Weakly modular graphs and nonpositive curvature

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Joint work with J. Chalopin, V. Chepoi, and D. Osajda

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Contents

- Weakly modular graphs (Chepoi 89)
- Connections to nonpositively curved spaces
- Some results

J. Chalopin, V. Chepoi, H. Hirai, D. Osajda
Weakly modular graphs and nonpositive curvature
arXiv:1409.3892, 2014

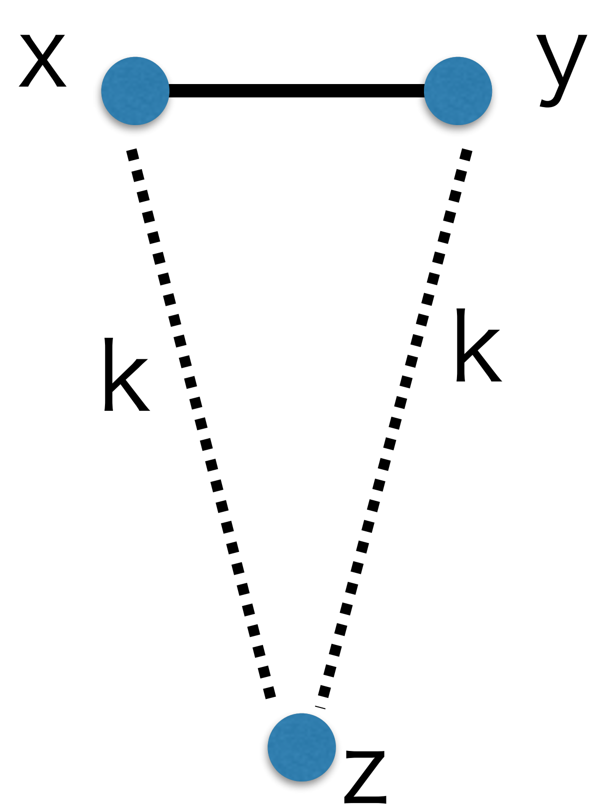
G is weakly modular (WM) $\Leftrightarrow$

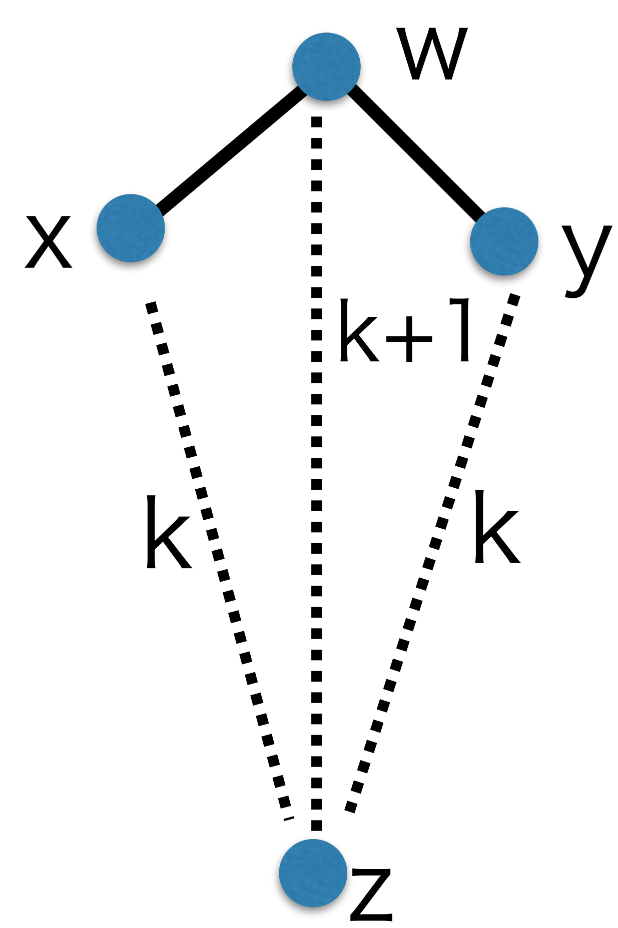
(TC) $\forall\, x,y,z : x\sim y,\ d(x,z) = d(y,z)$

$\Rightarrow \exists u: x \sim u \sim y,\ d(u,z) = d(x,z) - 1$

(QC) $\forall\, x,y,w,z : x\sim w\sim y,\ d(w,z)-1 = d(x,z) = d(y,z)$

$\Rightarrow \exists u: x \sim u \sim y,\ d(u,z) = d(x,z) - 1$





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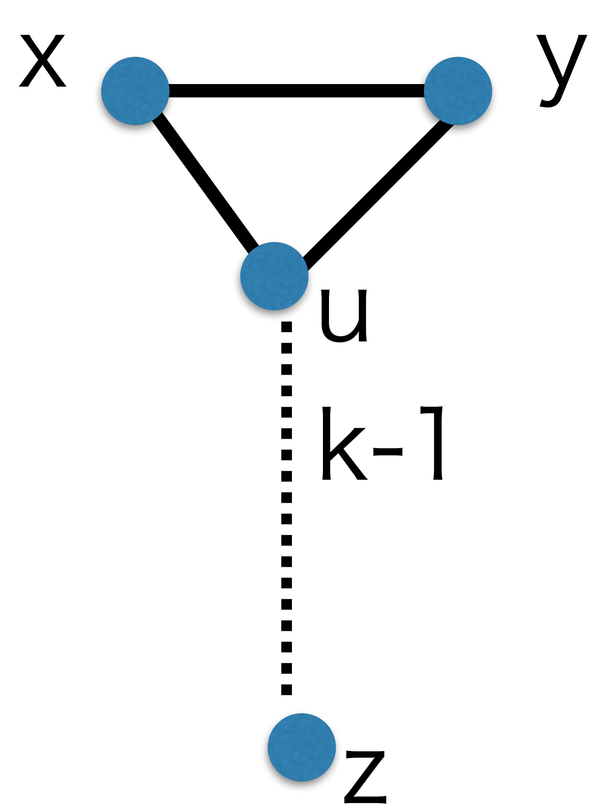
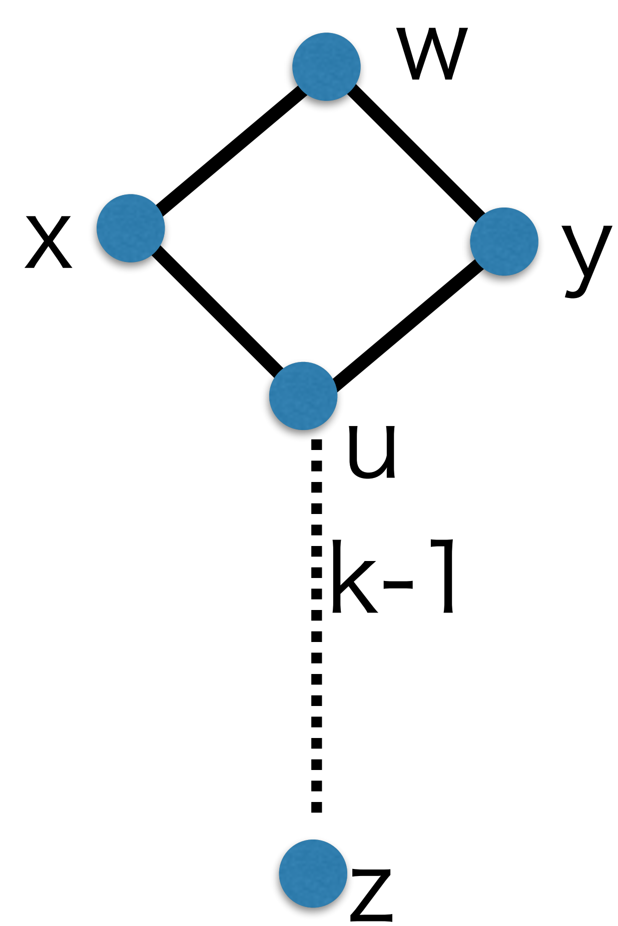
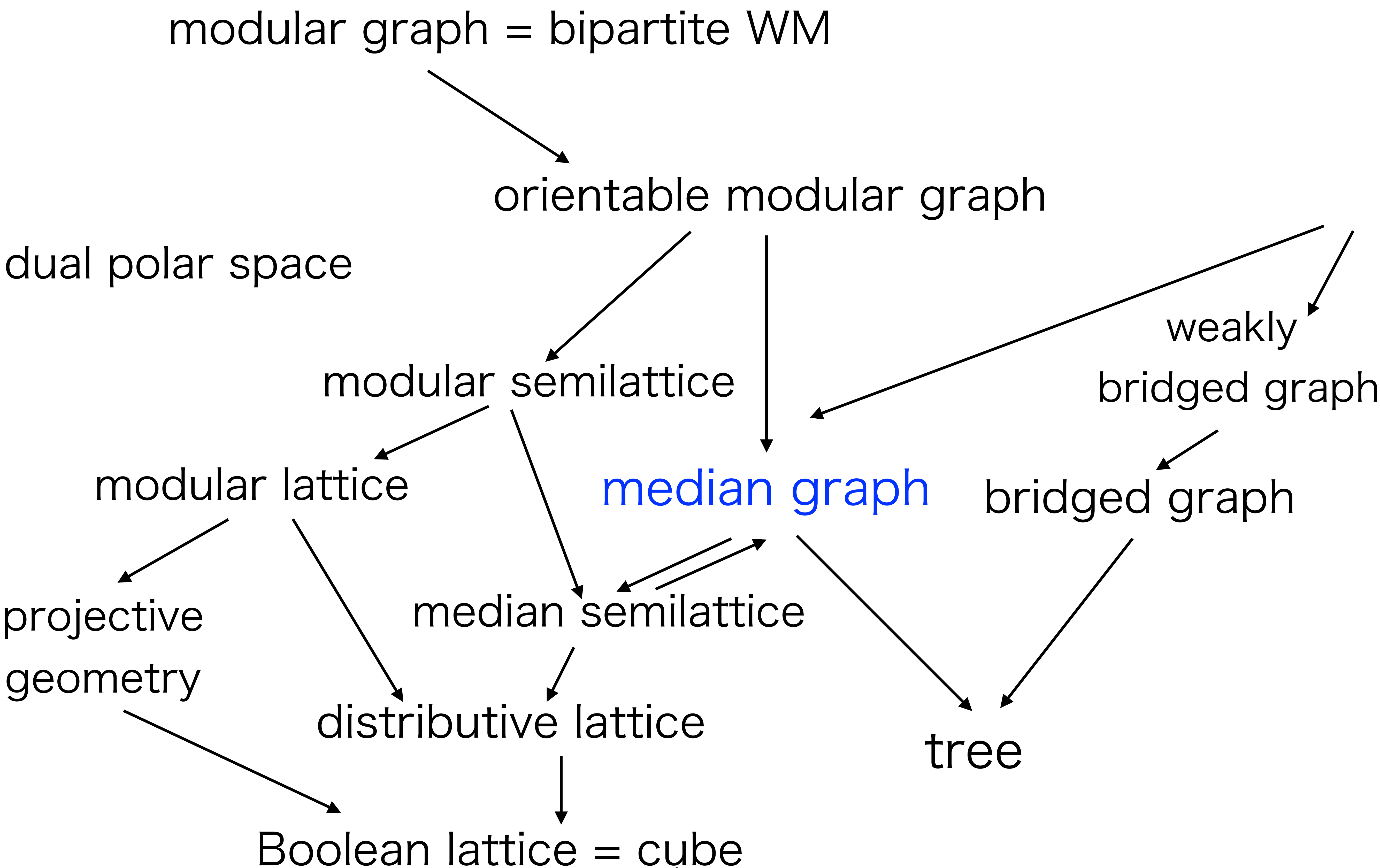




