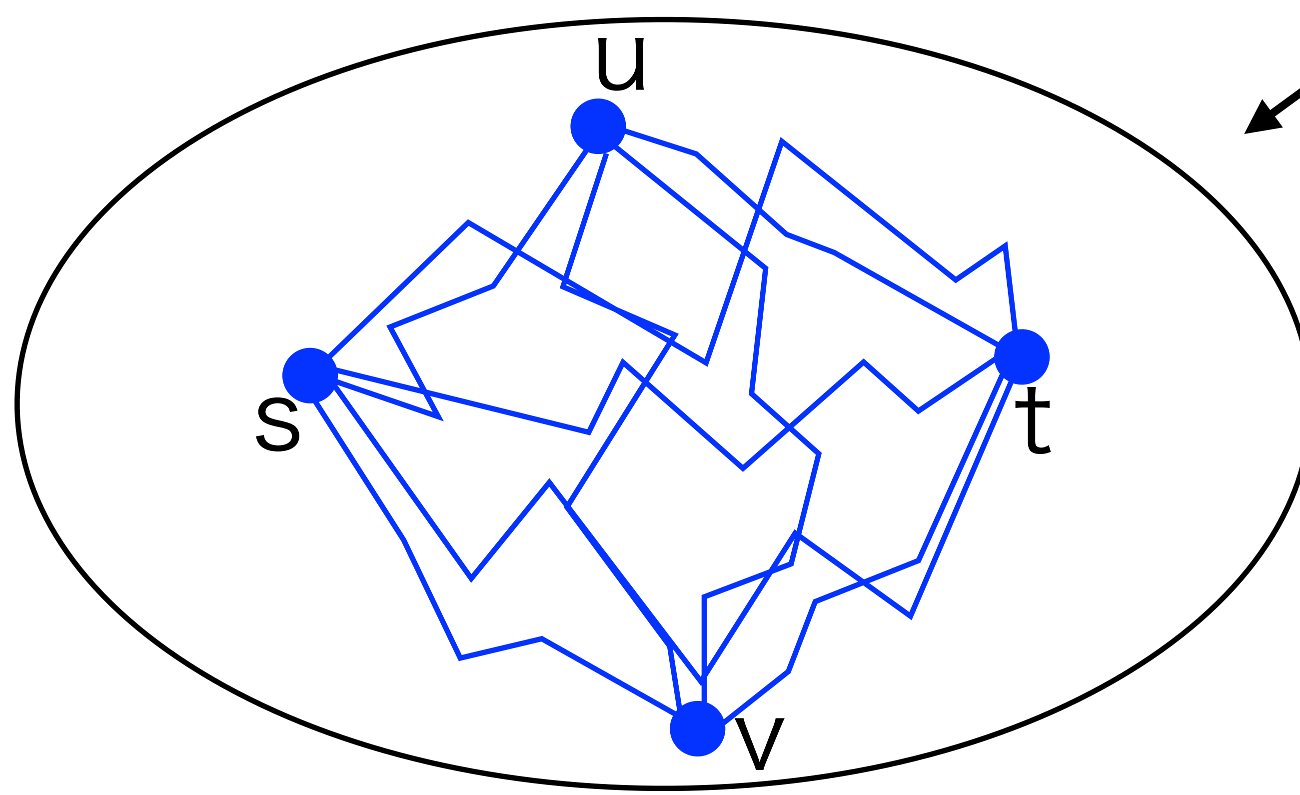
	s	t	u		
s		1	1	1	1
t	1		1	1	1
u	1	1		1	1
	1	1	1		1
	1	1	1	1	

free multiflow

Lovasz 76,

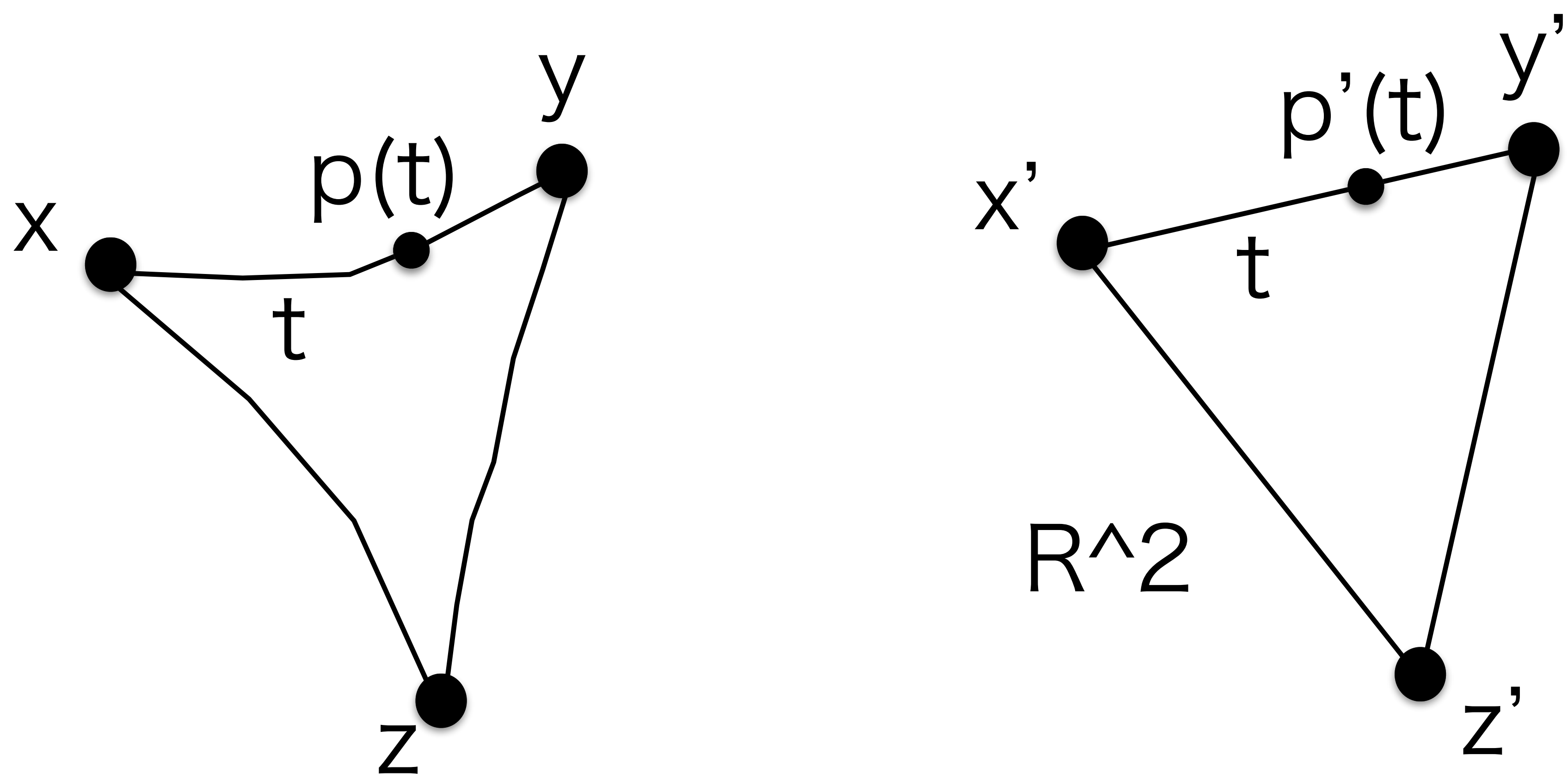
Cherkassky 77

k=2



CAT(0) space

~ geodesic metric space such that every geodesic triangle is "thin"



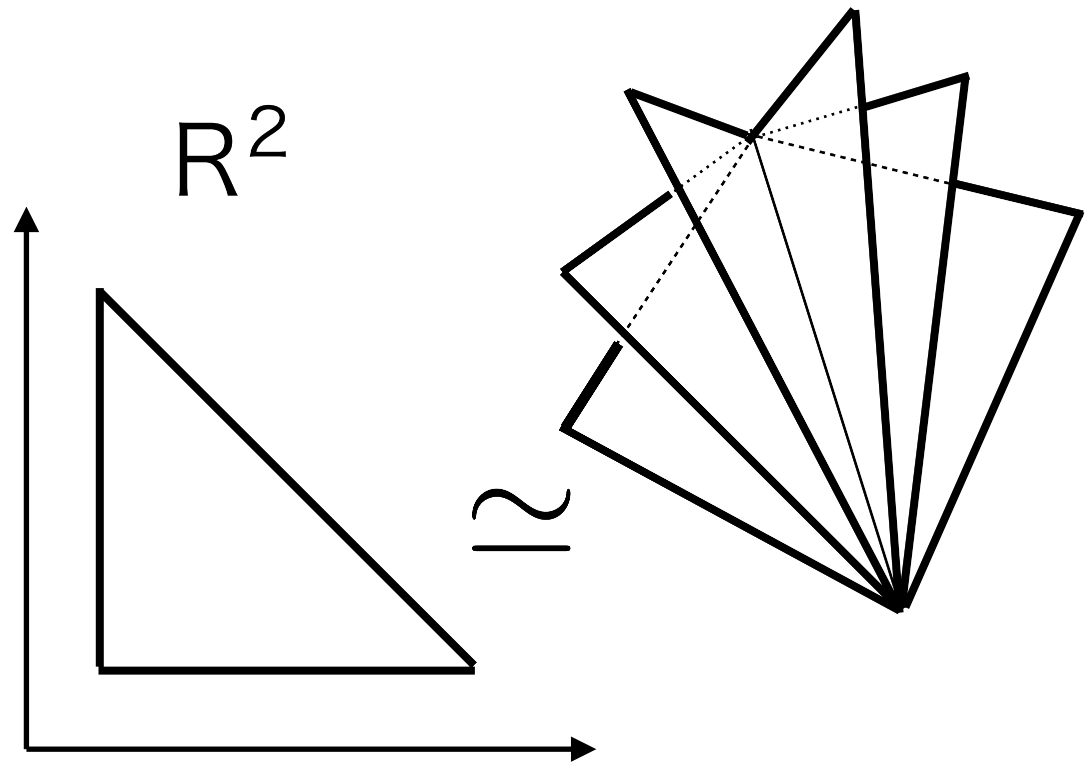
$d(y,z) = \| y' - z' \|$

$d(x,y) = \| x' - y' \|$

$d(z,x) = \| z' - x' \|$

$$d(p(t),z) \leqq \| p'(t) - z' \|$$

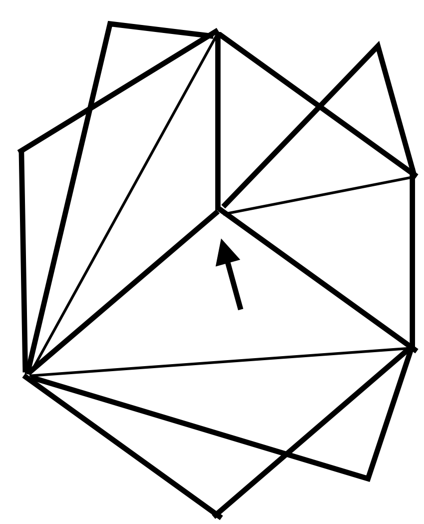
Folder

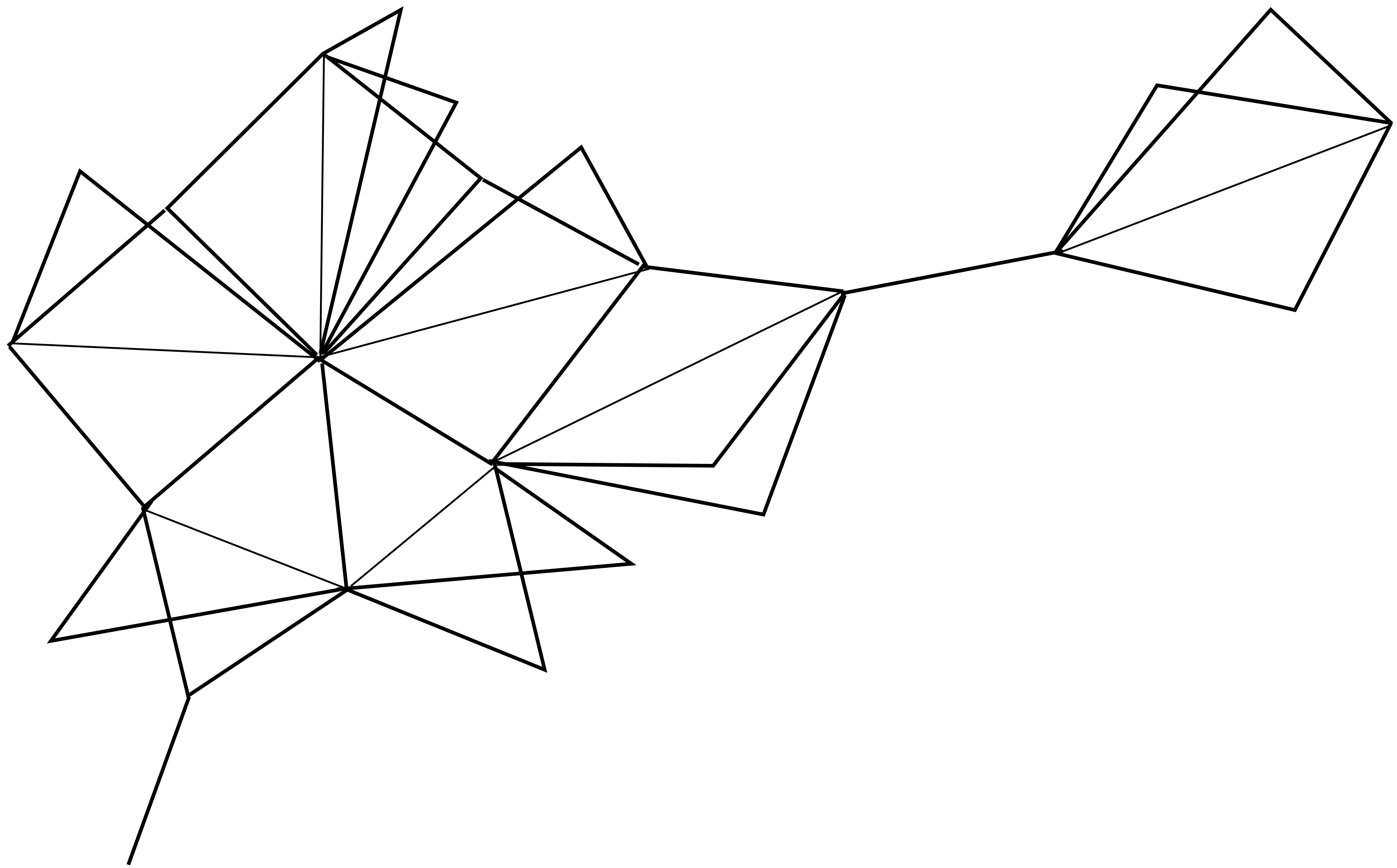


Folder complex

:= CAT(0) complex obtained by gluing folders

~ simply connected & without corner of cube





“μ embeds into a folder complex”

