Metric Graph Theory
in
Combinatorial Optimization

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Contents

I will explain some results in combinatorial optimization
closely related to Metric Graph Theory

- Tight spans (injective hulls) in multicommodity flow
- Median and modular graphs in multifacility locations
- Discrete convex analysis beyond $\mathbb{Z}^n$
(on orientable modular graphs)

$\fallingdotseq$ Introduction of my research from 2006

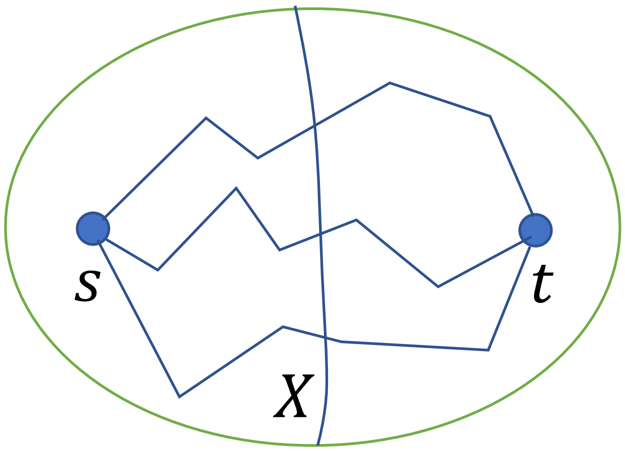
Median y of of vertices $x_1, x_2, x_3 \Leftrightarrow d(x_i, x_j) = d(x_i, y) + d(y, x_j)$

Modular graph $\Leftrightarrow$ every triple of vertices has a median

Median graph $\Leftrightarrow$ every triple of vertices has a unique median

Network flow --- Max-Flow Min-Cut Theorem

$G = (V, E)$: undirected
$c: E \to \mathbb{Z}_+$
$s, t \in V$: terminals



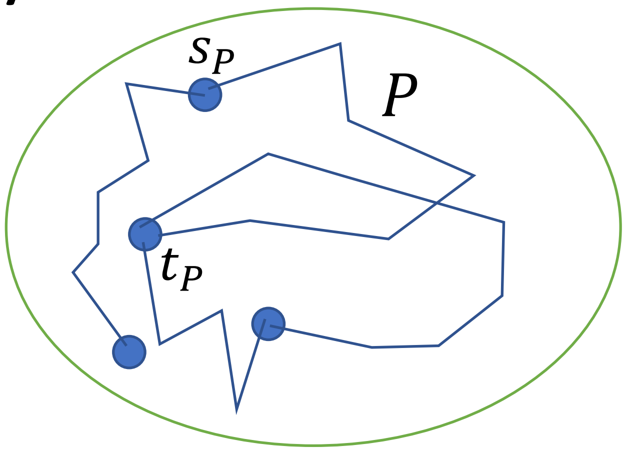
Flow $f: \{(s,t)\text{paths}\} \to \mathbb{Q}_+$ s.t. $\sum_{P: e \in P} f(P) \leq c(e) \ \ (e \in E)$

Flow-value of $f := \sum_P f(P)$

Max-Flow Min-Cut Theorem (Ford-Fulkerson)

- Max. flow-value of f $=$ $\text{Min} \ \ c(\delta X) := \sum_{e \in \delta X} c(e)$
  s.t. $s \in X \not\ni t$
- $\exists$ integral max flow

Multicommodity flow



$G = (V, E)$: undirected
$c: E \to \mathbb{Z}_+$
$S \subseteq V$: the set of terminals

$\mu: \binom{S}{2} \to \mathbb{Z}_+$: values of unit (s, t)-flows

Multiflow $f: \{S\text{−paths}\} \to \mathbb{Q}_+$ s.t. $\sum_{P: e \in P} f(P) \leq c(e) \ \ (e \in E)$

Maximize μ-flow-value of $f \ := \sum_P \mu(s_P, t_P) f(P)$

Question: Are there MFMC-type theorems for multiflows ?