Classes of WM graph



- Some classes of WM graphs are naturally associated with "metrized complex" of nonpositively-curvature-like property

- median graph ~~ CAT(0) cube complex

  (Gromov 87)

- bridged graph ~~ systolic complex

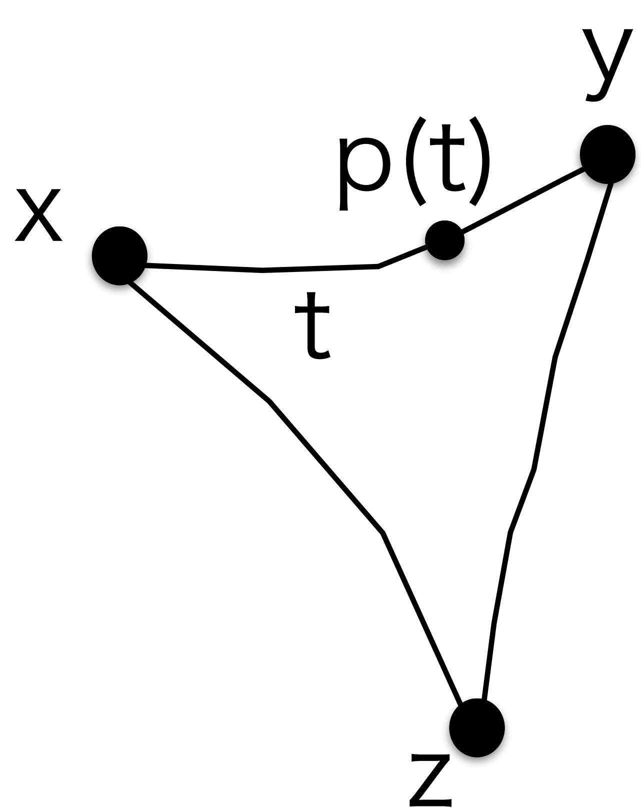
  (Haglund 03, Januszkiewicz-Swiantkowski 06)

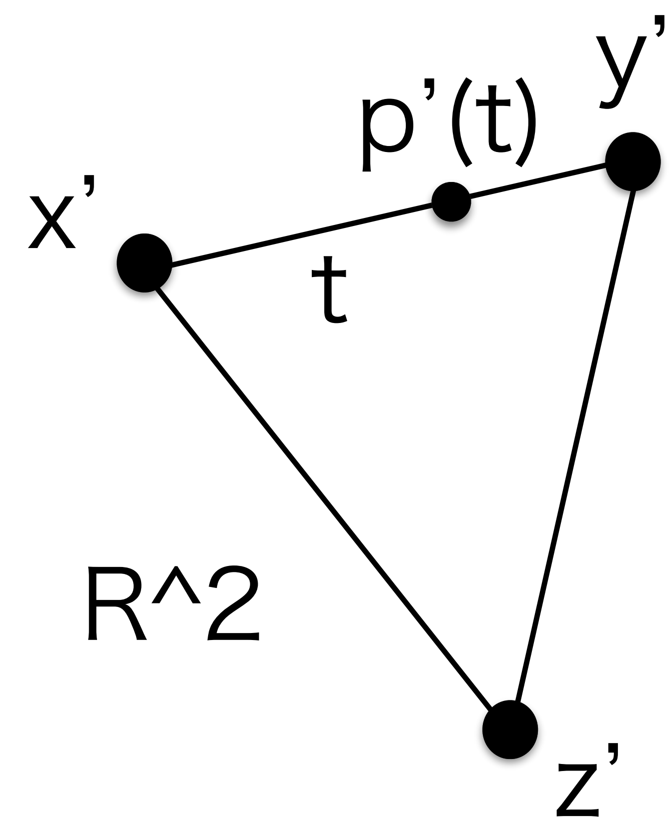
- modular lattice ~~> orthoscheme complex

  (Brady-McCammond 10)

CAT(0) space

~ geodesic metric space such that every geodesic triangle is “thin”