means

∃ K : folder complex,

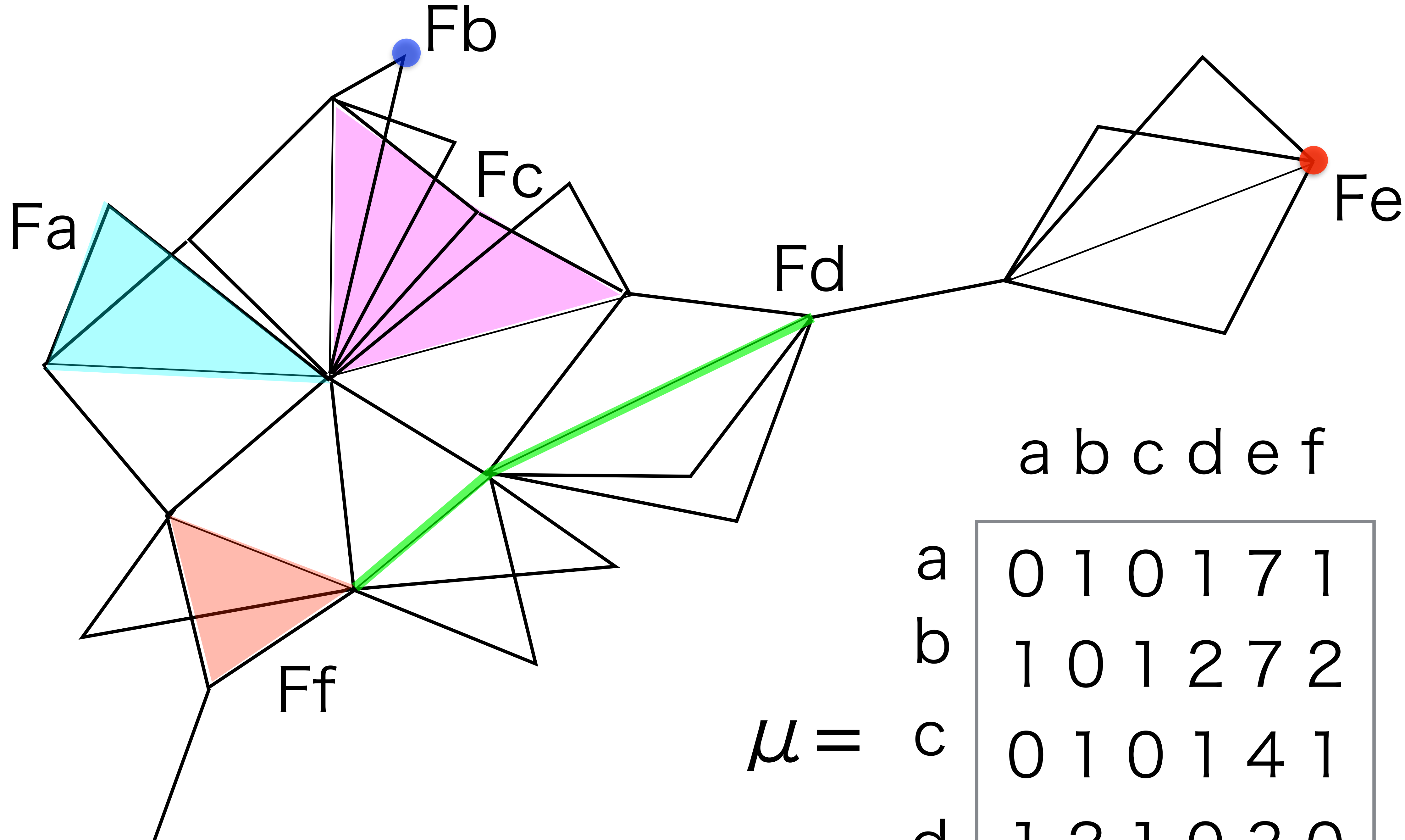
∃{ Fs : s in S }: convex sub-complexes with longer boundary edges

such that

μ(st) = D(Fs, Ft) (s,t in S)

D: ℓ_1-length metric on K

$$\mu = \begin{array}{c|cccccc|} & a & b & c & d & e & f \\ \hline a & 0 & 1 & 0 & 1 & 7 & 1 \\ b & 1 & 0 & 1 & 2 & 7 & 2 \\ c & 0 & 1 & 0 & 1 & 4 & 1 \\ d & 1 & 2 & 1 & 0 & 3 & 0 \\ e & 7 & 7 & 4 & 3 & 0 & 7 \\ f & 1 & 2 & 1 & 0 & 7 & 0 \\ \hline \end{array}$$



$\mu =$

	a	b	c	d	e	f
a	0	1	0	1	7	1
b	1	0	1	2	7	2
c	0	1	0	1	4	1
d	1	2	1	0	3	0
e	7	7	4	3	0	7
f	1	2	1	0	7	0

Thm: [H. SIDMA11]

K, { Fs }: embedding of μ

$$\text{Max } \Sigma \ \mu(st) \ f(P)$$

$$= \text{Min } \Sigma \ c(xy) \ D(p(x), p(y))$$

s.t. $p: V \to V(K)$,

$p(s)$ in Fs (s in S)

Thm: [H. SIDMA11]

K, { Fs }: embedding of μ

$$\text{Max } \Sigma \ \mu(st)\, f(P)$$

potential difference

$$= \text{Min } \Sigma \ c(xy)\, D(p(x), p(y))$$

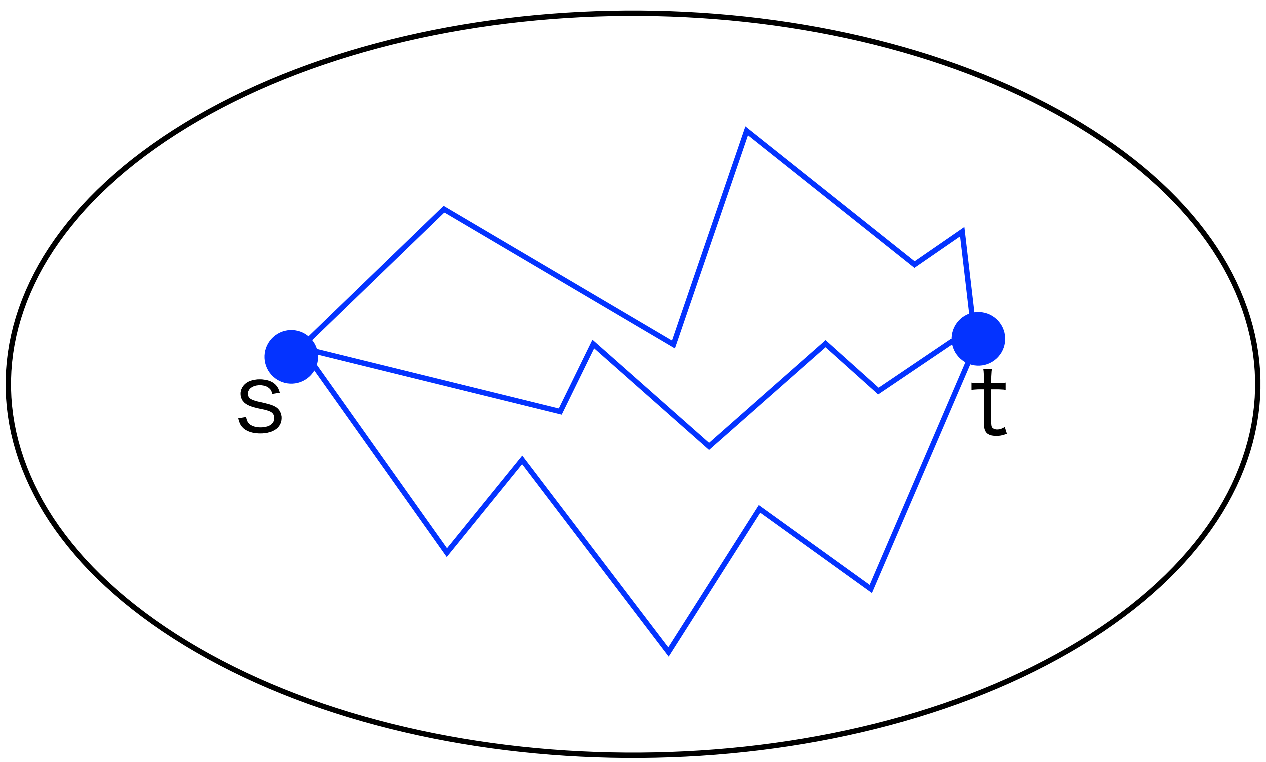
$$\text{s.t. } p: V \to V(K),$$

$$p(s) \text{ in } Fs \ (s \text{ in } S)$$

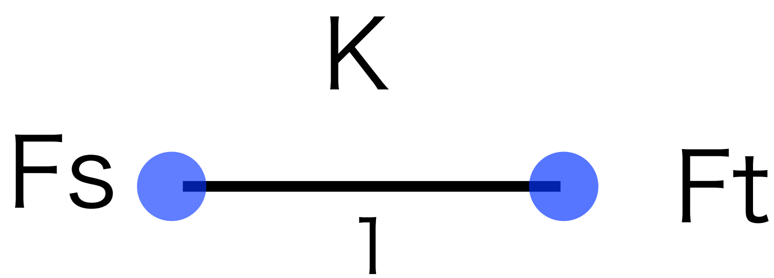
node potential

boundary condition

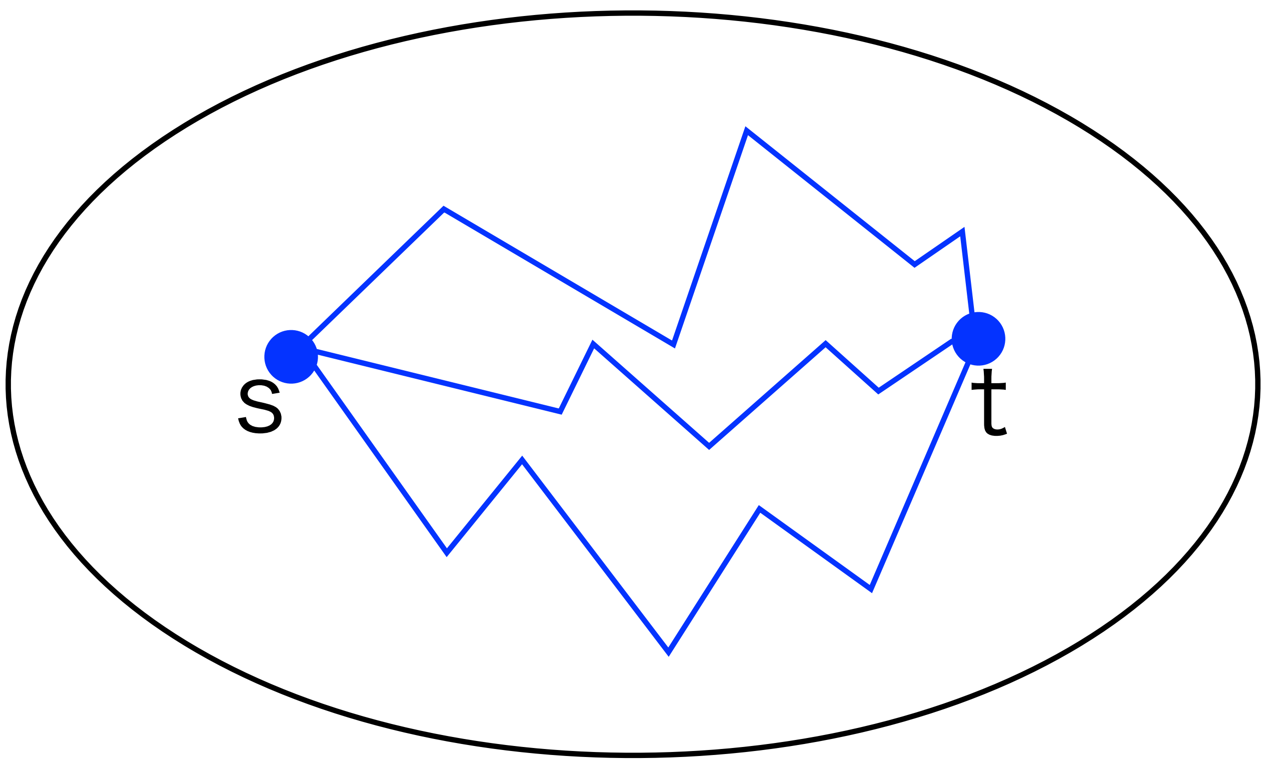
$$\mu \quad \begin{array}{c} \\ s \\ t \end{array} \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \end{array}$$



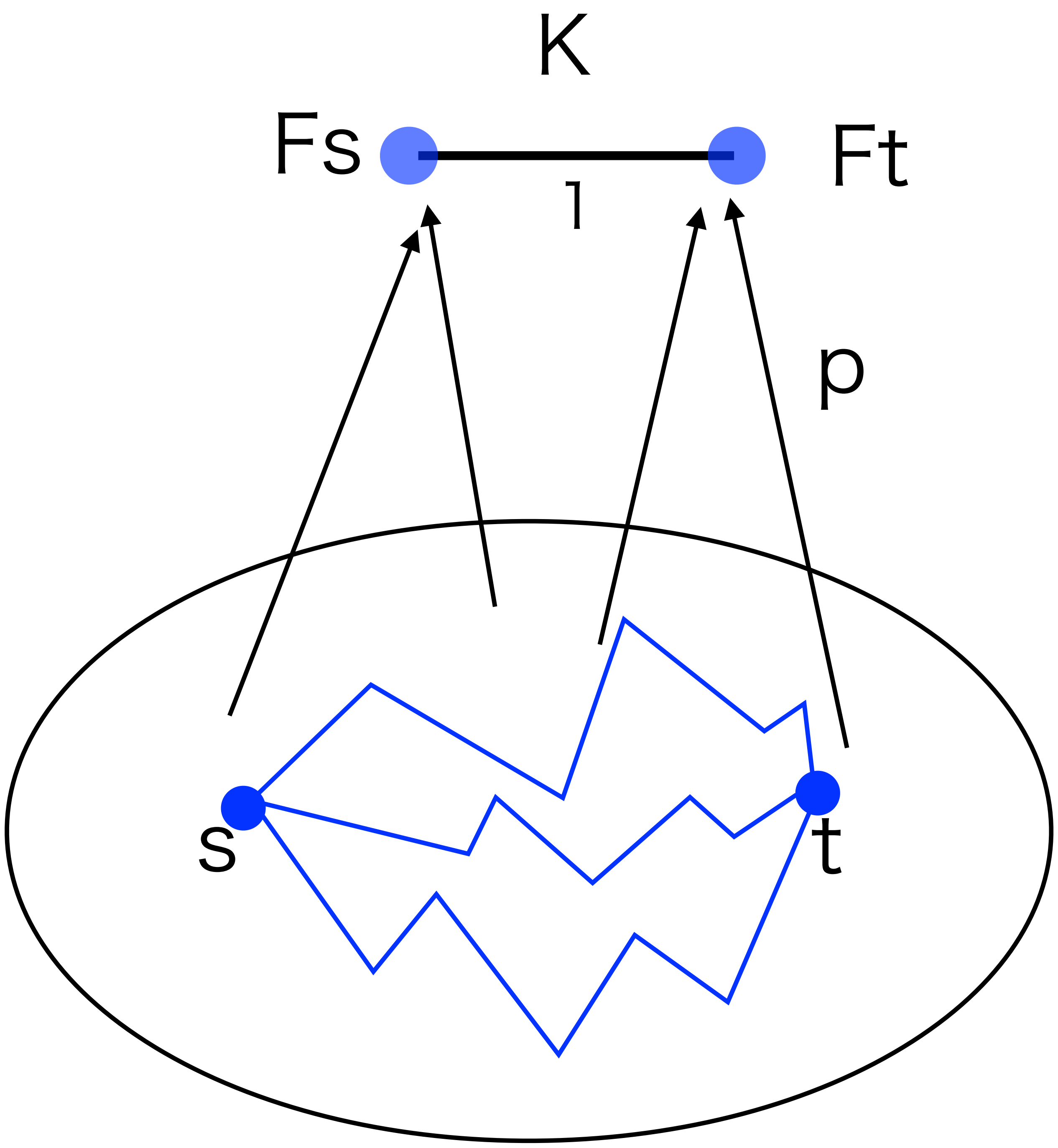
$$\text{Max } \Sigma \ \mu(st)\ f(P)$$



$$\mu \quad \begin{array}{ccc} & s & t \\ s & & 1 \\ t & 1 & \end{array}$$



Max Σ μ(st) f(P)



$$\mu \quad \begin{array}{c|cc} & s & t \\ \hline s & & 1 \\ t & 1 & \end{array}$$