- Combinatorial min-max relation
- Integrality of max flow

MFMC-type theorems for multiflows

$\mu(s,t') = \mu(s,s') = \mu(s',t) = \mu(t,t') = 0$

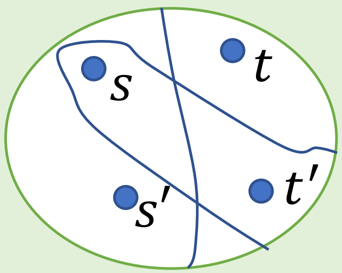
Thm (Hu) 2-commodity flow: $S = \{s,t,s',t'\}$, $\mu(s,t) = \mu(s',t') = 1$

- Max. μ-flow-value of f $=$ Min. $c(\delta X)$

  s.t. $s, s' \in X \not\ni t, t'$

  or $s, t' \in X \not\ni t, s'$

- $\exists$ half-integral max flow



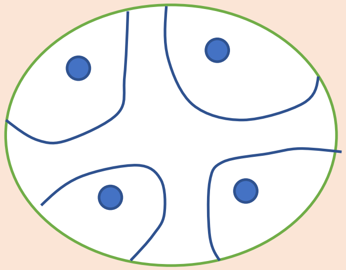
Thm (Lovasz, Cherkassky) Free multiflow: $\mu(s,t) = 1$ $(\forall s,t \in S)$

- Max. μ-flow-value of f $=$ Min. $\frac{1}{2}\sum_{s\in S} c(\delta X_s)$

  s.t. $\{X_s\}_{s\in S}$: disjoint

  $s \in X_s$

- $\exists$ half-integral max flow



On the other hand, 3-commodity flow
does not admit such a property, in the following sense:

$\forall k, \exists (V, E, c)$ there is no $\frac{1}{k}$-integral max flow

Question:
Which μ does admit MFMC-type theorem for multflow ?

Answer ← Geometry of the *tight span* associated with μ

Pioneering work:

A.V. Karzanov: Minimum 0-extensions of graph metrics, Europ. J. combin. 98.

V. Chepoi: A T_X-approach to cuts and metrics, Adv. in Applied Math. 98.

Multiflow-Metric Duality

Max. $\sum_P \mu(s_P, t_P) f(P)$ s.t. $\sum_{P: e \in P} f(P) \leq c(e)$ $(\forall e), f(P) \geq 0$ $(\forall P)$

|| LP-duality

Min. $\sum_e c(e) l(e)$ s.t. $\sum_{e \in P} l(e) \geq \mu(s_P, t_P)$ $(\forall P), l(e) \geq 0 (\forall e)$

|| $l \Rightarrow d_l :=$ shortest path metric

Min. $\sum_{e=xy} c(e) d(x,y)$ s.t. d: semi metric on V with $d|_S \geq \mu$

Obs. $c \geq 0$ → optimum = "minimal" metric d with $d|_S \geq \mu$

If μ is also a metric, then $d|_S = \mu$.

→ (V, d) is a *tight extension* of (S, μ)

→ (V, d) is a submetric of the *tight span* T_μ of (S, μ) (Isbell, Dress)

$$\exists \rho: V \to T_\mu \text{ s.t. } d(x,y) = d_{T_\mu}(\rho(x), \rho(y))$$

Sharpened multiflow duality by tight span (H. 09)

Max. $\sum_P \mu(s_P, t_P) f(P)$ s.t. f: multiflow

\\

Min. $\sum_{e=xy} c(xy) d_{T_\mu}\big(\rho(x), \rho(y)\big)$

s.t. $\rho: V \to T_\mu$

$\rho(s) \in T_{\mu,s}$ $(s \in S)$

Tight span (Isbell, Dress)

$T_\mu :=$ minimal set of $\{p \in \mathbb{R}_+^S \mid p(s) + p(t) \geq \mu(s,t)\ (s,t \in S)\}$

metrized by l_∞-metric (→ geodesic metric space)

$T_{\mu,s} := T_\mu \cap \{p(s) = 0\}$

c.f. nonmetric tight span (H. 06)