$d(y,z) = \| y' - z' \|$

$d(x,y) = \| x' - y' \|$

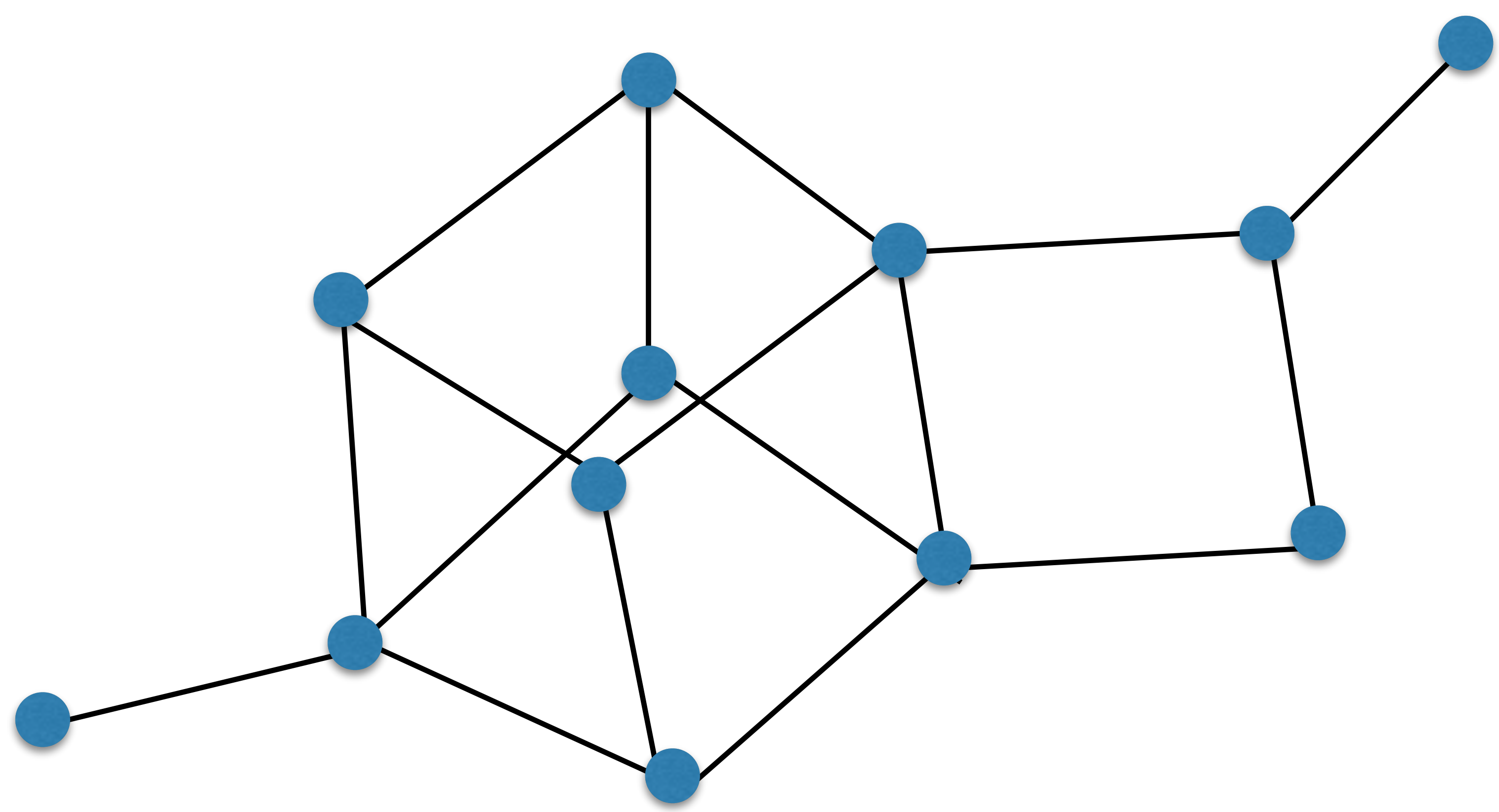
$d(z,x) = \| z' - x' \|$

$$d(p(t),z) \leqq \| p'(t) - z' \|$$

Median graph

⇔ every triple of vertices admits a unique median

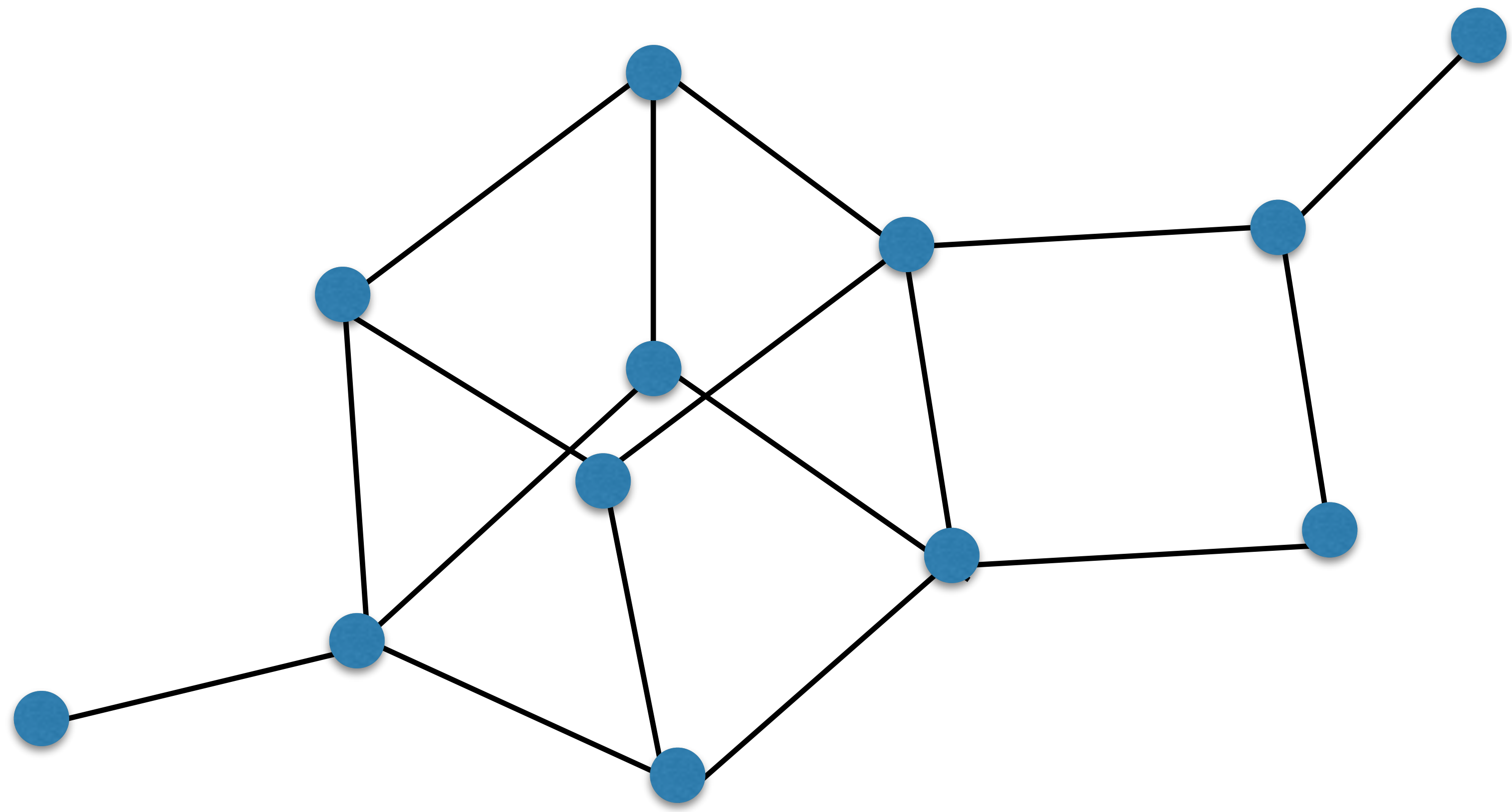
⇔ bipartite WM without K2,3



Median graph is obtained by “gluing” cubes

Median complex

:= cube complex obtained by filling "cube" to each cube-subgraph of median graph



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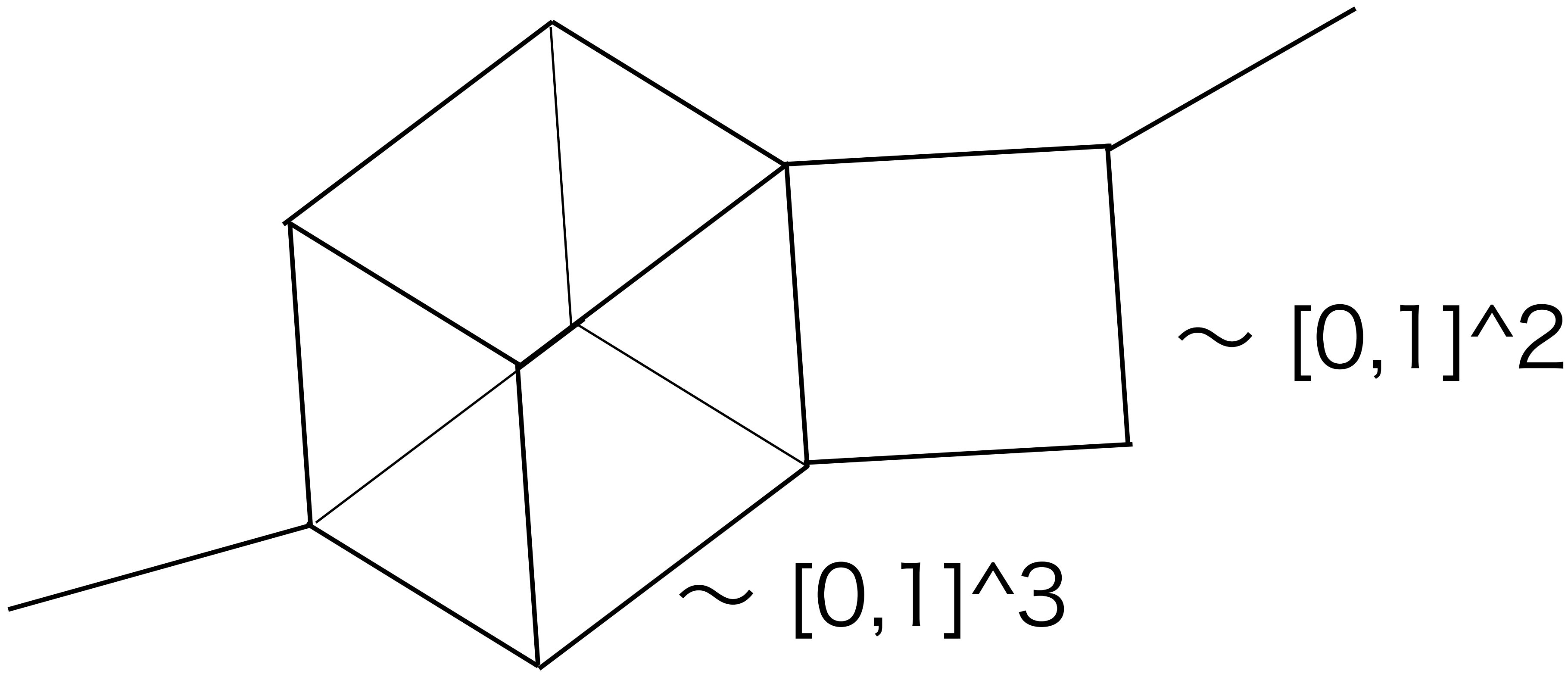

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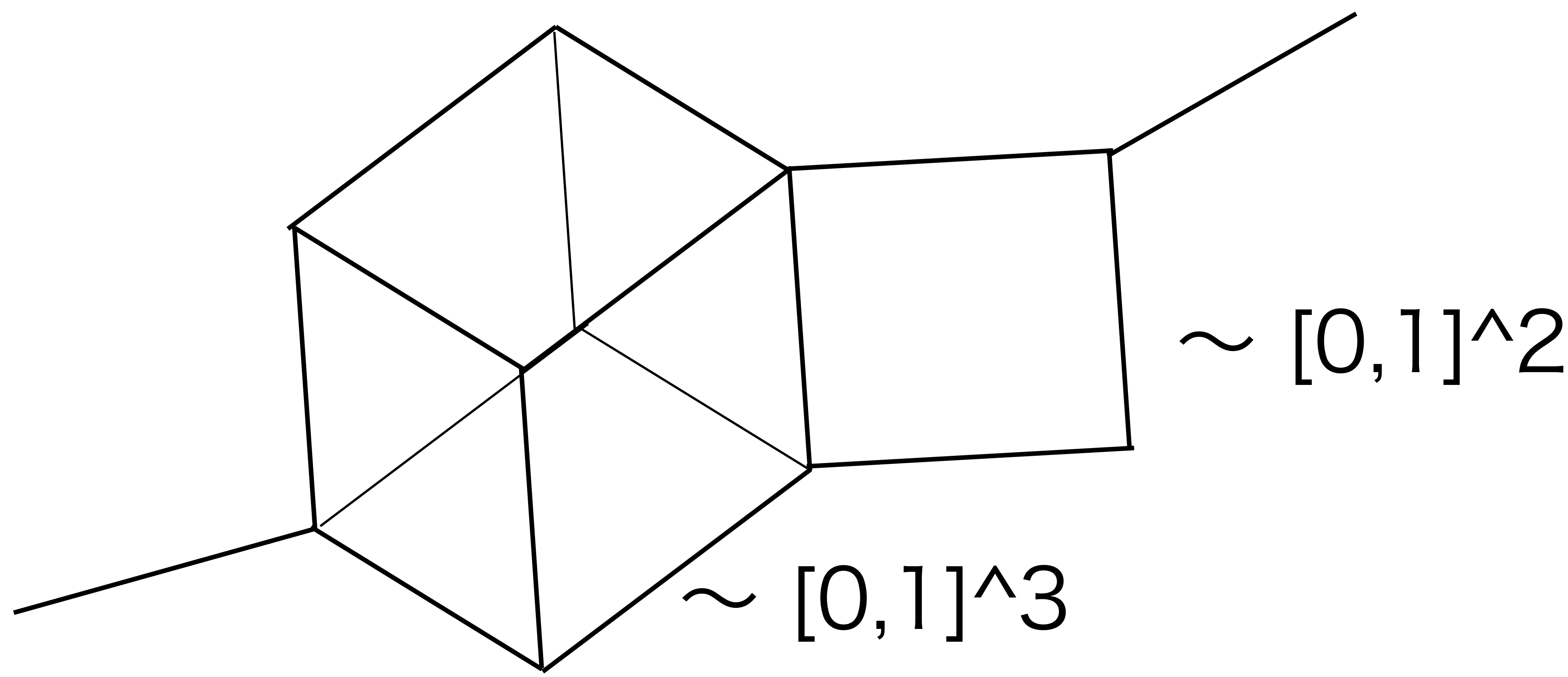


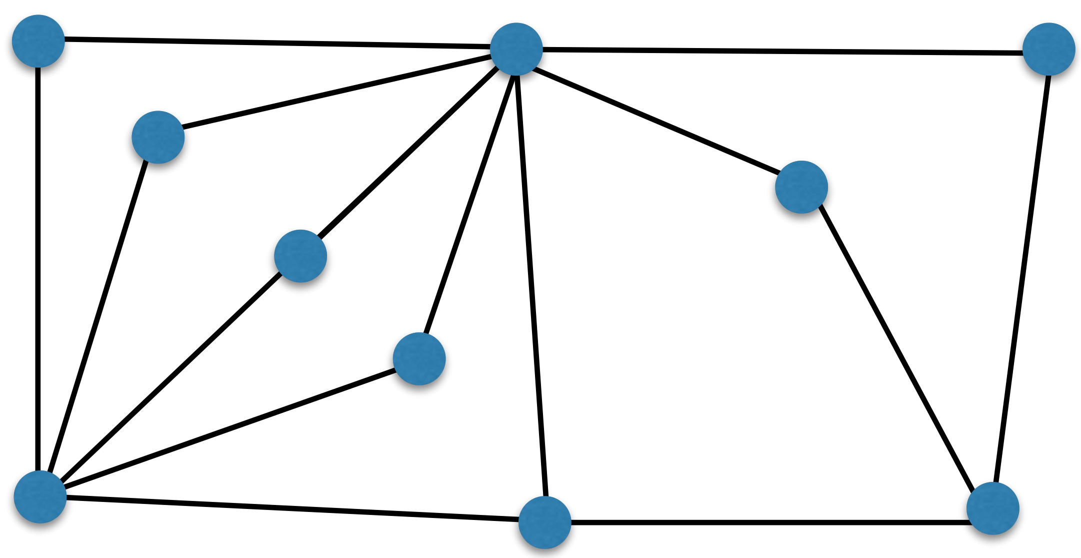
Thm (Chepoi, 2000)

Median complex ≡ CAT(0) cube complex

c.f. Gromov’s characterization of CAT(0) cube complex

Folder complex

:= B2-complex obtained by filling “folder” to each K2,m subgraph of bipartite WM without K3,3 and K3,3^-