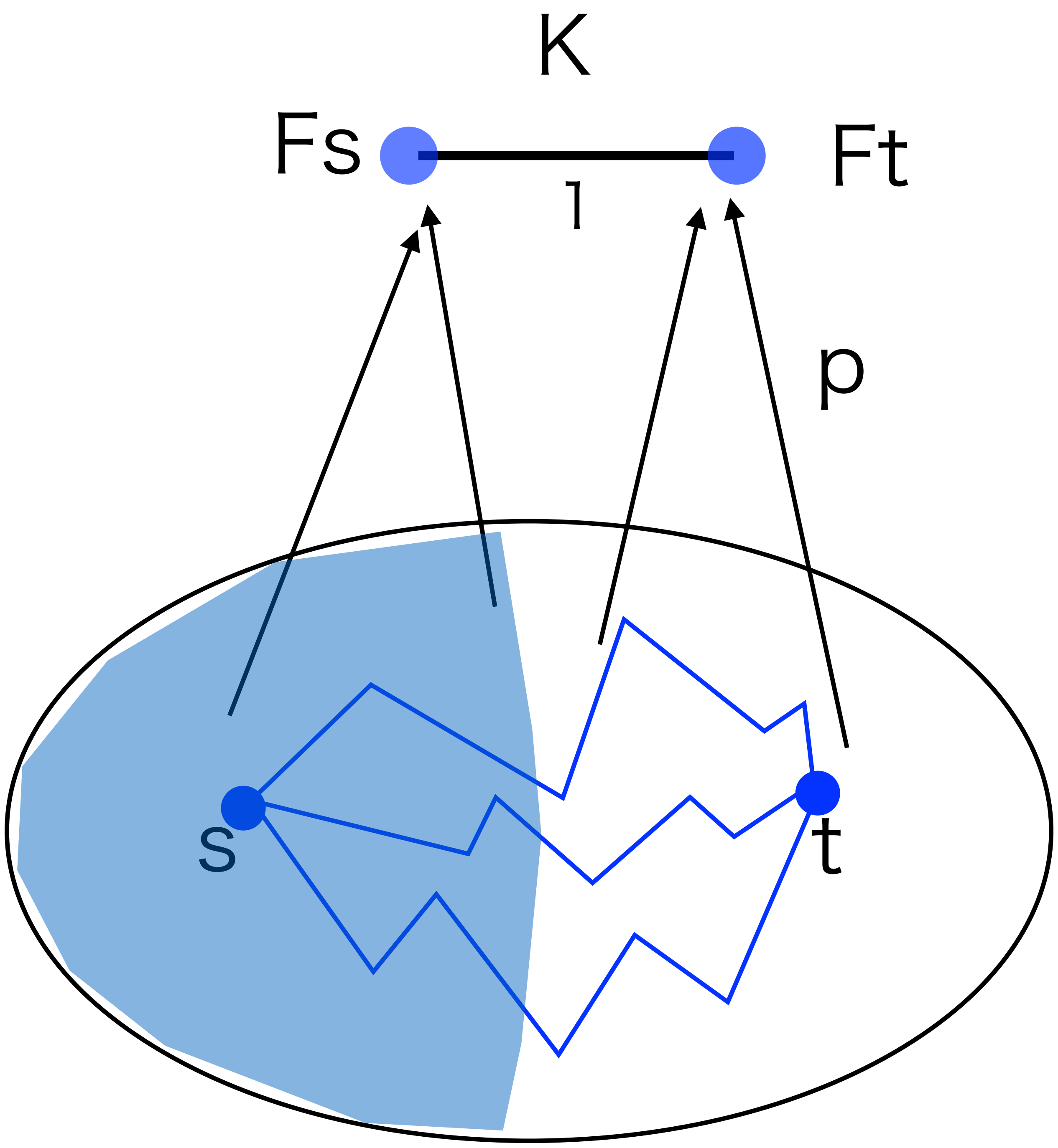
$$\text{Max } \sum \mu(st)\, f(P)$$

$$=$$

$$\text{Min } \sum c(xy)\, D(p(x), p(y))$$

s.t. $p: V \to V(K)$,

$p(s)$ in Fs (s in S)



$$\mu \quad \begin{array}{c|cc} & s & t \\ \hline s & & 1 \\ t & 1 & \end{array}$$

$$\text{Max } \Sigma \ \mu(st)\ f(P)$$

$$=$$

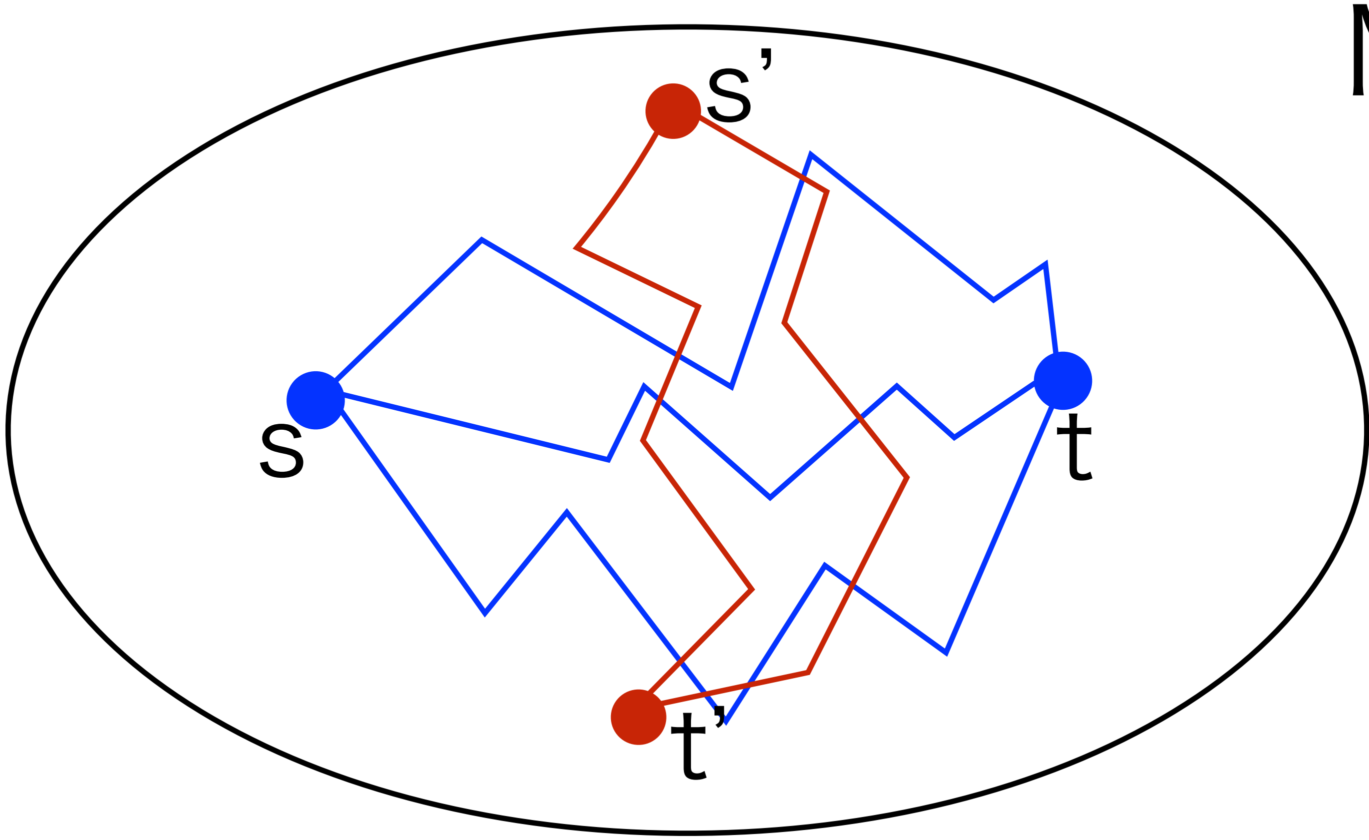
$$\text{Min } \Sigma \ c(xy)\ D(p(x), p(y))$$

s.t. $p: V \to V(K)$,

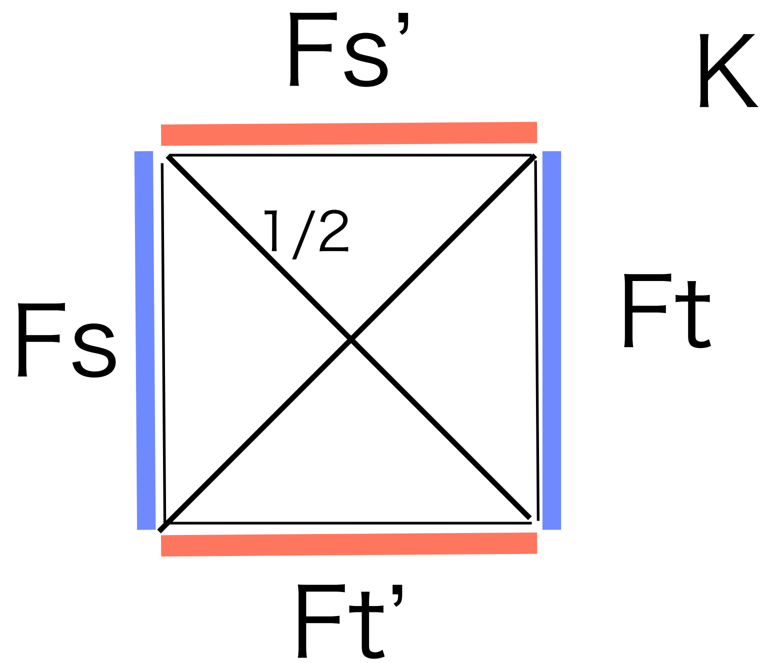
$p(s)$ in Fs (s in S)

μ

	s	t	s'	t'
s		1		
t	1			
s'				1
t'			1	

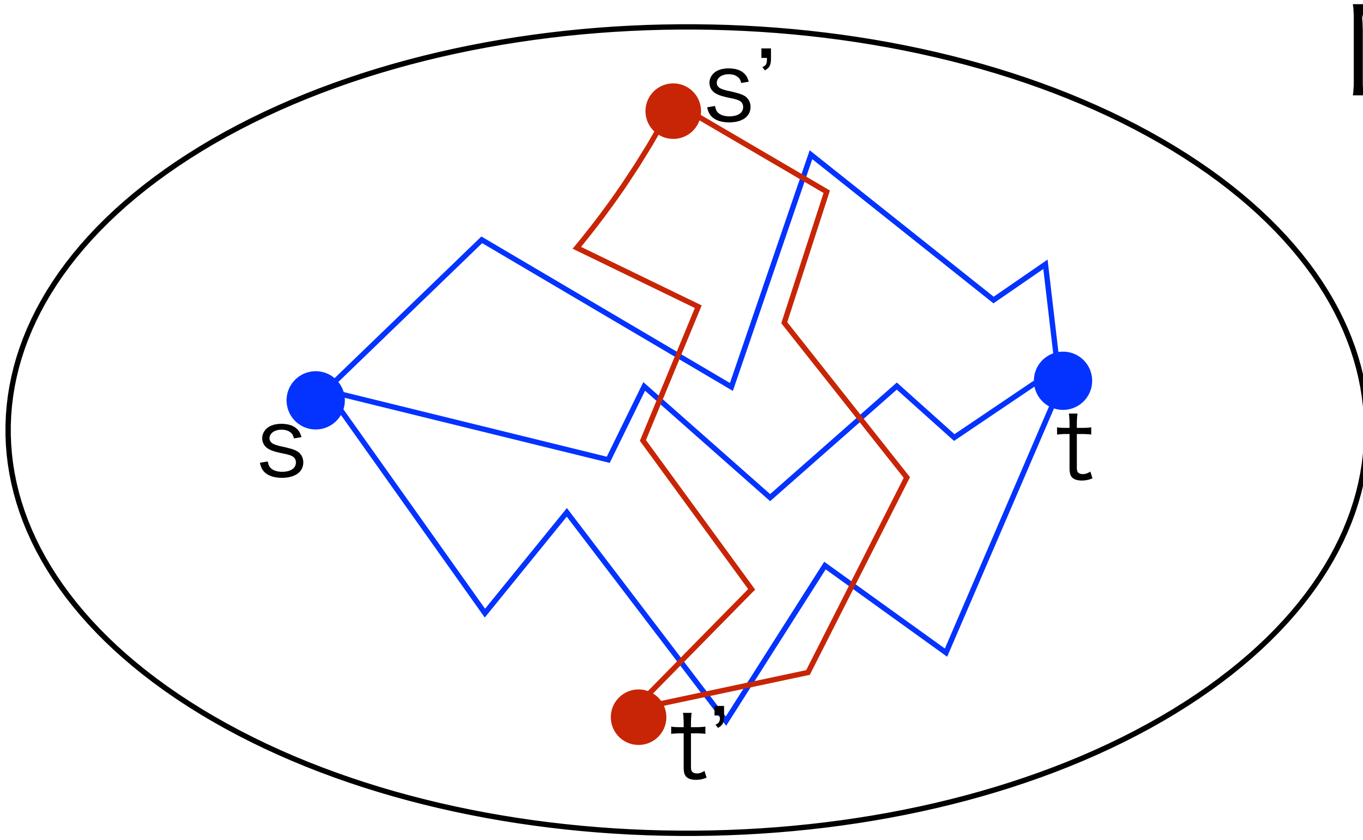


$\text{Max} \sum \mu(st)\, f(P)$

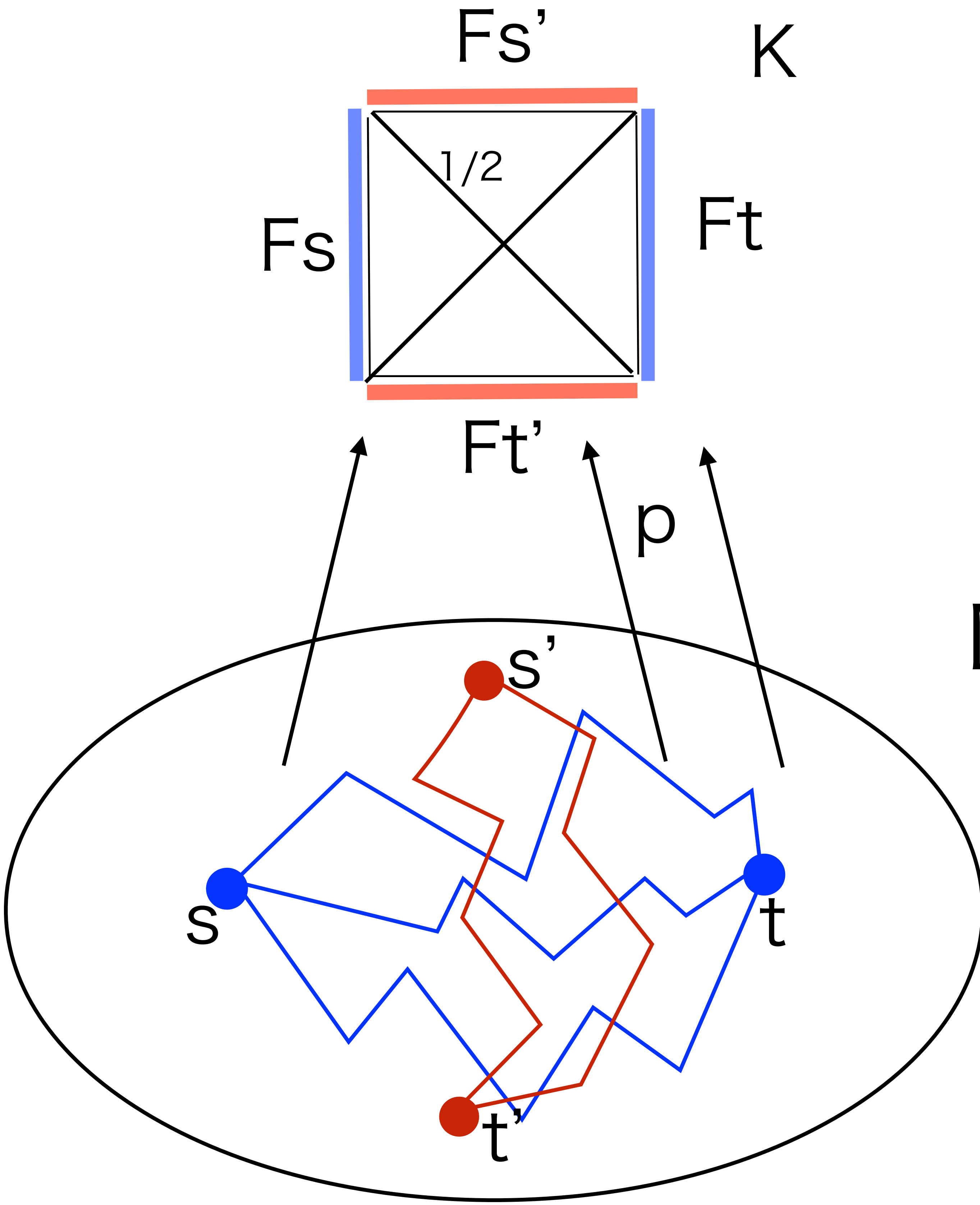


μ

	s	t	s'	t'
s		1		
t	1			
s'				1
t'			1	



Max $\Sigma\ \mu(st)\ f(P)$



$$\mu \quad \begin{array}{c|cccc} & s & t & s' & t' \\ \hline s & & 1 & & \\ t & 1 & & & \\ s' & & & & 1 \\ t' & & & 1 & \end{array}$$