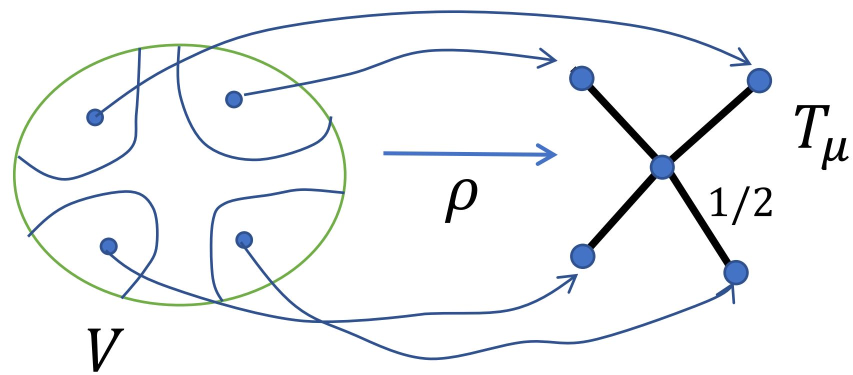
T_μ-derivation of Lovasz-Cherkassky formula: The case $\mu = 1$

Max. $\sum_P f(P)$ s.t. f: multiflow

$\frac{1}{2}\sum_{s\in S} c(\delta X_s)$

Min. $\sum_{e=xy} c(xy)\, d_{T_\mu}\big(\rho(x), \rho(y)\big)$

s.t.



Such a discretization works if $\dim T_\mu \leq 2$

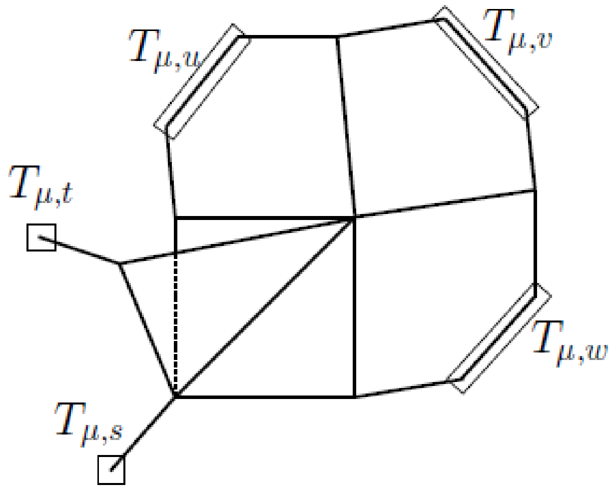
$\because$ T_μ is gluing of polygons in l_∞-plane $\simeq$ l_1-plane

Folder-complex decomposition of 2-dim tight span

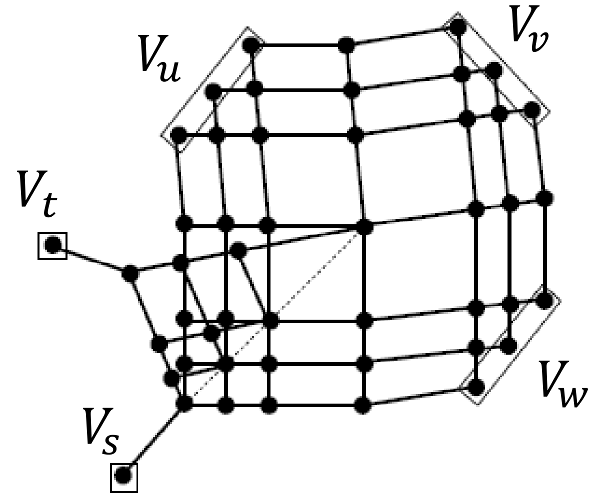
(Karzanov 98, H. 09, H. 11)

$\mu =$

	s	t	u	v	w
s	0	2	3	4	2
t	2	0	3	3	3
u	3	3	0	1	3
v	4	3	1	0	1
w	2	3	3	1	0

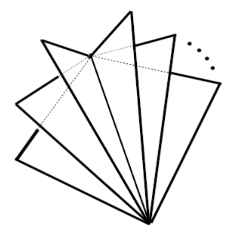


T_μ



Γ

Folder complex = CAT(0) B_2-complex

$\Downarrow$ $\Uparrow$ Filling "folder"  to each $K_{2,m}$-subgraph

Its 1-skeleton Γ = orientable modular graph without isometric C_k for $k \geq 6$

V. Chepoi: Graphs of some CAT(0) complexes, Adv. in Applied Math. 2000

2-dim tight span completely characterizes "discreteness phenomena" of multiflows

Thm (Karzanov 98, H.09, H.11)