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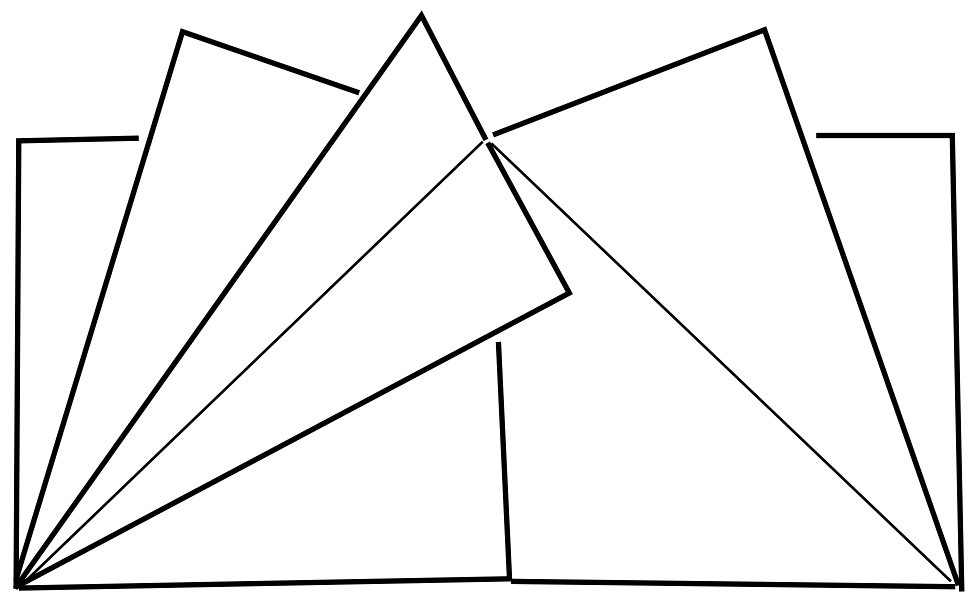

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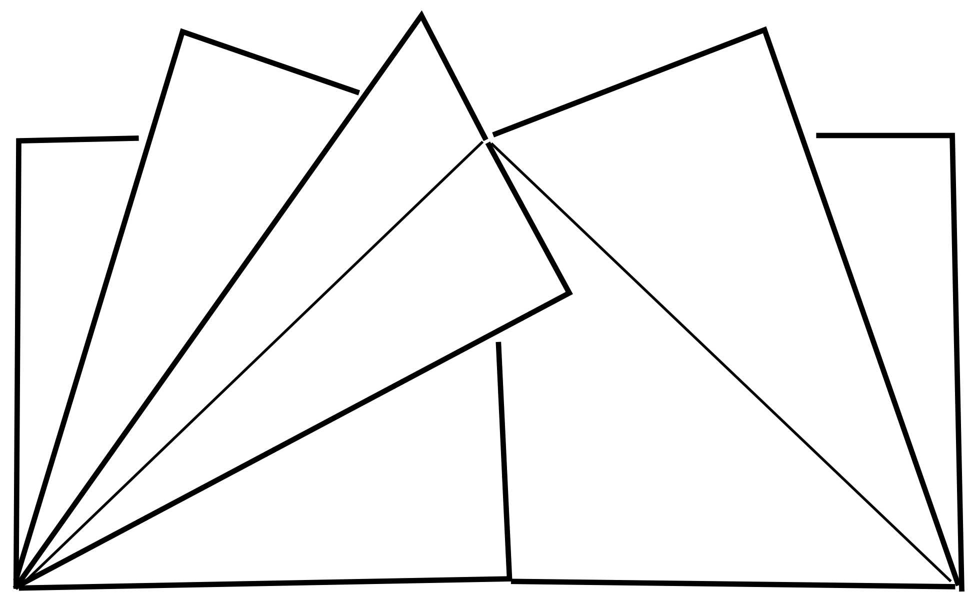


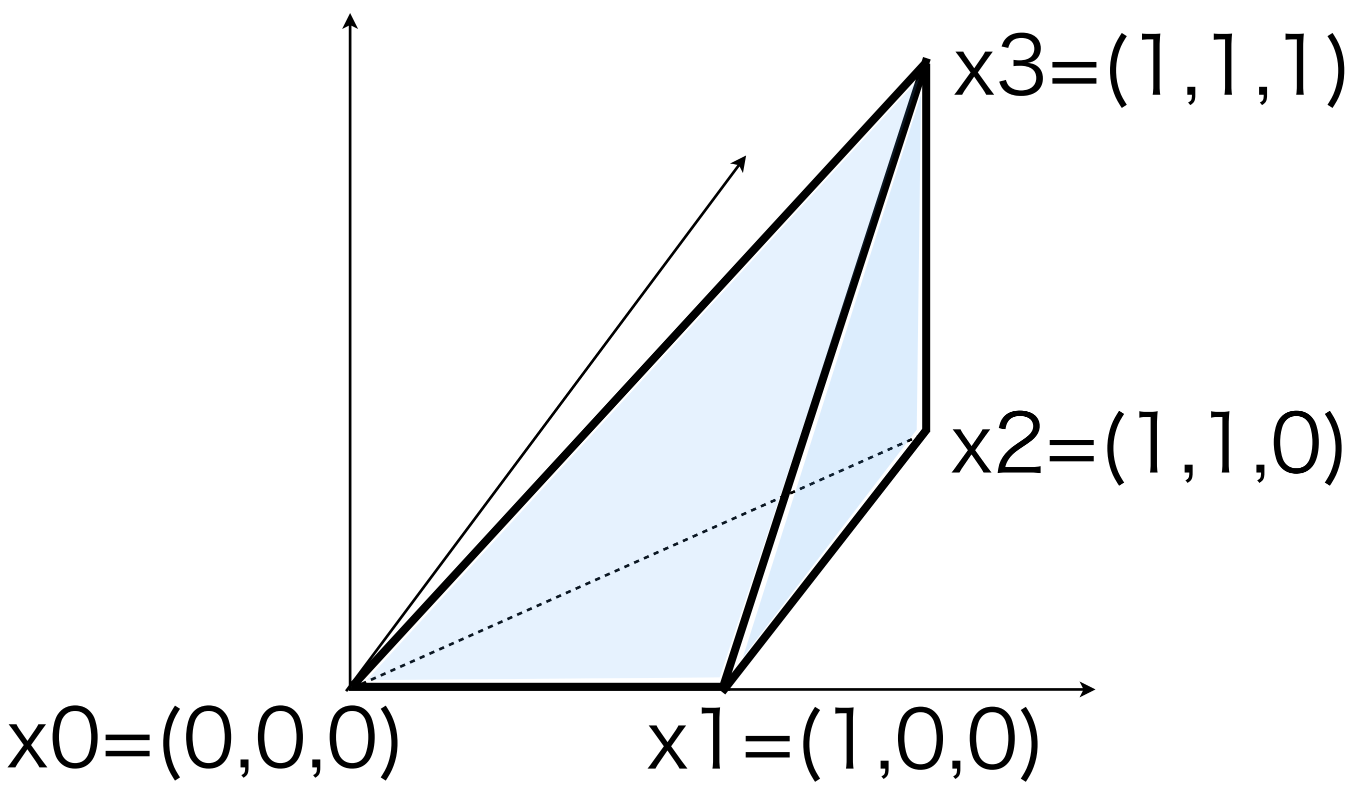
Thm (Chepoi 2000)

Folder complex ≡ CAT(0) B2-complex

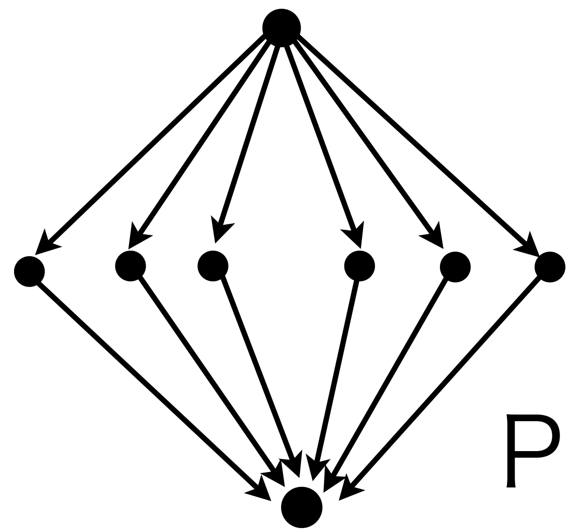
Orthoscheme complex (Brady-McCammond10)

P: graded poset

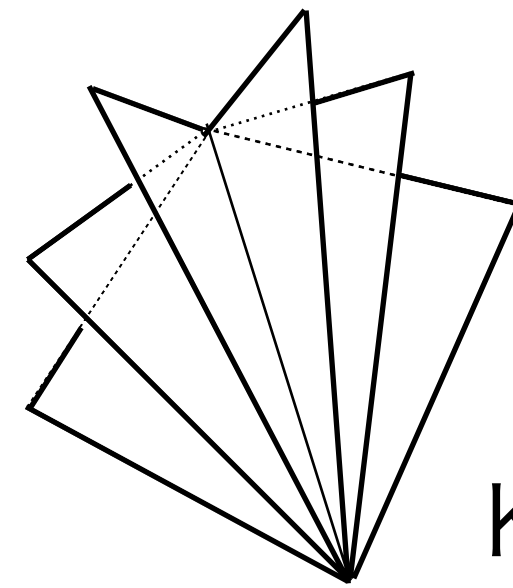
K(P): = complex obtained by filling



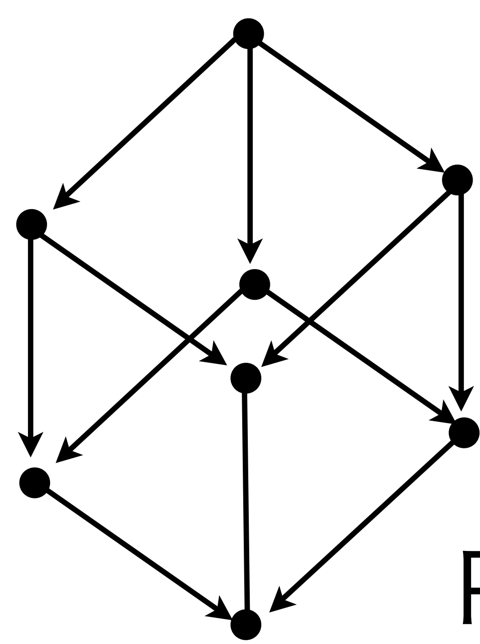
to each maximal chain x0 < x1 < ・・・< xk, k=1,2,3..