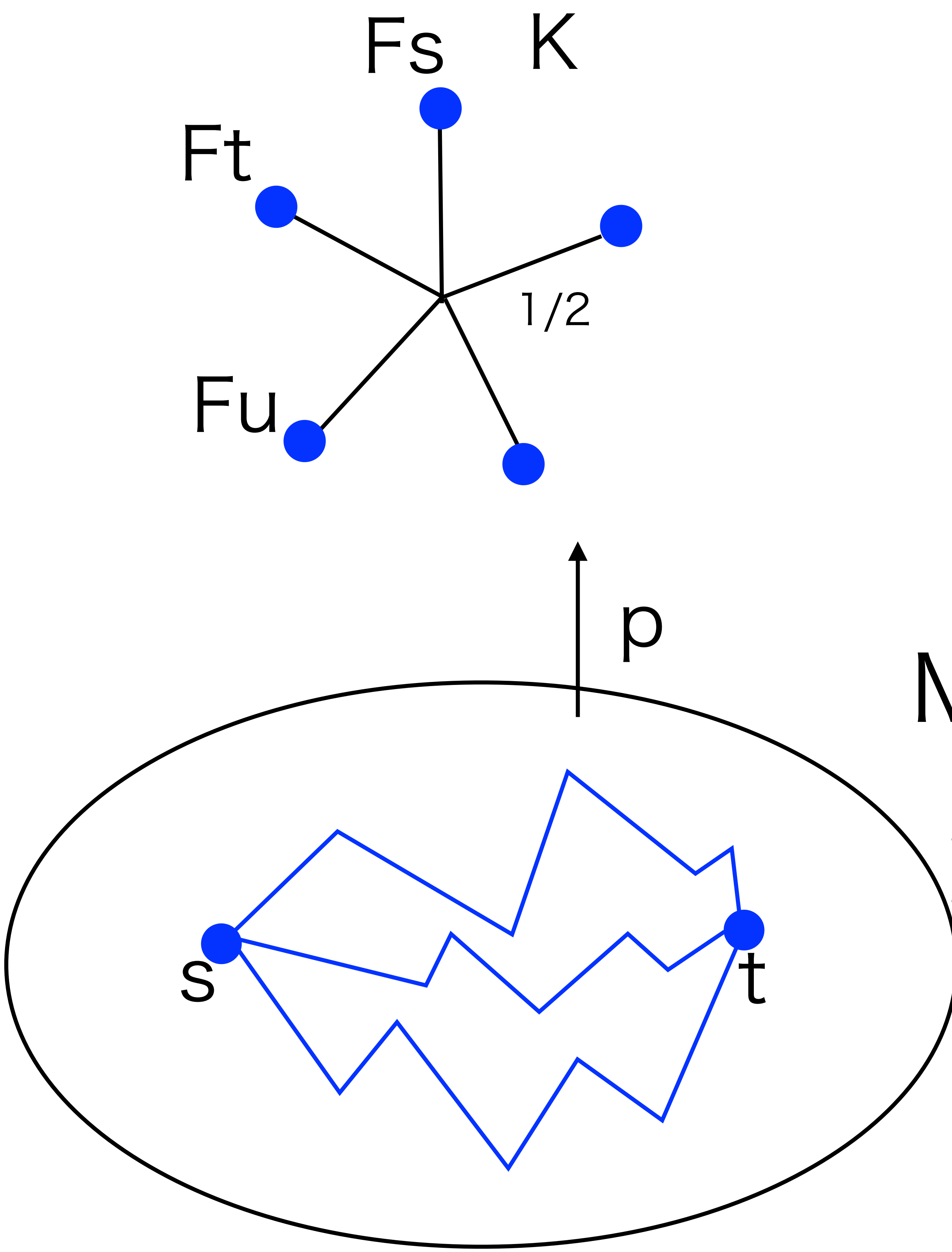
Max Σ μ(st) f(P)

=

Min Σ c(xy) D(p(x), p(y))

s.t. p: V → V(K),

p(s) in Fs (s in S)



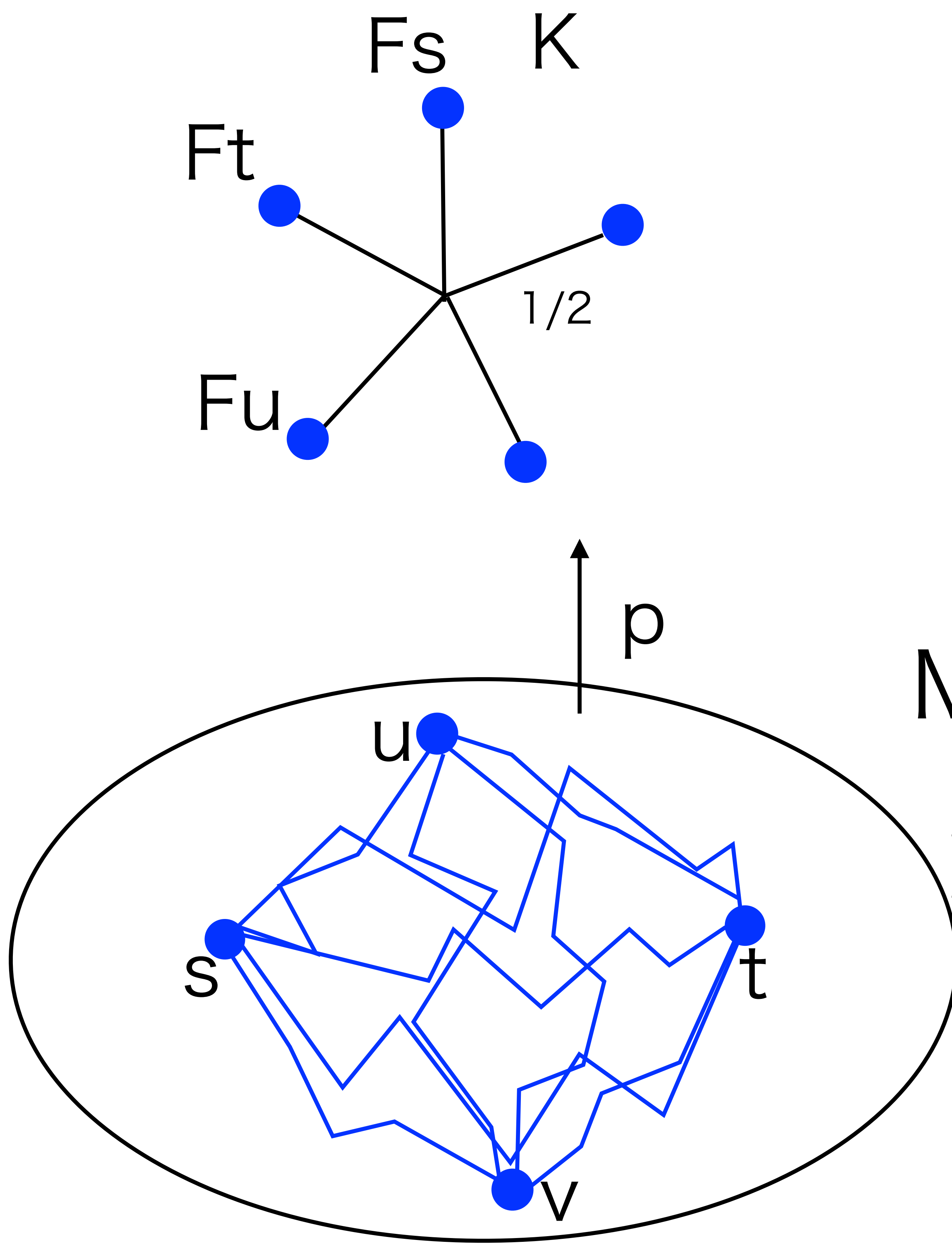
$$\mu \quad \begin{array}{c} \\ s \\ t \\ u \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} s & t & u & & \end{array} \\ \begin{bmatrix} & 1 & 1 & 1 & 1 \\ 1 & & 1 & 1 & 1 \\ 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & 1 & \end{bmatrix} \end{array}$$

Max Σ μ(st) f(P) =

Min Σ c(xy) D(p(x), p(y))

s.t. p: V → V(K),

p(s) in Fs (s in S)



$$\mu \quad \begin{array}{c} \\ s \\ t \\ u \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} s & t & u & & \end{array} \\ \begin{bmatrix} & 1 & 1 & 1 & 1 \\ 1 & & 1 & 1 & 1 \\ 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & 1 & \end{bmatrix} \end{array}$$

Max Σ μ(st) f(P) =

Min Σ c(xy) D(p(x), p(y))

s.t. p: V → V(K),

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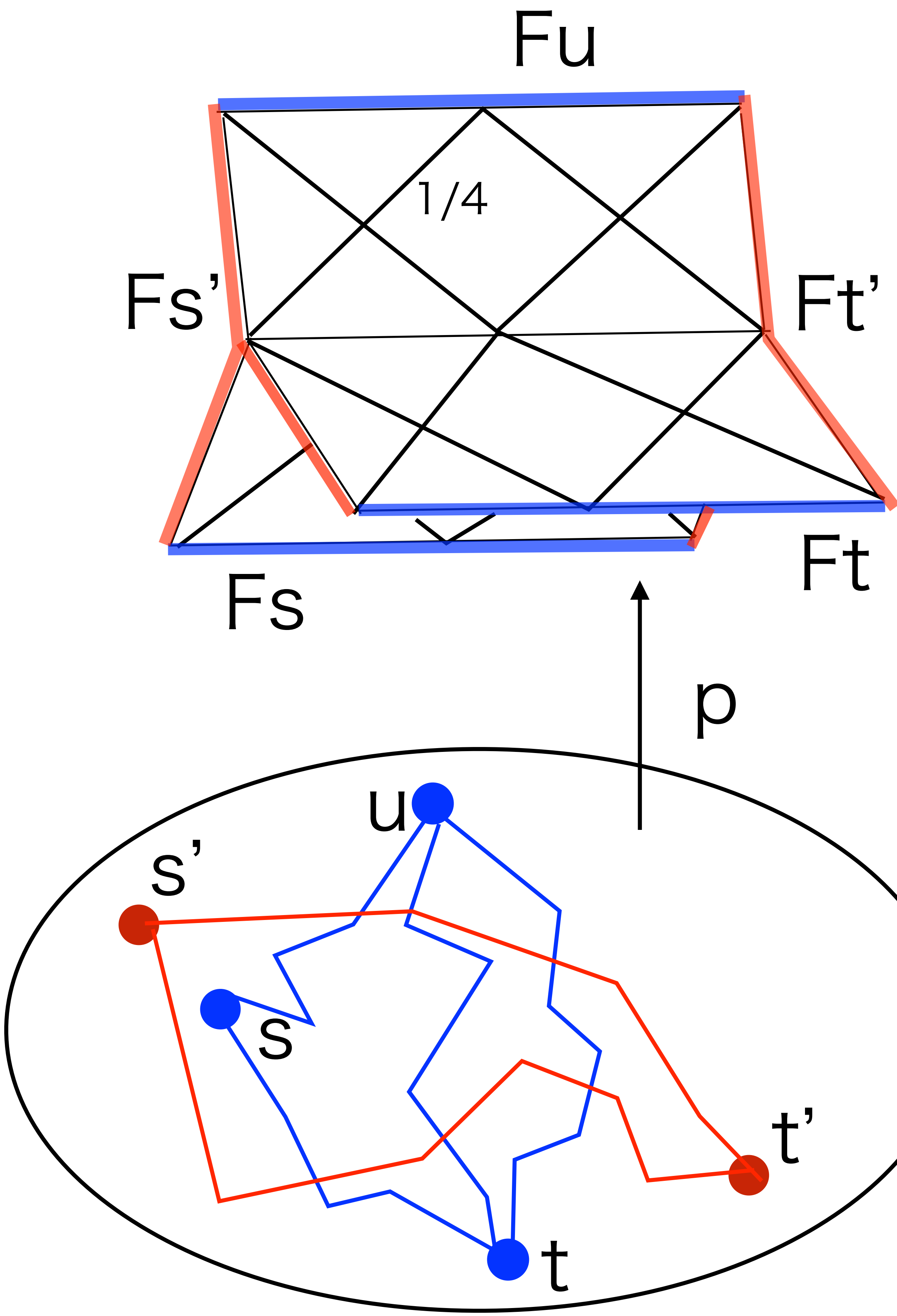


$$\mu \quad \begin{array}{c} \\ s \\ t \\ u \\ s' \\ t' \end{array} \begin{array}{ccccc} s & t & u & s' & t' \\ \hline & 1 & 1 & & \\ 1 & & 1 & & \\ 1 & 1 & & & \\ & & & & 1 \\ & & & 1 & \end{array}$$

Max Σ μ(st) f(P) =

Min Σ c(xy) D(p(x), p(y))

s.t. p: V → V(K),

p(s) in Fs (s in S)

Thm [H. STOC10, MOR14]

μ embeds into a folder complex

$\Rightarrow$ $\exists$ 1/24-integral max-flow

Thm [H. JCTB09, SIDMA11]

Otherwise, no k:1/k-integral max-flow

Thm [H. STOC10, MOR14]

μ embeds into a folder complex

$\Rightarrow$ $\exists$ 1/24-integral max-flow

Thm [H. JCTB09, SIDMA11]

Otherwise, no k:1/k-integral max-flow

Proof tools: linear programming duality + α

LP-dual = LP over (semi)metrics on V

(Onaga-Kakusho, Iri 71),

Tight span (Isbell 64, Dress 84)

Splitting-off technique, $\cdots$

Multifacility Location Problem

G: graph (city), d: path-metric

We are going to locate n facilities on V(G) such that the communication cost is minimum.

$$\Sigma\, b(iv)\, d(p(i), v) + \Sigma\, c(ij)\, d(p(i),p(j))$$

cost between facilities and places

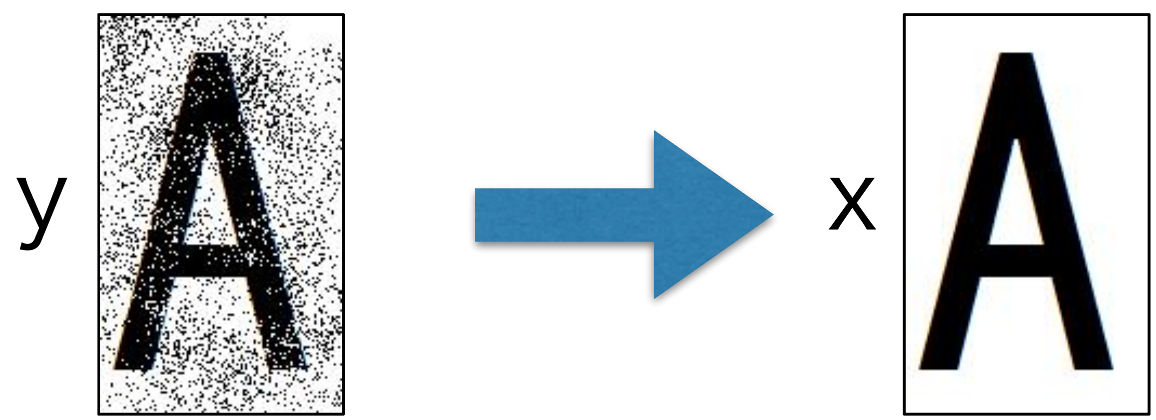
cost between facilities

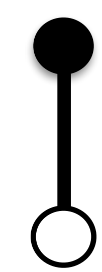
p(i): location of facility i

Formulated in 70's

Extending minimum-cut problem

Recent applications: Labeling tasks
in machine learning, computer vision, …



G = 

$$\text{Min. } \Sigma\, b\, d(y(i), x(i)) + \sum_{i,j:\text{adjacent}} c\, d(x(i),x(j))$$