If $\dim T_\mu \leq 2 \Longrightarrow$ Max. μ-flow-value of multiflow

||

Min. $\sum_{e=xy} c(e) d_\Gamma(x,y)$

s.t. $\rho: V \to V(\Gamma)$, $\rho(s) \in V_s$ $(s \in S)$

1-skeleton of folder-complex decomposition of T_μ

Thm (H. 14)

If $\dim T_\mu \leq 2 \Longrightarrow \exists\ 1/24$-integral max flow

1/4 conjectured

Thm (Karzanov 98, H. 09)

If $\dim T_\mu \geq 3 \Longrightarrow \forall k, \exists (V, E, c)$ s.t. no $1/k$-integral max flow

Multifacility location problem = 0-extension problem

Γ: undirected graph (with unit edge length)

0-Ext$[\Gamma]$: Given a set $V \supseteq V(\Gamma)$, weight $c: \binom{V}{2} \to \mathbb{Q}_+$

Min. $\sum_{x,y \in V} c(xy) d_\Gamma(\rho(x), \rho(y))$

s.t. $\rho: V \to V(\Gamma)$ retraction

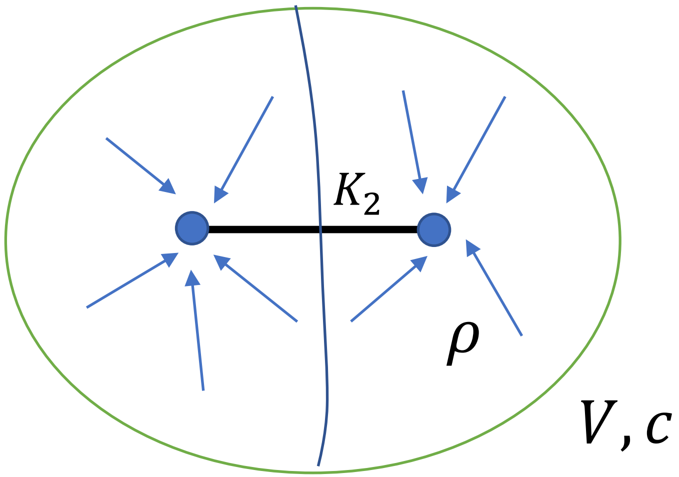
$\Updownarrow \quad V - V(\Gamma) =: \{1,2, \dots, n\}$

Given $n \in \mathbb{Z}_+$ $b_{i,v}, c_{ij} \in \mathbb{Q}_+$ $(1 \leq i,j \leq n, v \in V(\Gamma))$

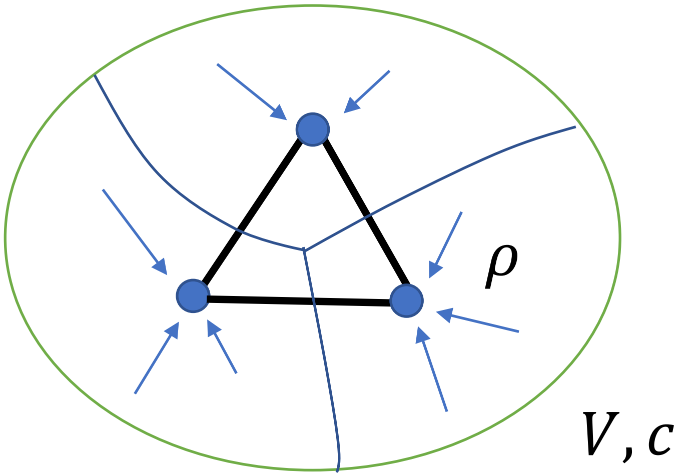
Min. $\sum_{i,v} b_{i,v} d_\Gamma(x_i, v) + \sum_{i,j} c_{ij} d_\Gamma(x_i, x_j)$

s.t. $(x_1, x_2, \dots, x_n) \in V(\Gamma)^n$

$\Gamma = K_2$ → 0-Ext$[\Gamma]$ = Minimum cut problem **P**



$\Gamma = K_m \ (m \geq 3)$ → 0-Ext$[\Gamma]$ = Multiway cut problem **NP-hard**



Question: What is Γ for which 0-Ext$[\Gamma] \in$ **P** ?