P



K(P) ~ folder



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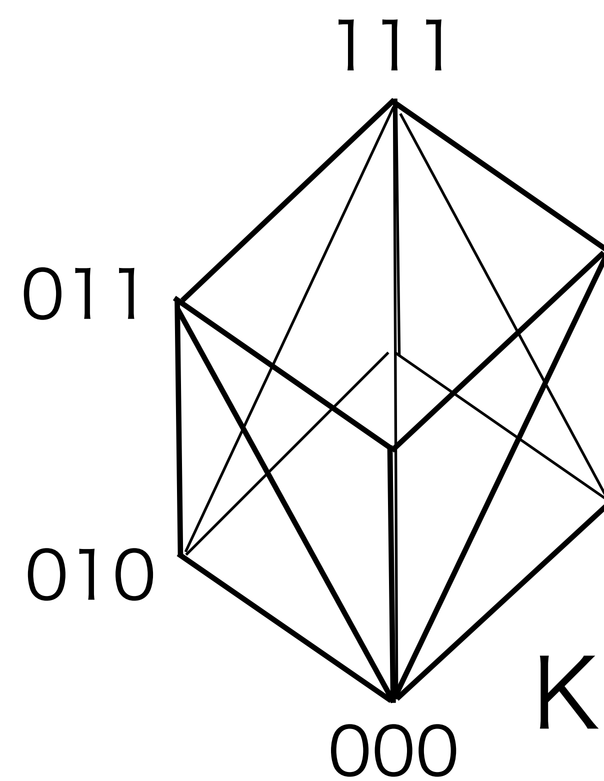

K(P) ~ [0,1]^3

What are posets P for which K(P) is CAT(0) ?

Conjecture (Brady-McCammond 10)
K(P) is CAT(0) for modular lattice P.

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Theorem (Haettel, Kielak, and Schwer 13)
K(P) is CAT(0) for “complemented” modular lattice P.
~ lattice of subspaces of vector space

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~ lattice of subspaces of vector space

Theorem (CCHO14)
K(P) is CAT(0) for modular lattice P.

Idea for proof

Obs: If P is distributive, then K(P) = order polytope.

Thm [Birkhoff-Dedekind]
For two chains in modular lattice,
there is a distributive sublattice containing them.

plus standard proof technique of
"spherical building is CAT(1)"

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K(P) is CAT(0) for modular semilattice P.

Modular semilattice
= semilattice whose covering graph is bipartite WM

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= semilattice whose covering graph is bipartite WM

Median semilattice
= semilattice whose covering graph is median graph

Theorem [CCHO14]
K(P) is CAT(0) for median semilattice P.

$\leftarrow$ Gluing construction (Reshetnyak's gluing theorem)

We introduced a new class of WM graph,

SWM graph

:= WM without K_4^- and isometric $K_{3,3}^-$

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SWM graph

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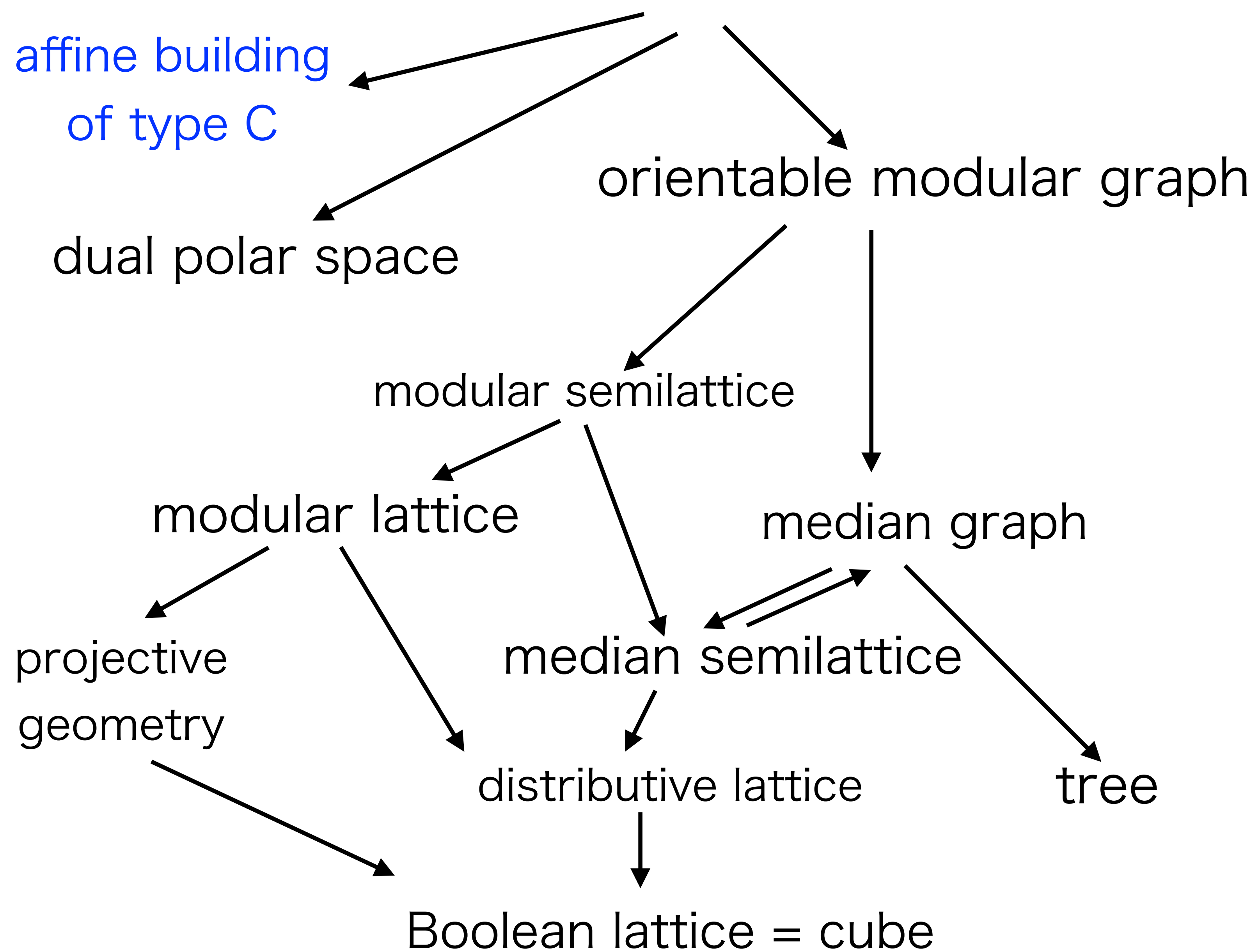

Metrized complex K(G) from SWM-graph G

B(G):= the set of all Boolean-gated sets of G

X: Boolean-gated $\Leftrightarrow$

$x,y \in X$, $x \sim u \sim y \Rightarrow u \in X$,

$x,y \in X$: $d(x,y) = 2 \Rightarrow \exists$ 4-cycle $\ni x,y$

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→ Boolean-gated set induces dual polar space

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$\rightarrow$ Boolean-gated set induces dual polar space

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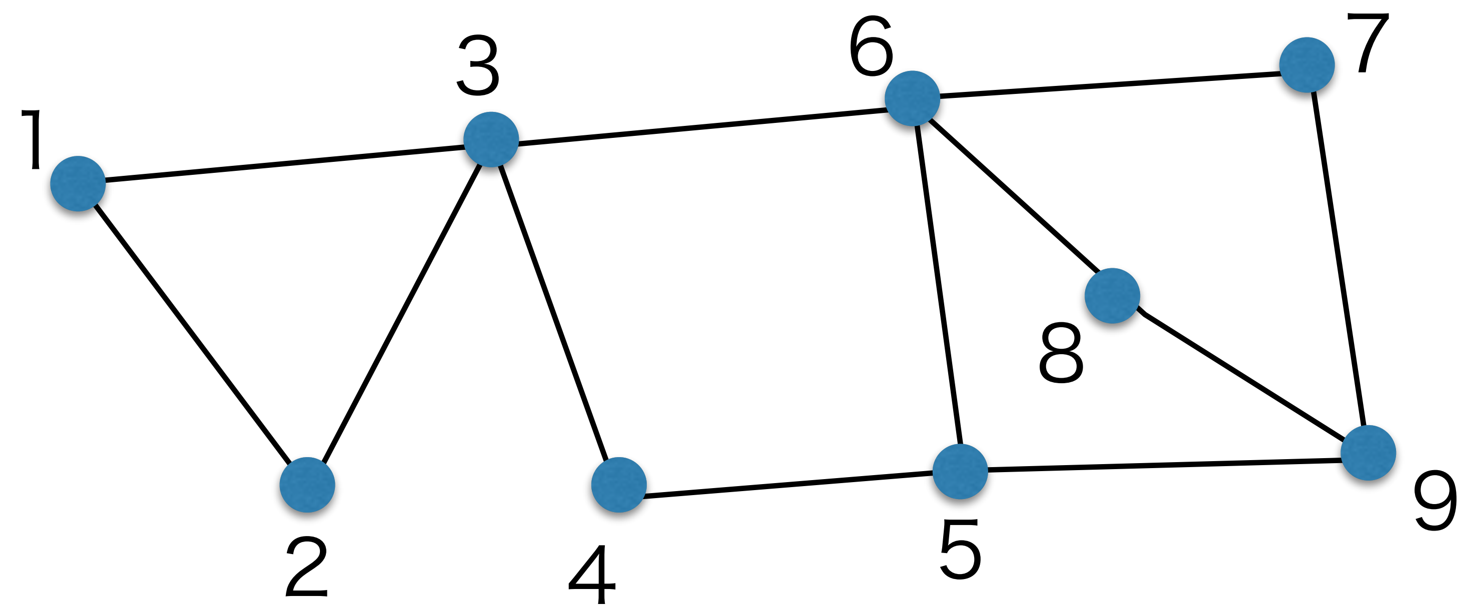
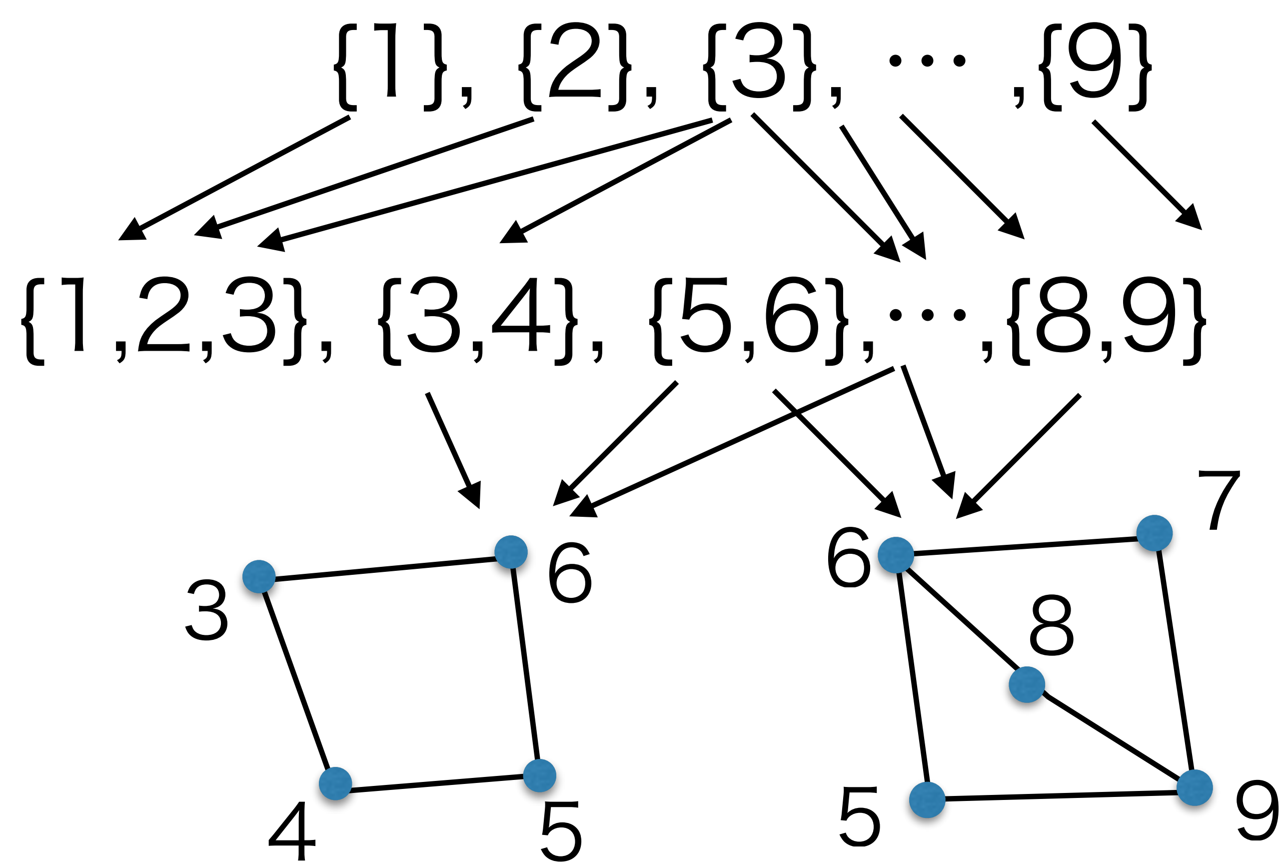
K(G):= orthoscheme complex of B(G)