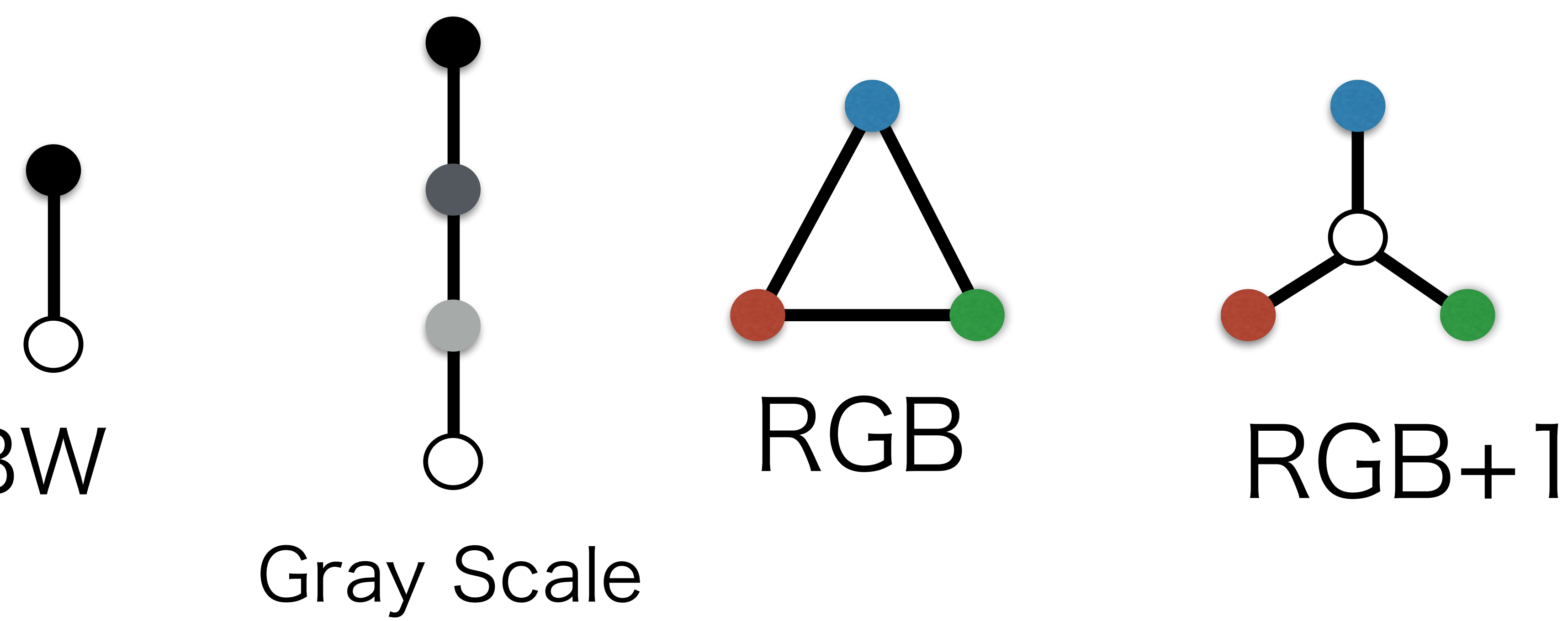
s.t. x(i) in { white, black} (i: pixel)

Multifac[G]:

min. $\Sigma b(iv)\, d(p(i), v) + \Sigma c(ij)\, d(p(i),p(j))$

s.t. $p(i)$ in $V(G)$ $(i=1,2,\cdots,n)$

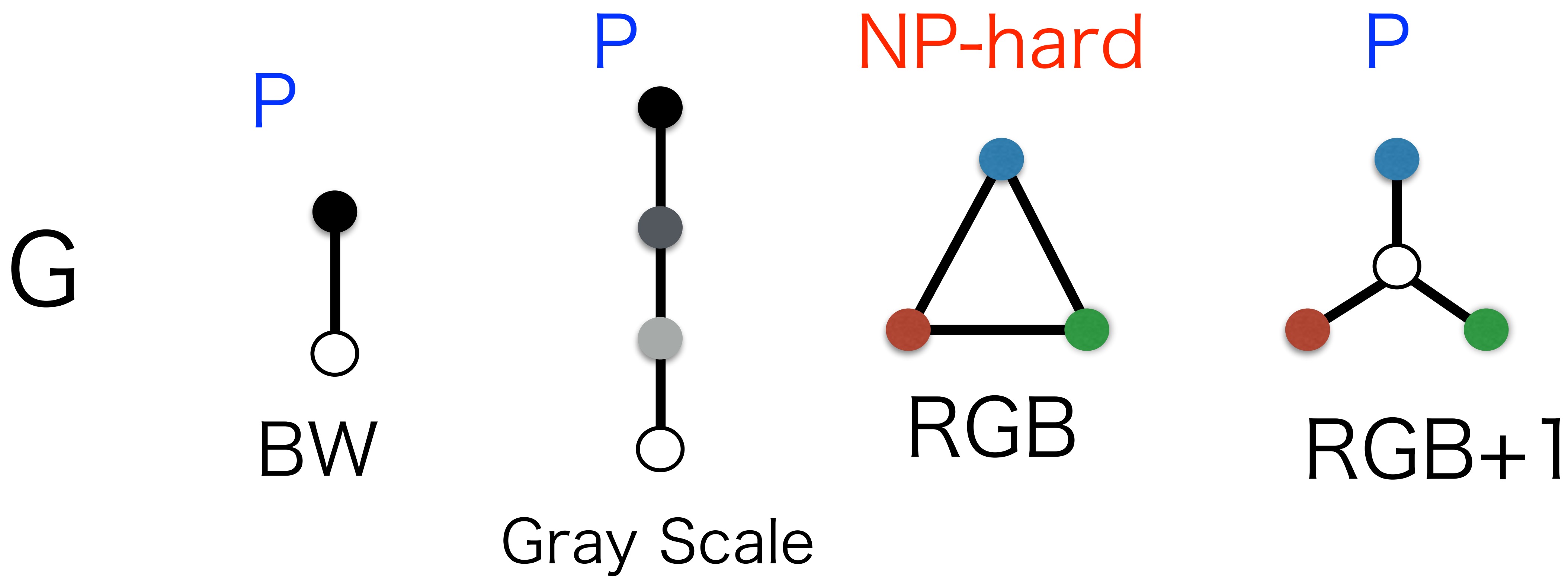
G



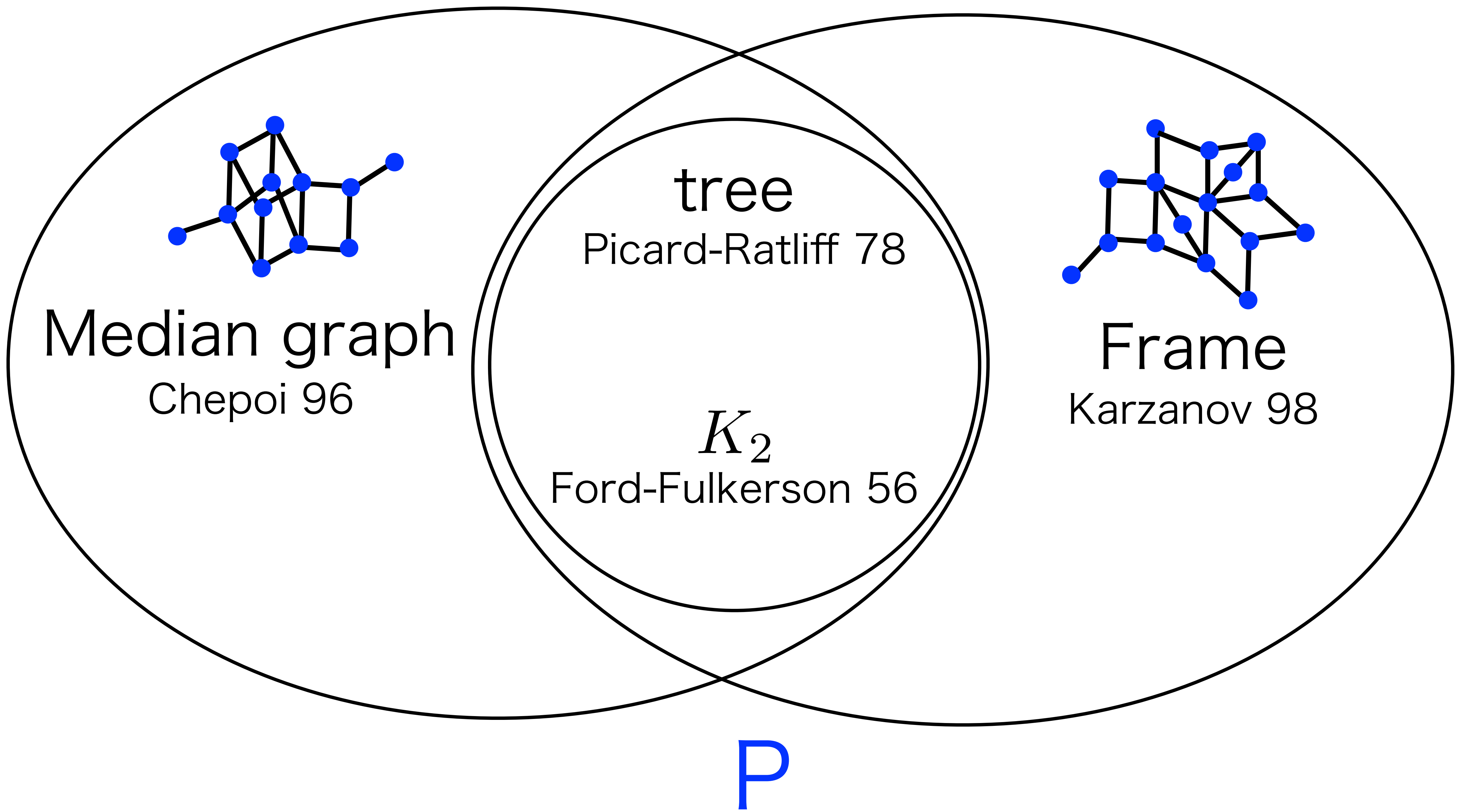
Multifac[G]:

$$\text{min. } \Sigma\, b(iv)\ d(p(i), v) + \Sigma\, c(ij)\ d(p(i), p(j))$$

$$\text{s.t. } \quad p(i) \text{ in } V(G)\ (i=1,2,\cdots,n)$$



What is G for which Multifac[G] is in P ?
(Karzanov 98)



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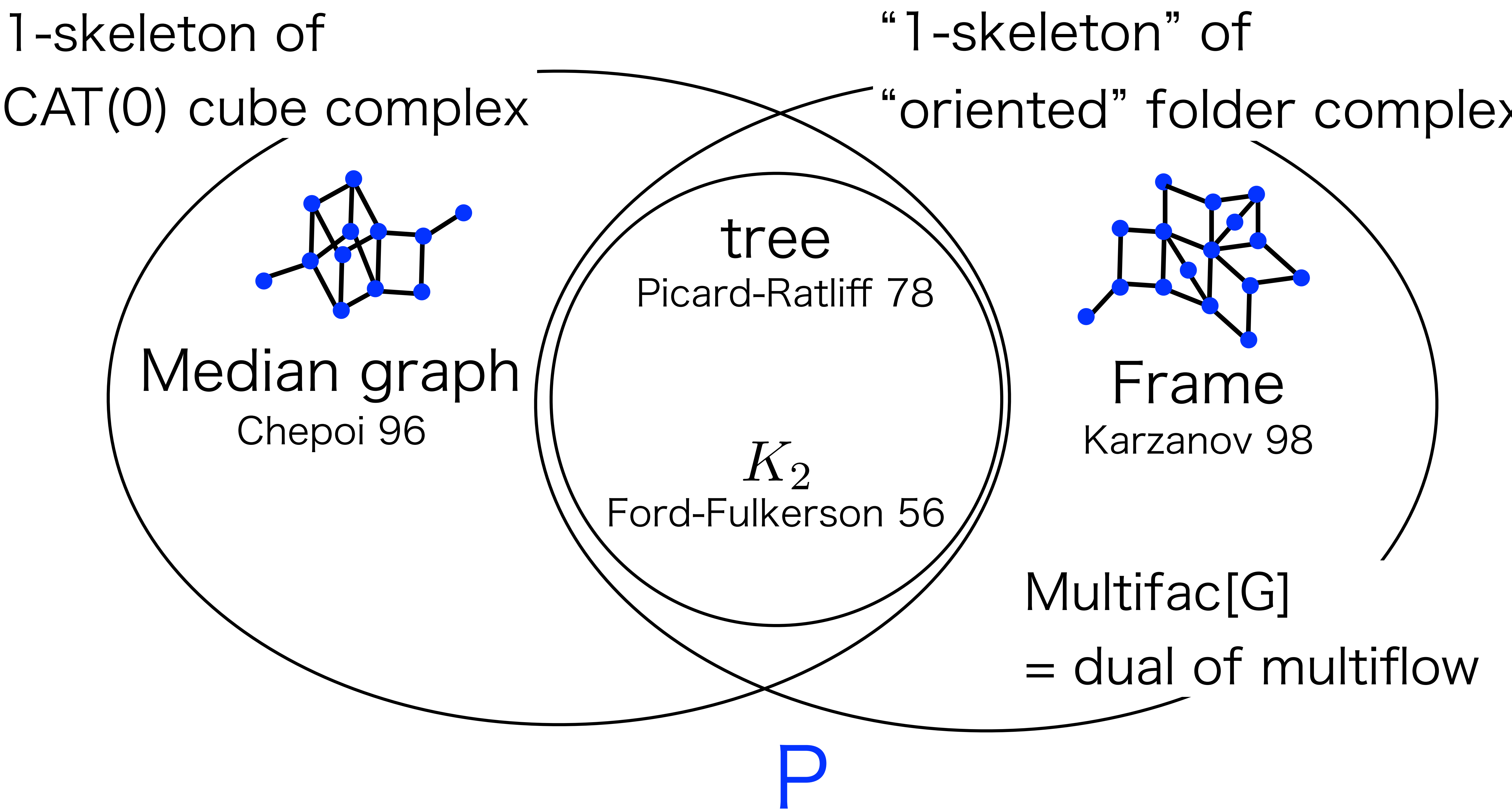


A dichotomy

Thm [H. SODA13, MPA to appear]

If G is orientable modular, then Multifac[G] is in P.

Thm [Karzanov 98]

Otherwise Multifac[G] is NP-hard.