Chepoi 96, Karzanov 98

Thm (Picard-Ratliff 78) Γ : tree → 0-Ext$[\Gamma] \in$ **P**

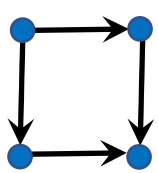
Fact: Γ, Γ': **P** → $\Gamma \times \Gamma'$: **P**

Thm (Chepoi 96) Γ : median graph → 0-Ext$[\Gamma] \in$ **P**

further generalization Karzanov 04

Thm (Karzanov 98)
Γ : orientable modular graph without $C_k (k \geq 6)$
→ 0-Ext$[\Gamma] \in$ **P**

Orientability
∃orientation, ∀4-cycle



Weighted generalization
Bandelt-Chepoi-Karzanov 00

Dichotomy Theorem

Thm (Karzanov 98)
$\Gamma \neq$ orientable modular graph → 0-Ext$[\Gamma]$ is **NP-hard**

Thm (H. 16)
$\Gamma =$ orientable modular graph → 0-Ext$[\Gamma]$ is in **P**

I presented this result at CIRM, 2013 and met with
Victor, Jeremie, Damian to start CCHO paper [CCHO 20].
It turned out that om-graph is very nice graph from MGT perspective:

- Euclidean building of type C → om-graph
- Thickening of om-graph → Helly graph
- Euclidean building of type A → om-graph [H.20]
- om-graph → CAT(0)-orthoscheme complex [H. 21]

Proof idea inspired by *Discrete Convex Analysis*

Discrete Convex Analysis (Murota 96 ∼)

∼ A theory of "convex" function on $\mathbb{Z}^n$
for well-solvable discrete optimization problems

Discrete convex functions

∼ *Nice* functions in term of computational complexity
(matroids, submodular func, M-convex / L-convex func.)

Connection to 0-Ext: $\Gamma =$ path of length L

Min. $\sum_{i,v} b_{i,v} \; |x_i - v| \; + \sum_{i,j} c_{ij} \left| x_i - x_j \right|$

s.t. $(x_1, x_2, \dots, x_n) \in \{0,1,2,\dots,L\}^n \subseteq \mathbb{Z}^n$

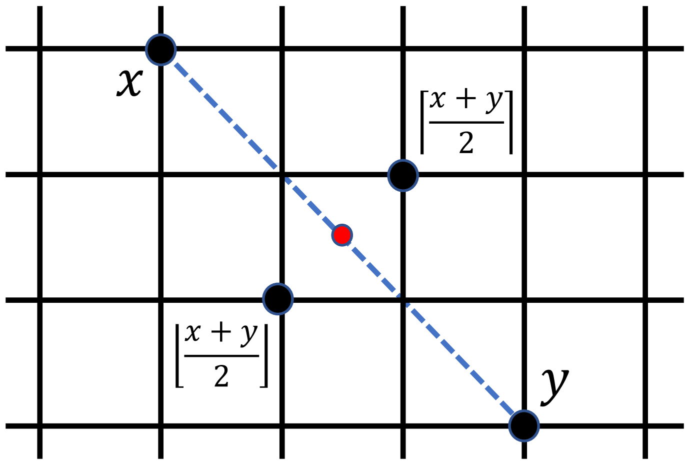
→ This is L♮-convex function minimization

L$^\natural$-convex function on $\mathbb{Z}^n$

(Favati-Tardella 90, Murota 98, Fujishige-Murota 00)

Def. $g: \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$: L$^\natural$-convex

$$\Leftrightarrow \quad g(x) + g(y) \geq g\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + g\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) \quad (x, y \in \mathbb{Z}^n)$$