G

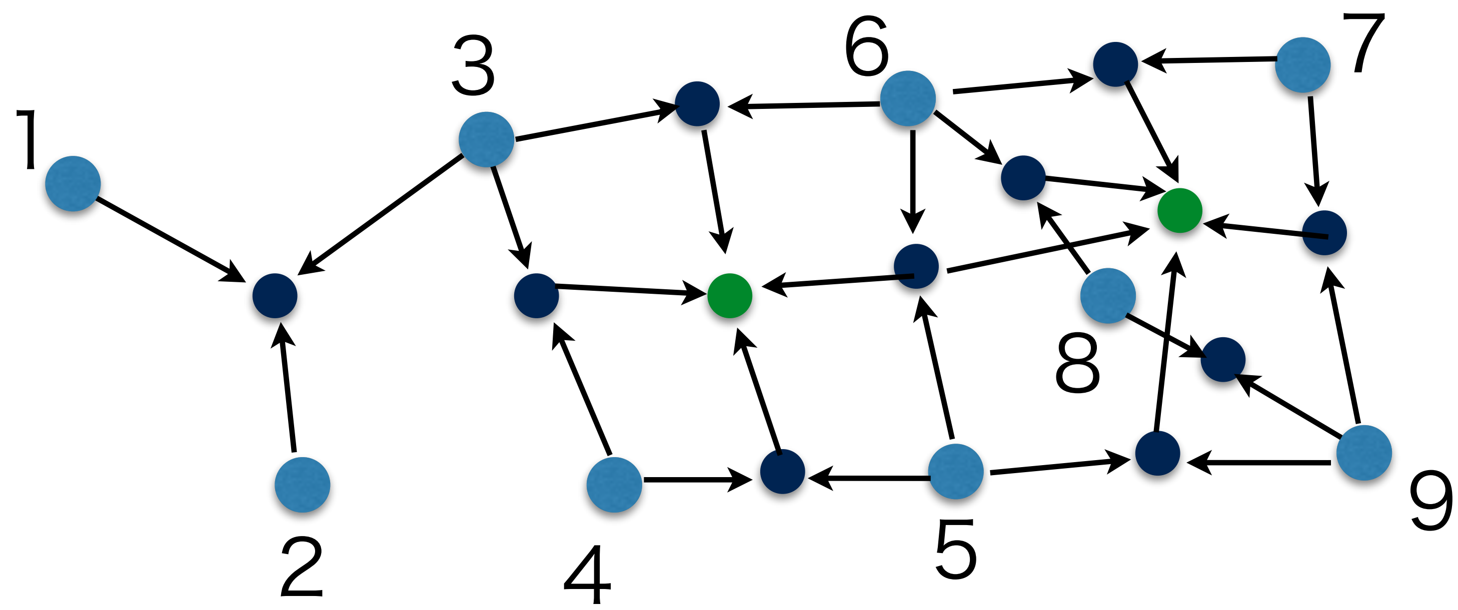
B(G)