Def: G is modular

⇔ every triple of vertices has a median

median u of x,y,z:

⇔ $d(x,y) = d(x,u) + d(u,y)$,

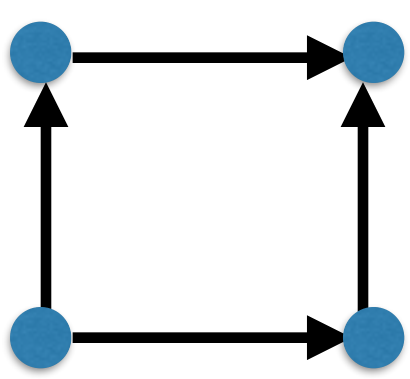
$d(y,z) = d(y,u) + d(u,z)$,

$d(z,x) = d(z,u) + d(u,x)$

Def: G is orientable

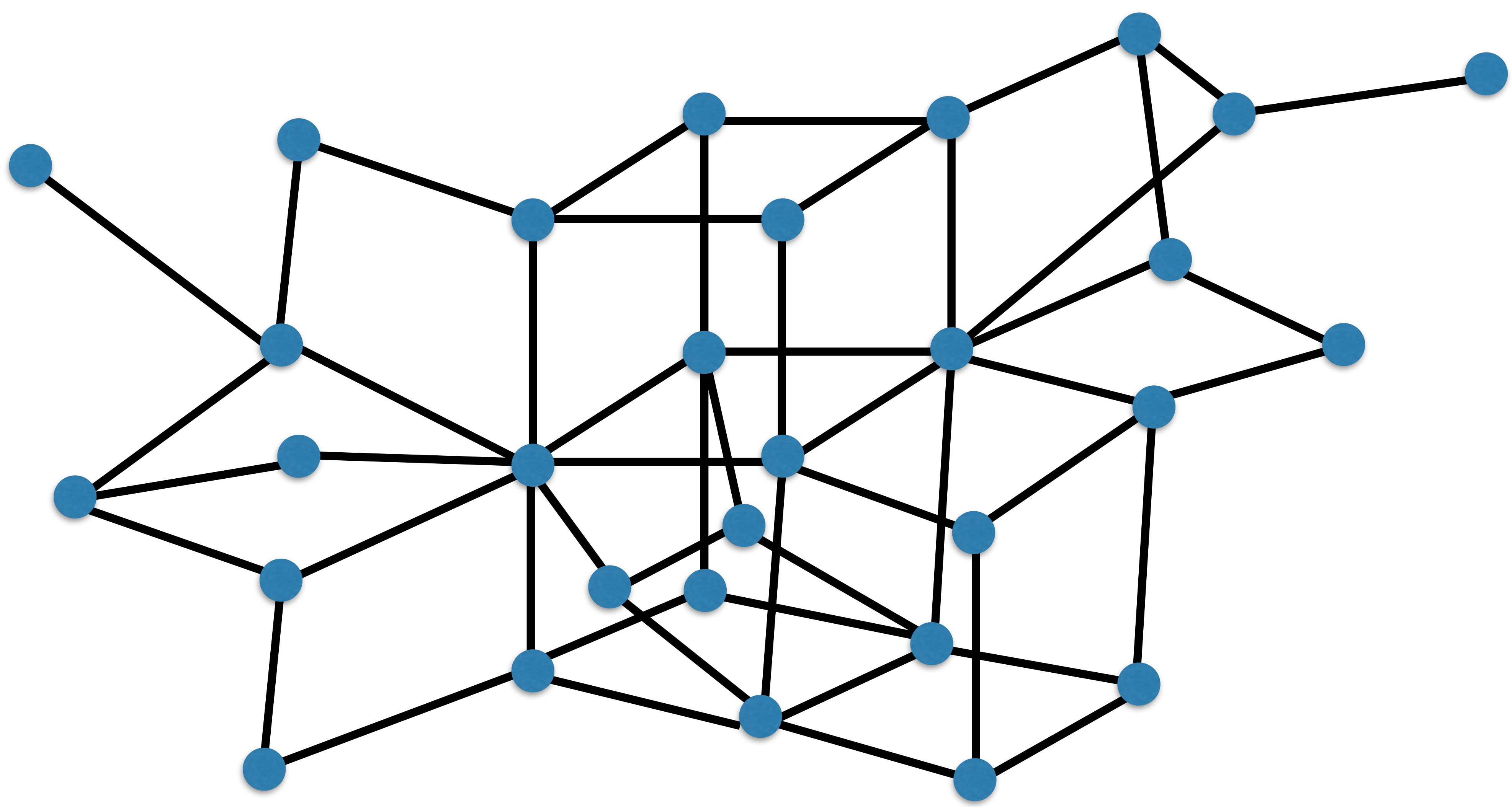
⇔ ∃ orientation such that

∀ 4-cycle is oriented as



Orientable modular graph

Ex: tree, cube, grid graph, modular lattice, median graph, their products and "gluing"



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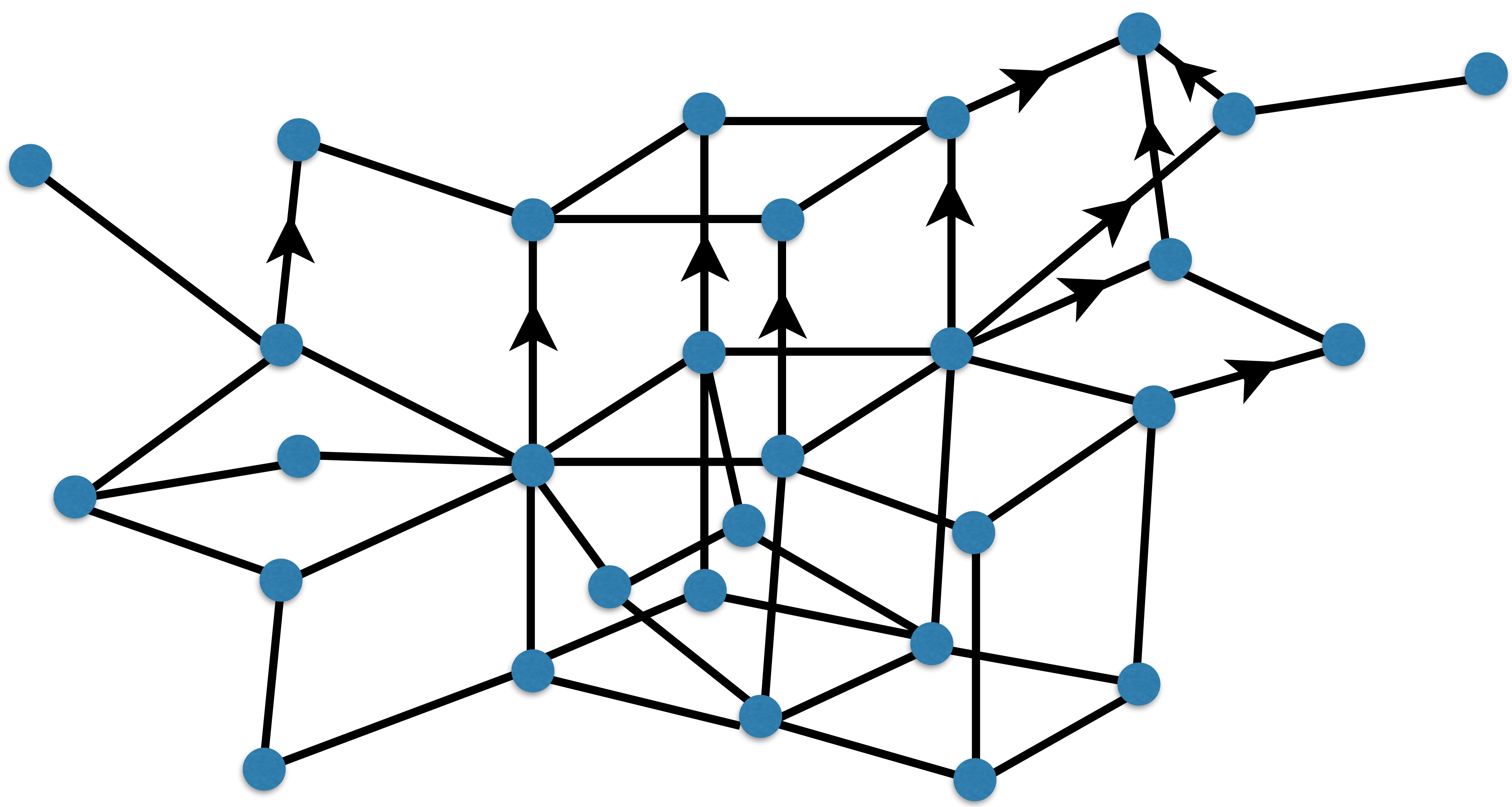
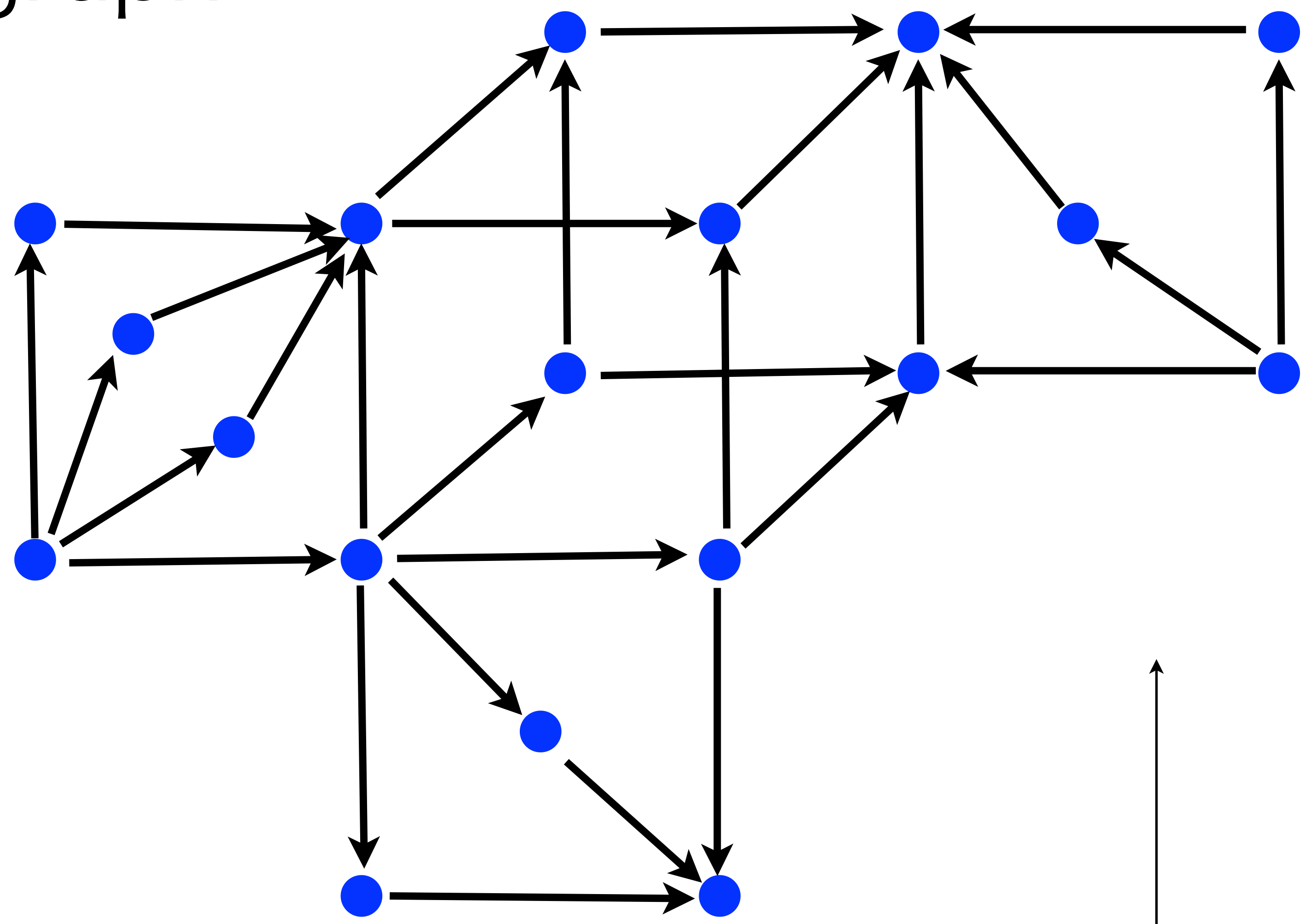


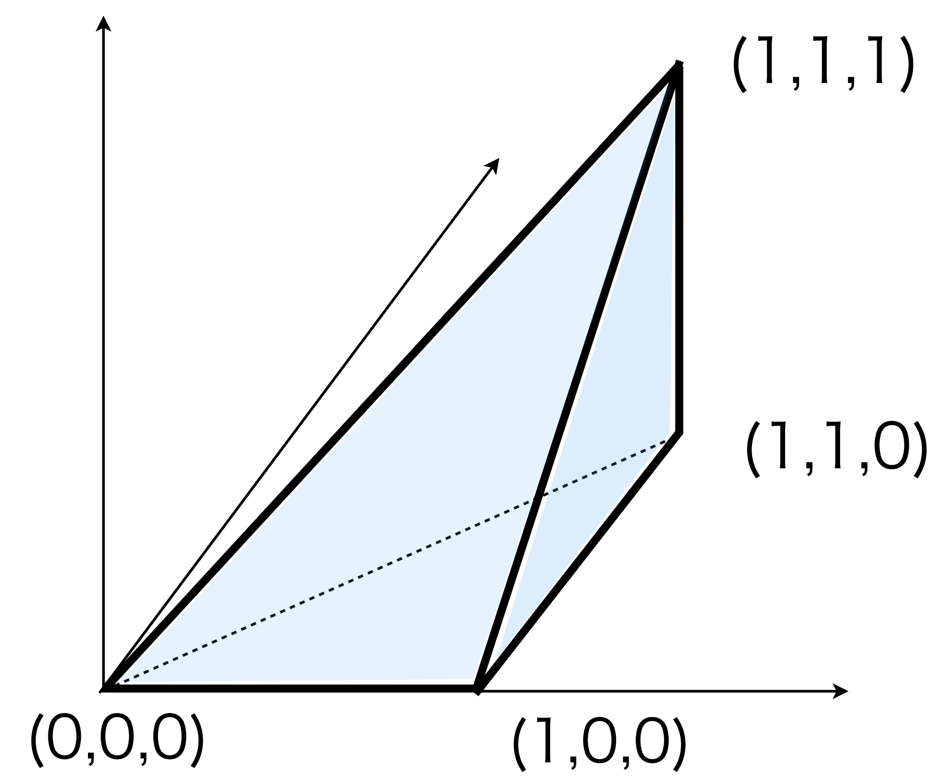
Proof tools:
lattice theory, metric graph theory (Bandelt-Chepoi)
discrete convex analysis (Murota),
valued CSP (Thapper-Zivny),⋯

My intuition behind the proof:
View Multifac[G] as an optimization over
a complex associated with om-graph $G \times G \times \cdots \times G$

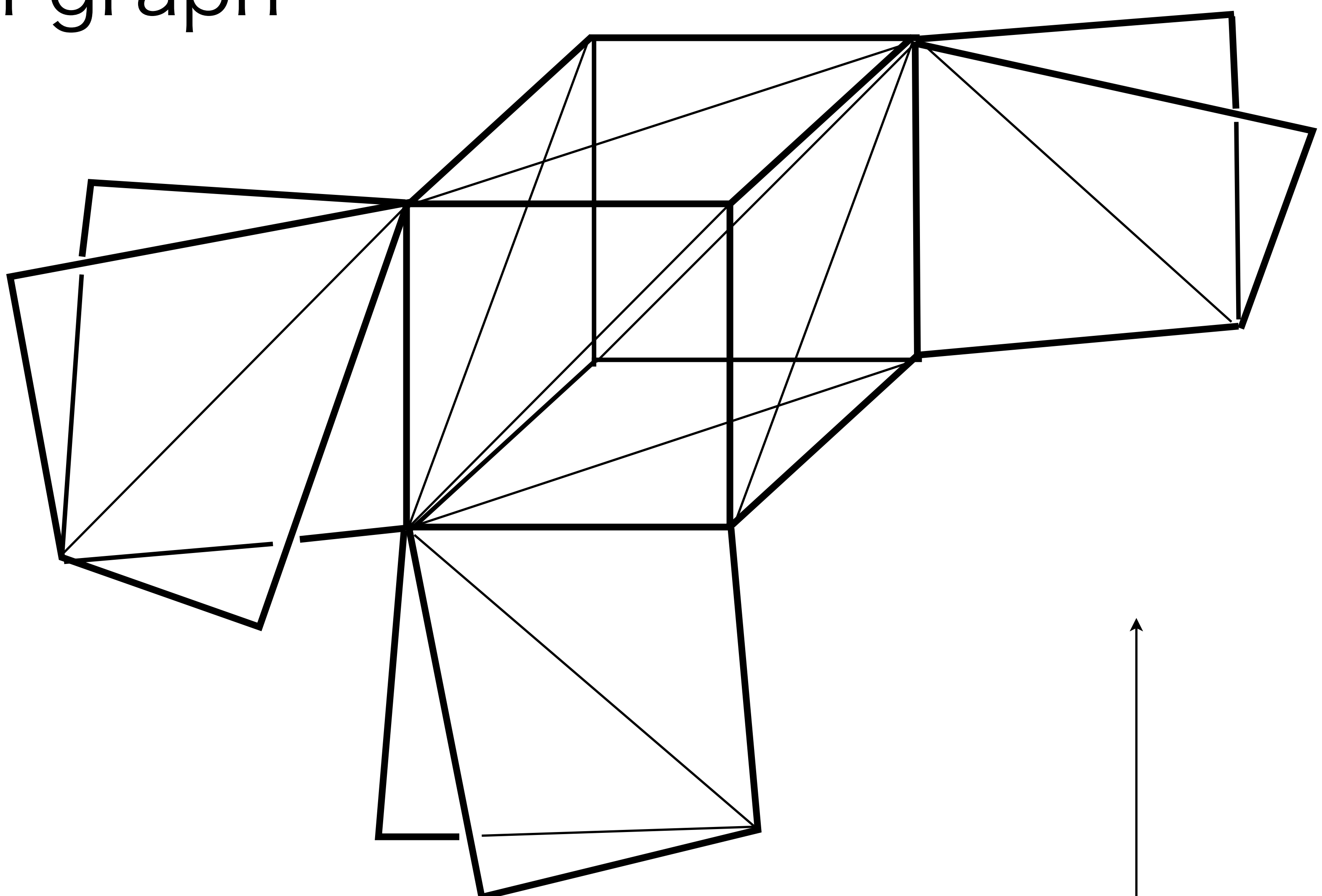
G: om-graph



K(G): complex obtained by filling to each maximal chain of a cube subgraph (= Boolean lattice)