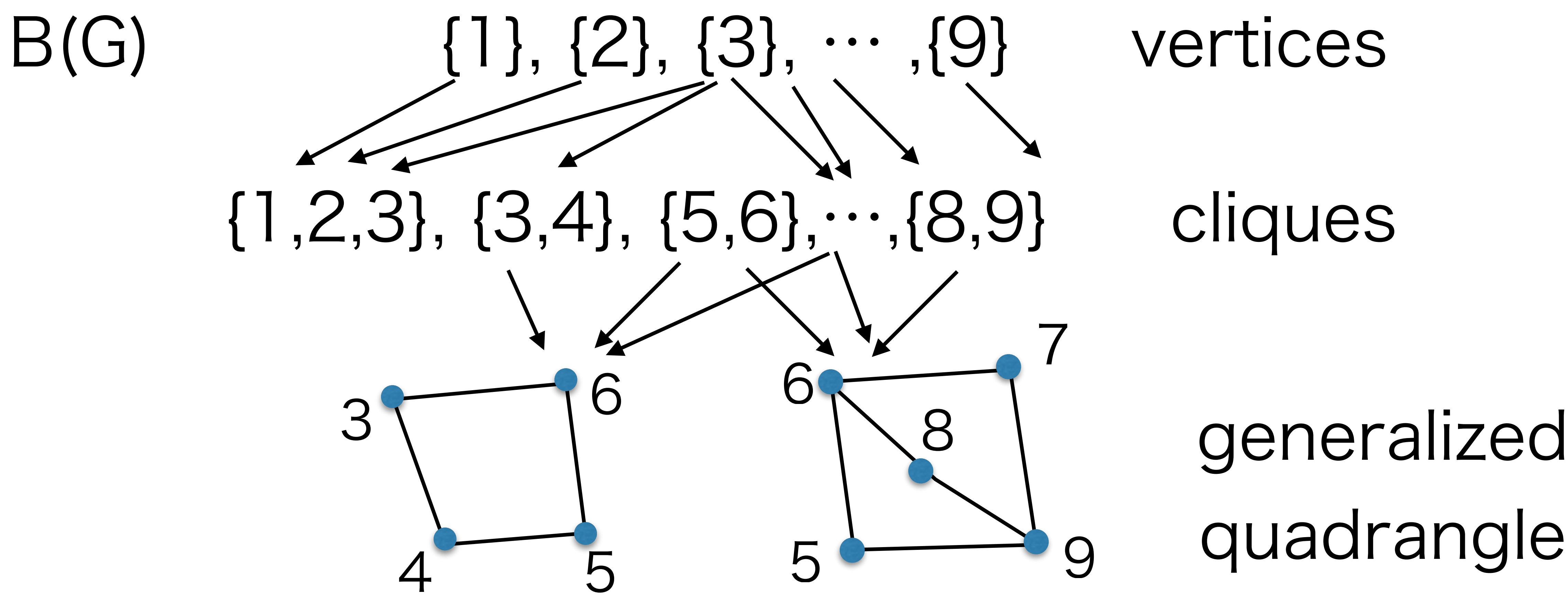
vertices

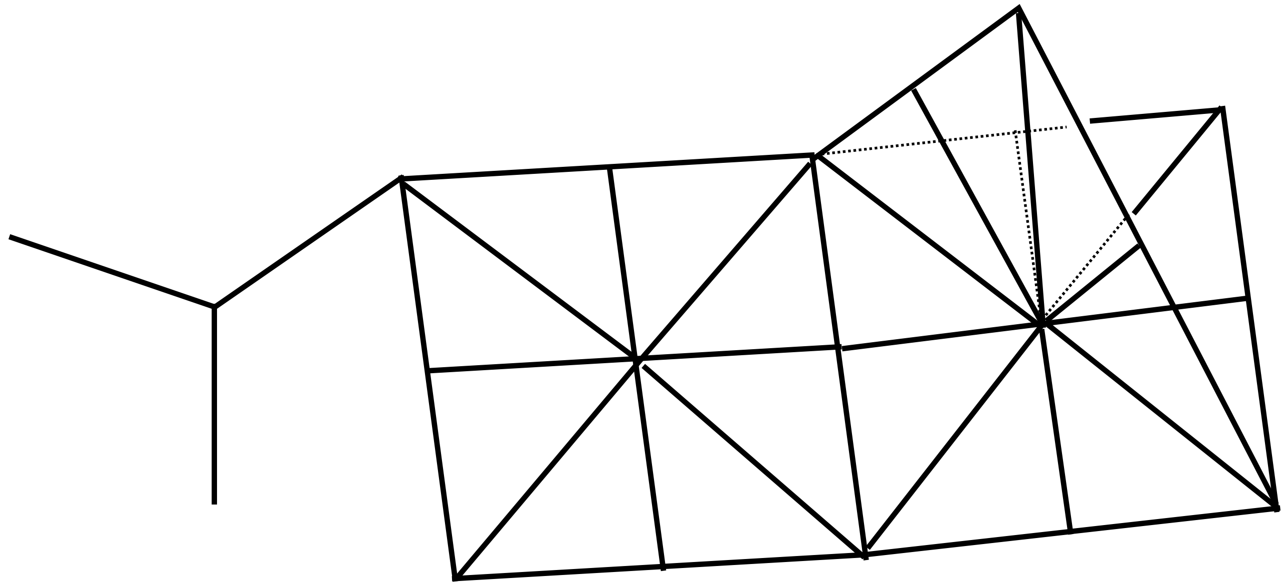
cliques

generalized quadrangle


1
2
3
4
5
6
7
8
9

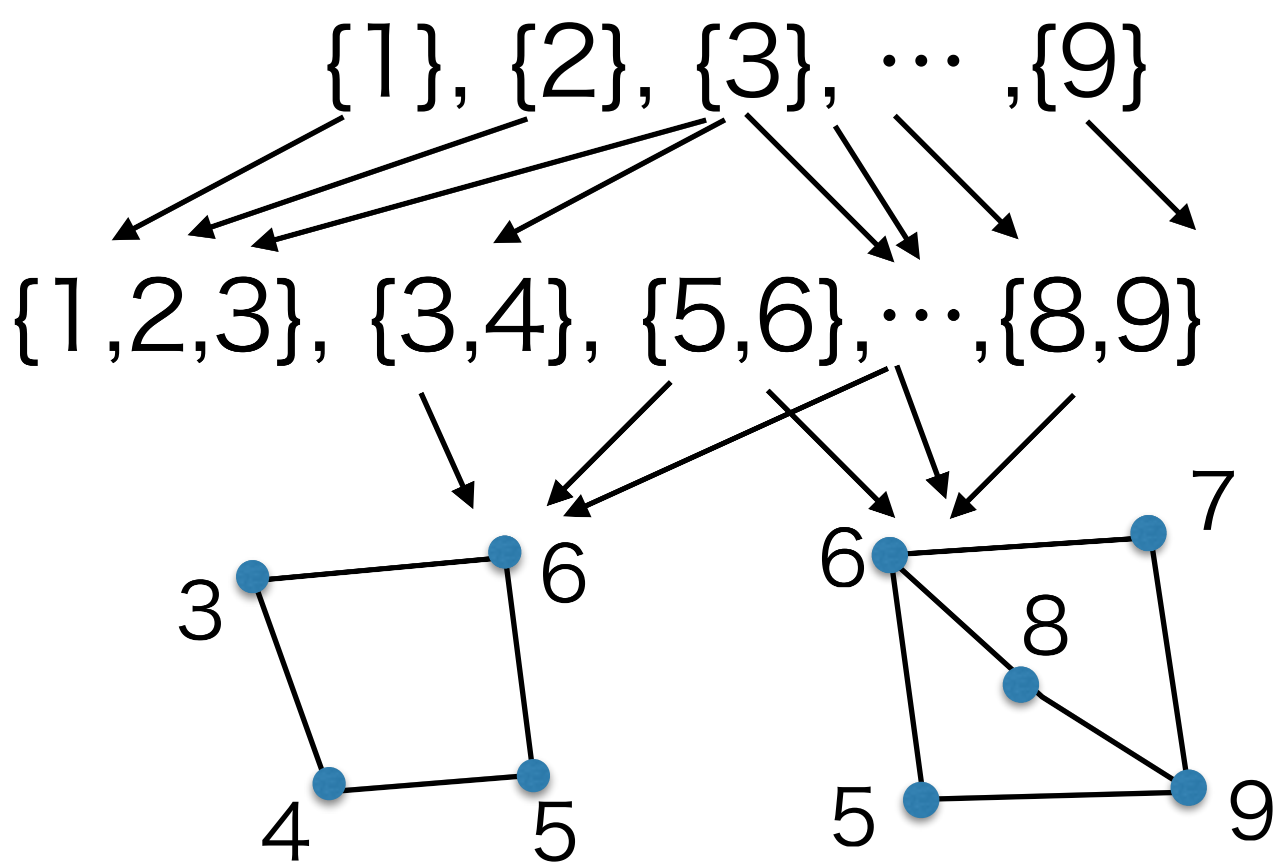
B(G)


B(G)
{1}, {2}, {3}, ⋯ ,{9}
vertices
{1,2,3}, {3,4}, {5,6},⋯,{8,9}
cliques
3
6
4
5
6
7
8
5
9
generalized
quadrangle



K(G)

B(G)



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G: median graph $\rightarrow$ B(G): set of cube-subgraphs

$\rightarrow$ K(G) subdivides median complex

G: median graph → B(G): set of cube-subgraphs

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G: bipartite WM without K3,3 and K3,3^-

→ B(G): set of maximal K2,m subgraphs

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G: SWM from affine building Δ of type C

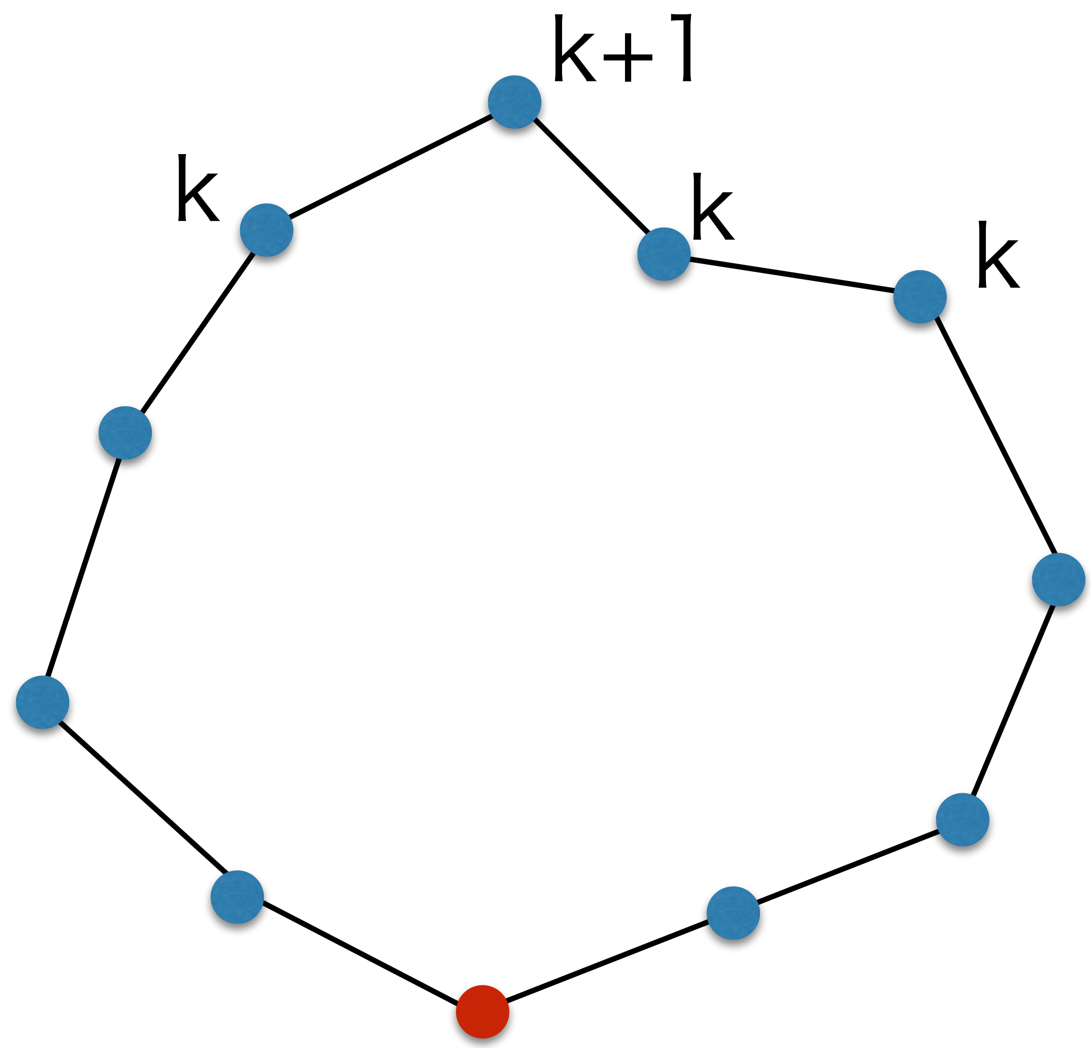
→ K(G) = the standard metrication of Δ

Conjecture (CCHO14)
K(G) is CAT(0) for SWM-graph G.

Some Topological Graph Theory result

Lemma (CCHO14)

Triangle-Square complex of WM-graph is simply-connected



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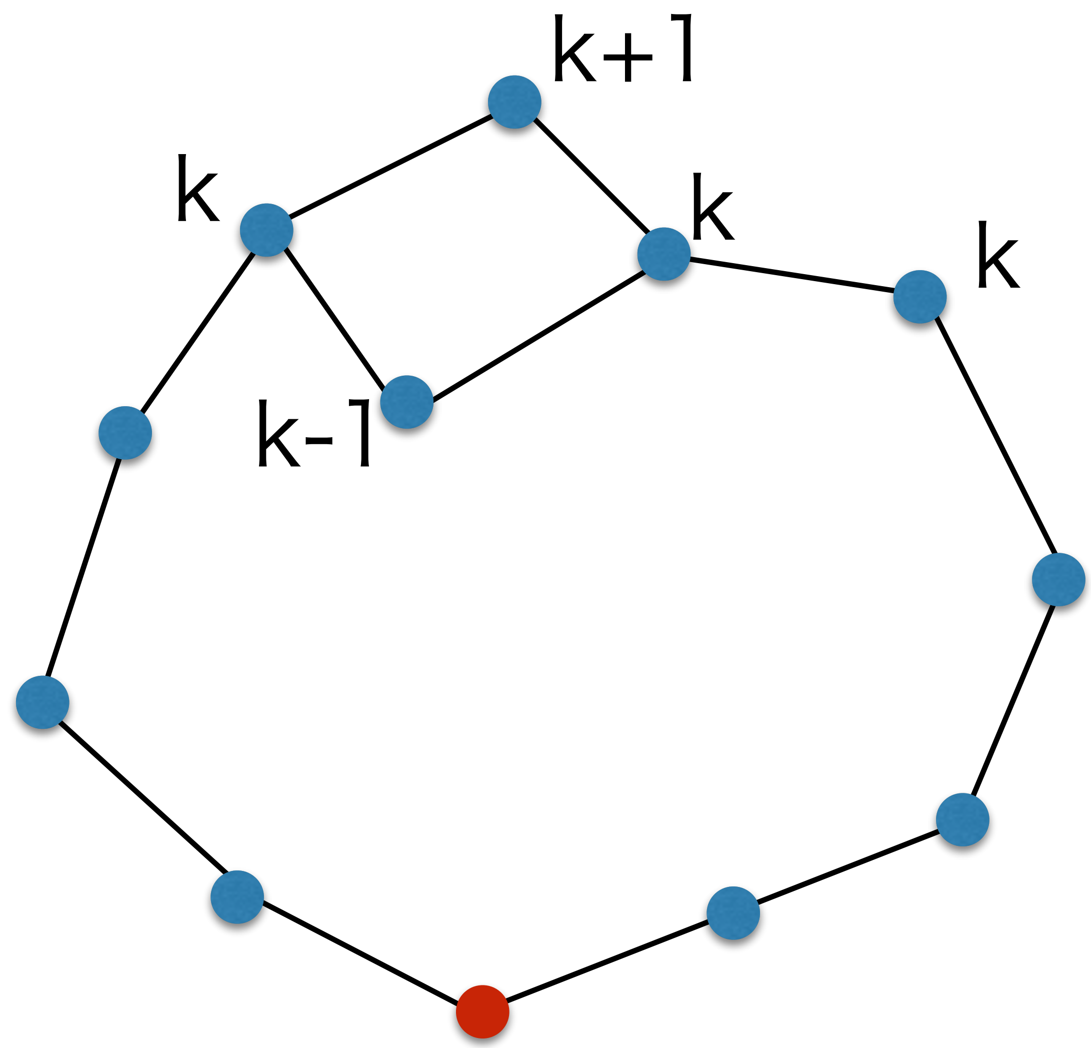

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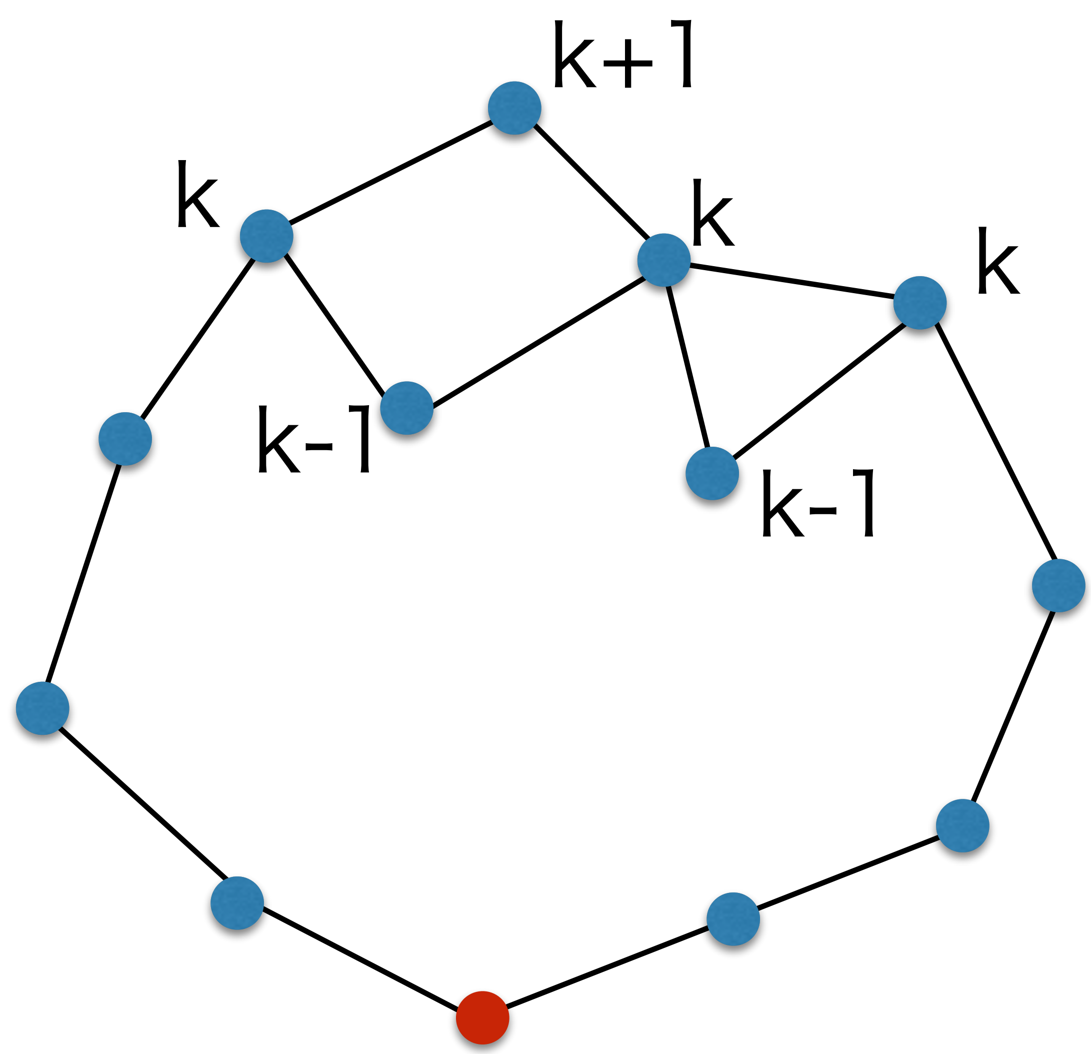

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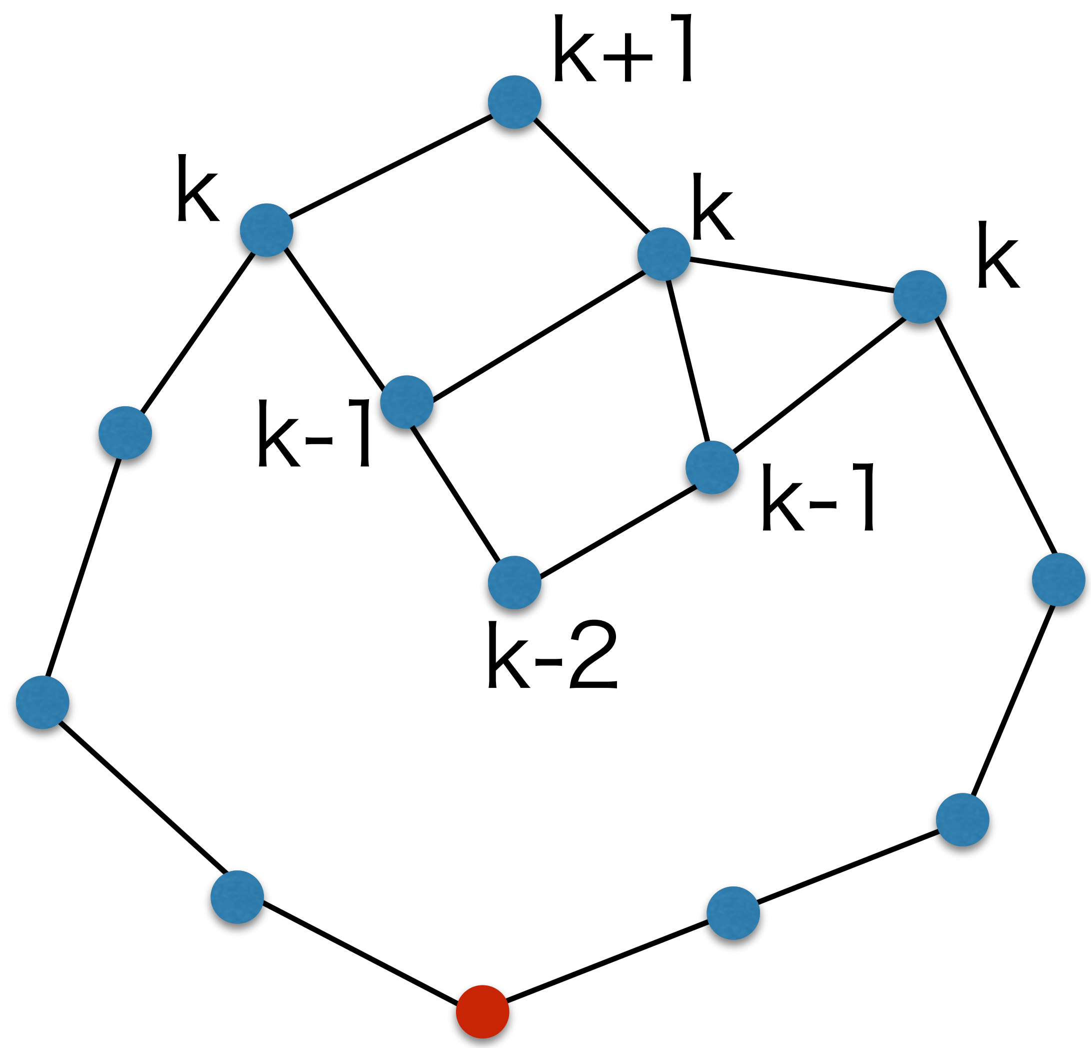

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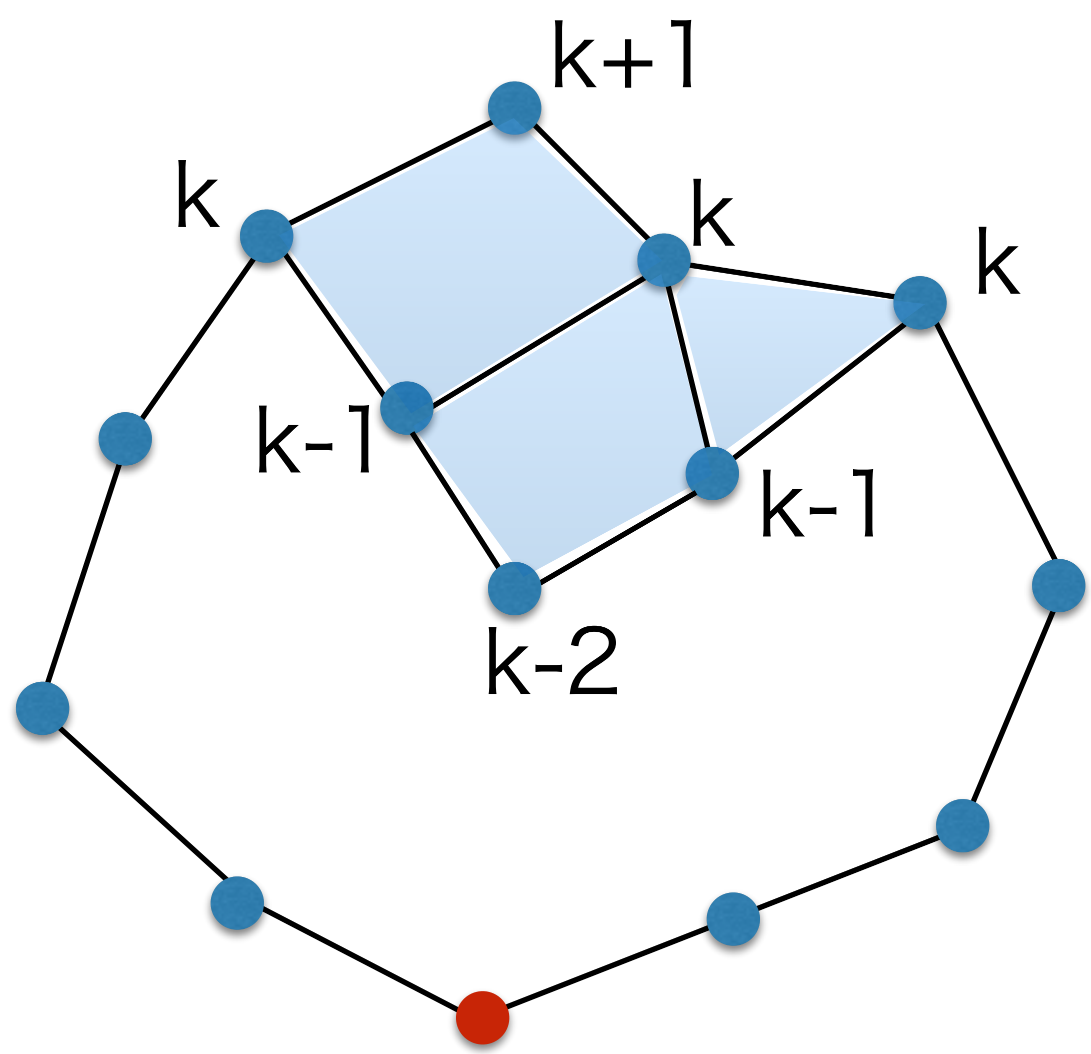

A Local-to-Grobal characterization of WM-graph (analogue of Cartan-Hadamard theorem ?)

Theorem (CCHO14)
If G is locally-WM and TS-complex of G is simply-connected, then G is WM.

Locally-WM: (TC) & (QC) with $d(x,z) = d(y,z) = 2$

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Locally-WM: (TC) & (QC) with $d(x,z) = d(y,z)= 2$

Theorem (CCHO14)
The 1-skeleton of the universal cover of TS-complex of locally-WM-graph is WM.

Thank you for your attention !