G: om-graph



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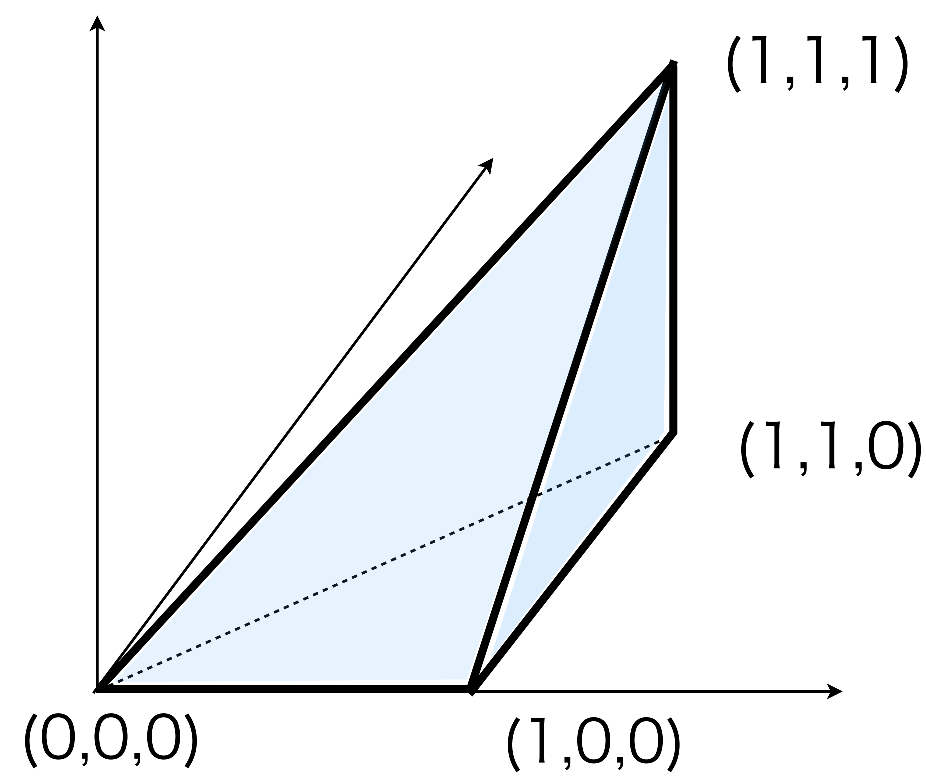
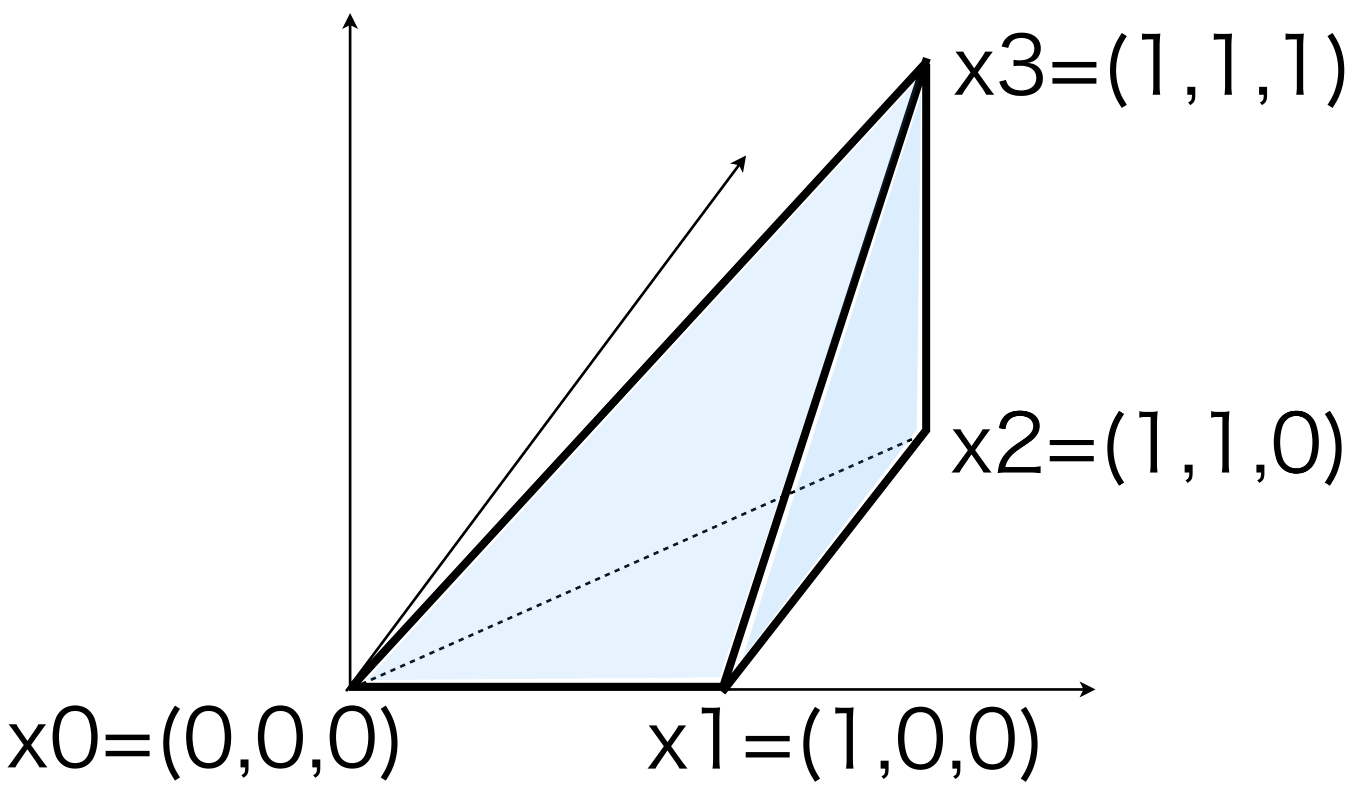


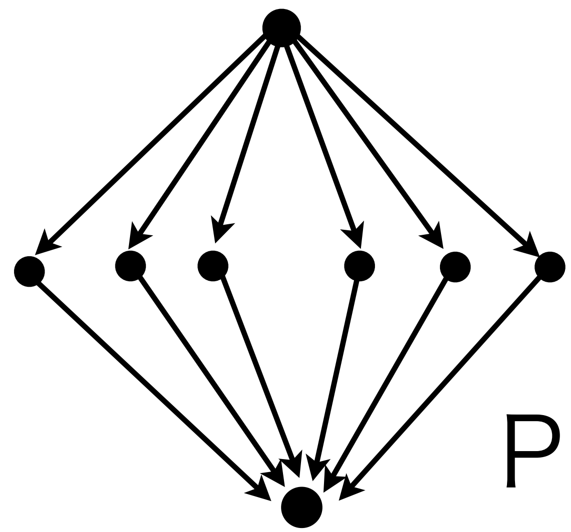
Orthoscheme complex (Brady-McCammond10)

P: graded poset

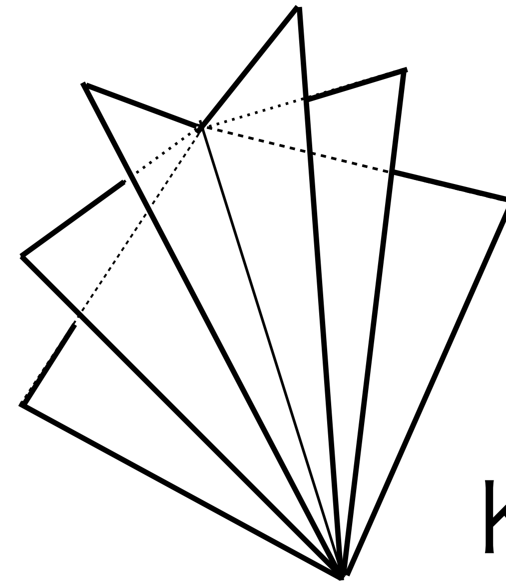
K(P): = complex obtained by filling



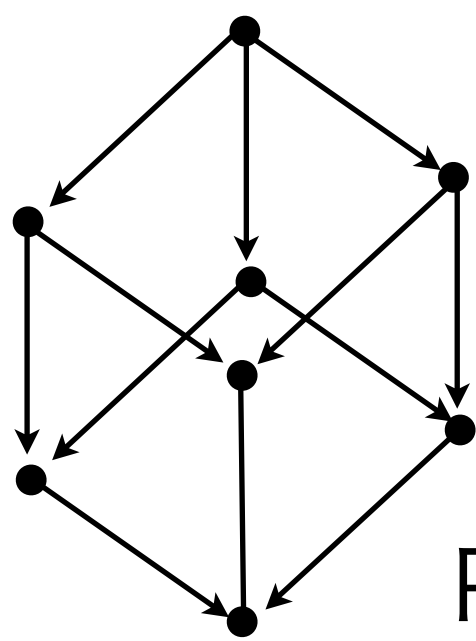
to each maximal chain x0 < x1 < ・・・< xk



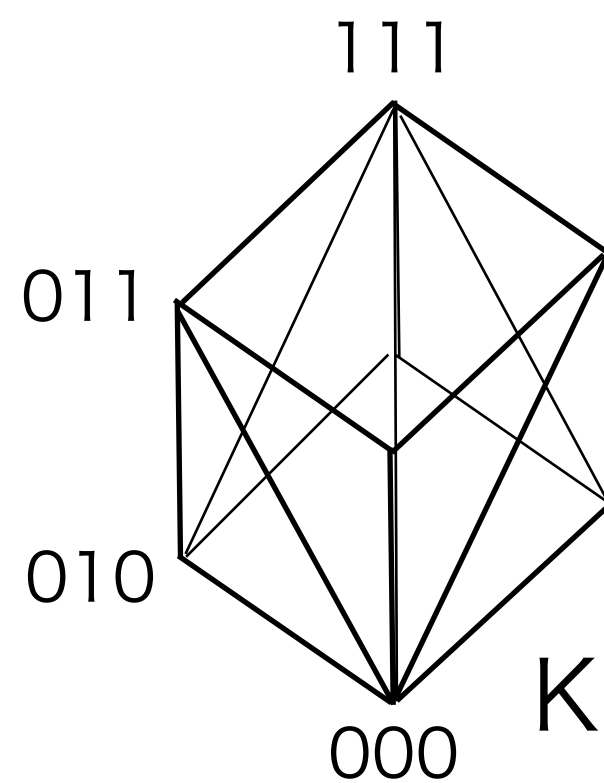
P



K(P) ~ folder



P



K(P) ~ 3-cube

BM are interested with P such that K(P) is CAT(0)

Thm [Chalopin, Chepoi, H, Osajda 14; conjecture of BM10]
P: modular lattice → K(P) is CAT(0).

Om-graph G is a gluing of modular lattices.
K(G) is a gluing of K(P) for modular lattices P.

Conj [CCHO14]

G: om-graph → K(G) is CAT(0)

G: median graph

→ K(G) subdivides CAT(0) cube complex

G: frame → K(G) = folder complex

G: om-graph from Euclidean building Δ of type C

→ K(G) = the standard metrization of Δ

proved at August 2015

Conj [CCHO14]

G: om-graph → K(G) is CAT(0)

G: median graph

→ K(G) subdivides CAT(0) cube complex

G: frame → K(G) = folder complex

G: om-graph from Euclidean building Δ of type C

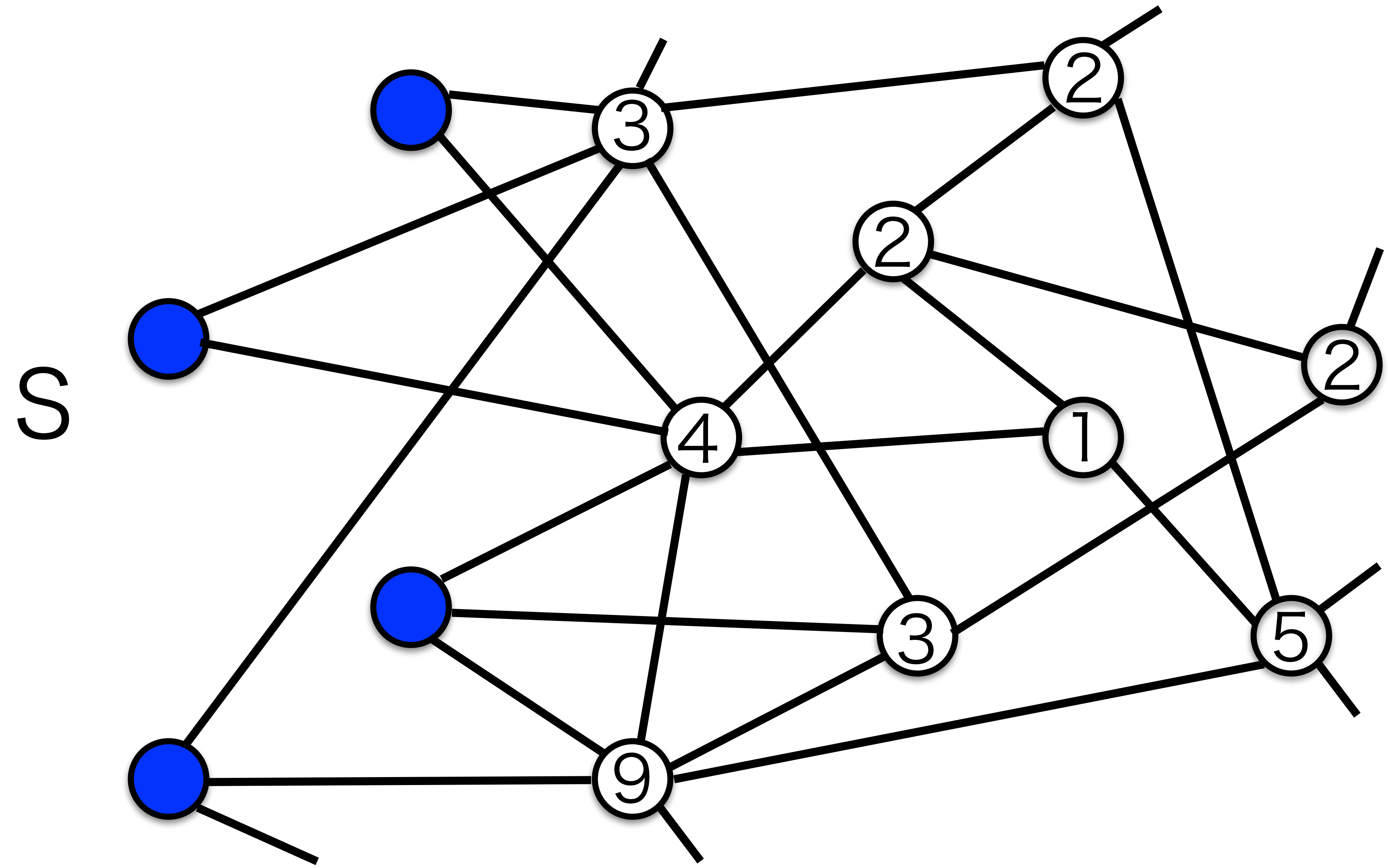
→ K(G) = the standard metrization of Δ

Concluding remarks

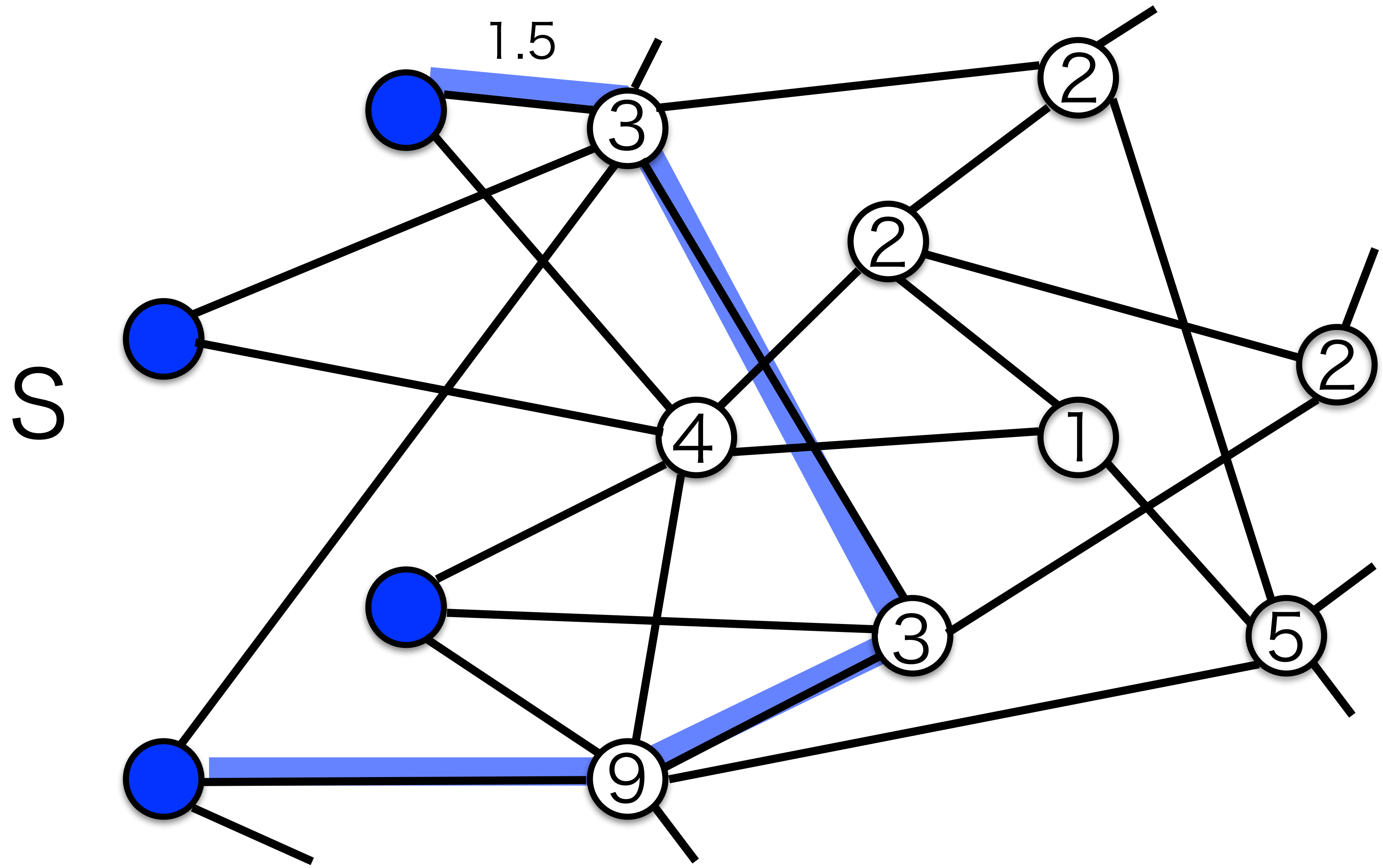
- Nonpositive curvature property $\overset{?}{\sim}$
  tractability in combinatorial optimization

- Convex optimization over CAT(0) space:
  - Phylogenetic distance in tree space [Owen, Bacak,⋯]
  - Dual of min-cost free multiflow problem
    = convex optimization over product of stars
    —> efficient combinatorial algorithm [H.14]
  - Dual of max. node-cap. free multiflow problem
    ⋯

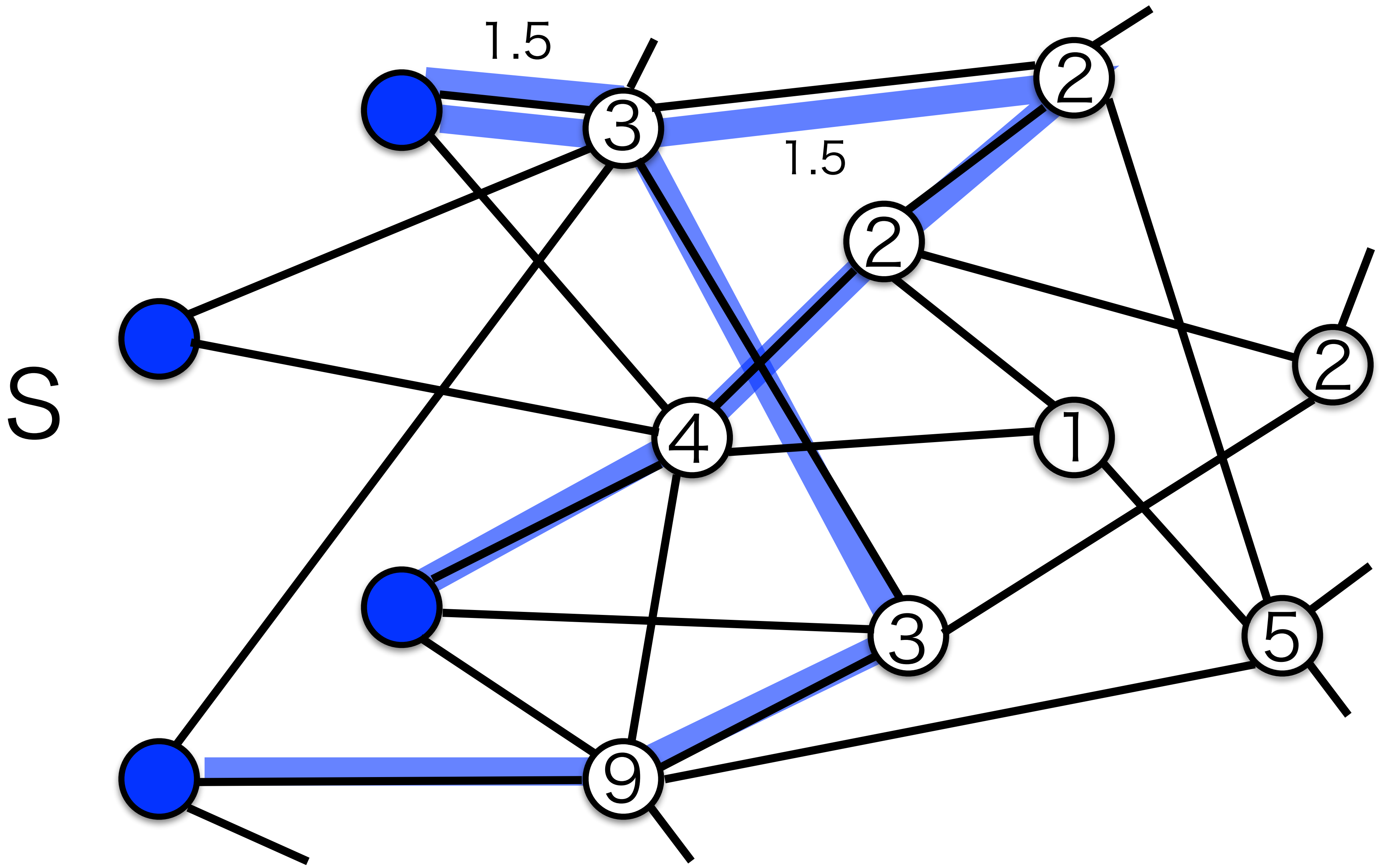
Maximum node-capacitated free multiflow problem



Maximum node-capacitated free multiflow problem



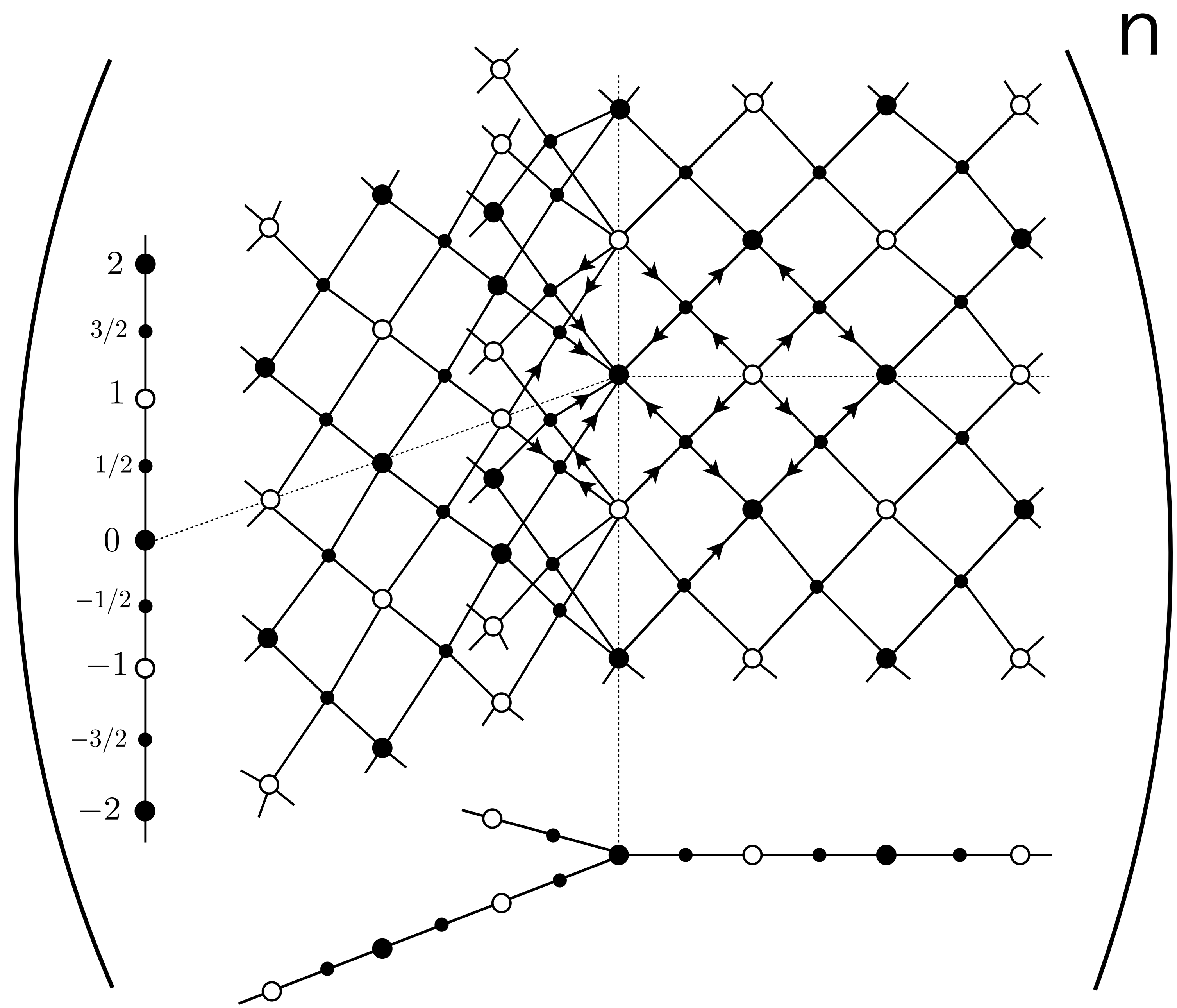
Maximum node-capacitated free multiflow problem



Max total flow value

= Min   g(p)

over   p ∈



—> discrete convex optimization on Euclidean building

Max total flow value

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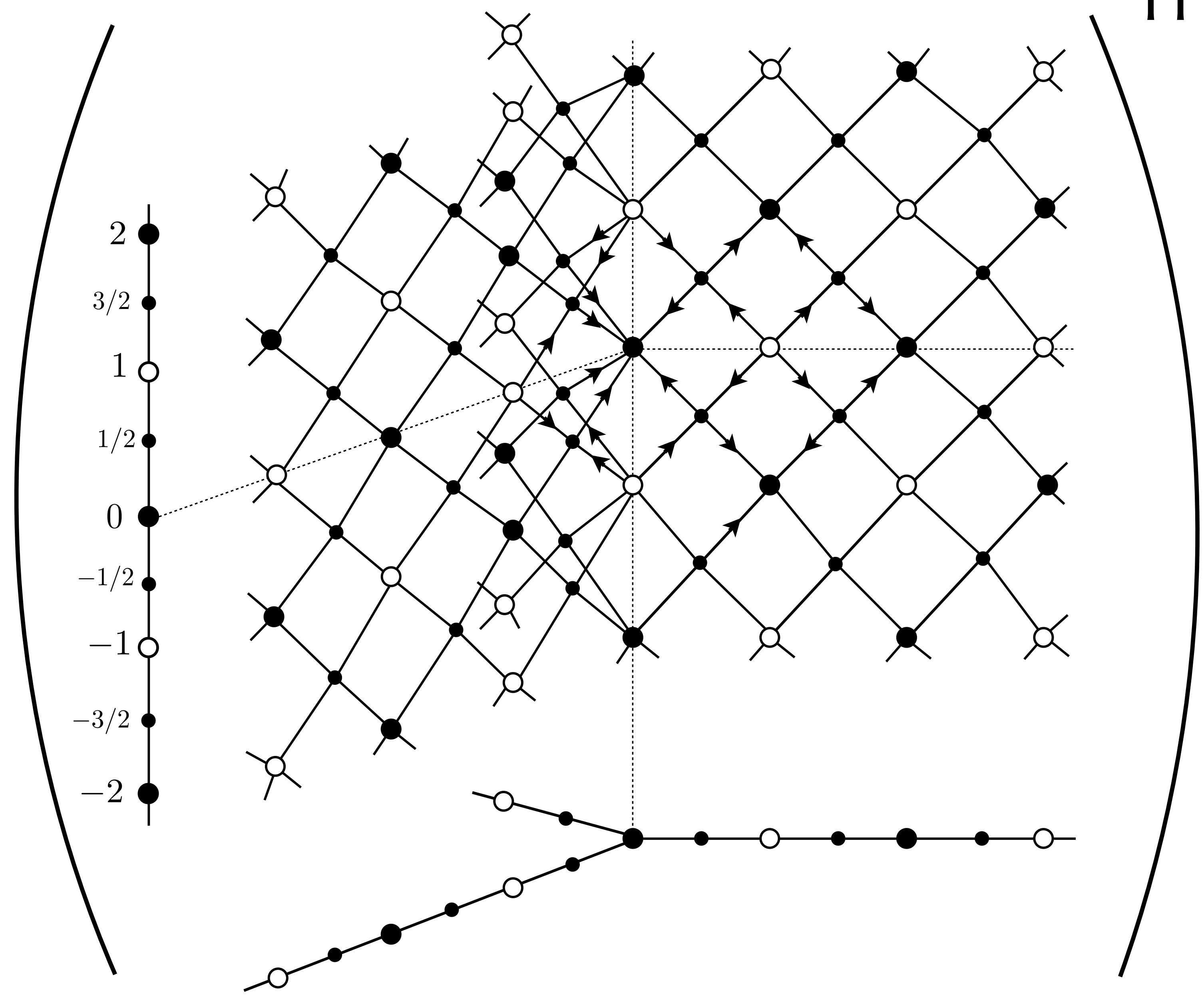

n

—>

the first combinatorial strongly polynomial time algo.

[H.15]

Thank you for your attention !

My papers are available at

http://www.misojiro.t.u-tokyo.ac.jp/~hirai/