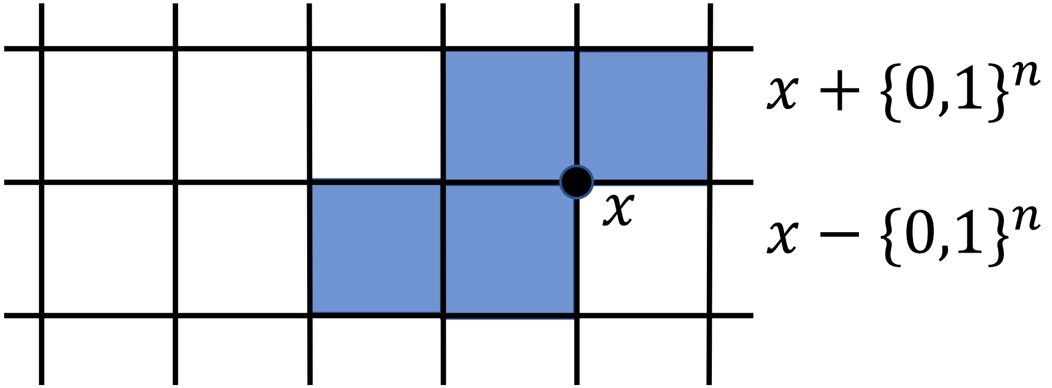


Rem. $\{0,1\}^n \ni u \mapsto g(x+u)$ is submodular

$$\text{i.e.,}\ g(x+u) + g(x+v) \geq g(x + \min(u,v)) + g(x + \max(u,v))$$

Steepest Descent Algorithm (SDA) framework for minimizing an L-convex function

- Local-to-global optimality condition



- Optimality check ← Submodular Function Minimization (SFM)

Lovasz-Grotchel-Schrijver 1981, Schrijver 2000, Iwata-Fleischer-Fujishige 2001,...

- # iterations = l_∞-distance between **opt** and initial x

Kolmogorov-Shioura 2009, Murota-Shioura 2014

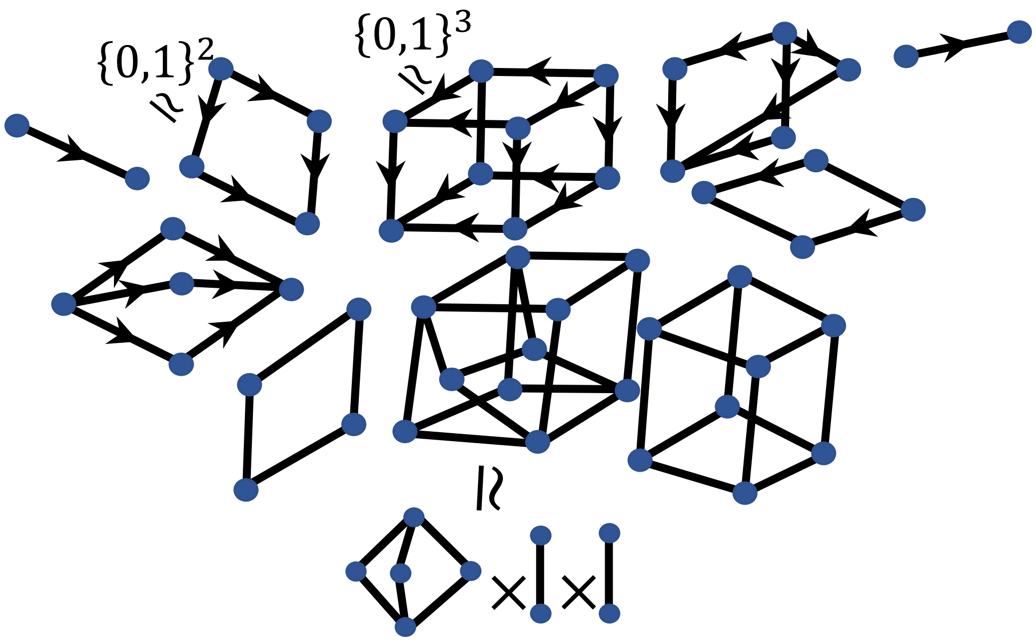
Viewing $\mathbb{Z}^n$ as a grid-graph, can we extend theory of L-convex functions to more general graphs ?

-- Philosophy of Discrete Convex Analysis beyond $\mathbb{Z}^n$--

L-convex function on orient**ed** modular graph & submodular function on modular semilattice [H.16, H.18]

Oriented modular graph = gluing of modular lattices & modular semilattices

Bandelt-Van de Vel-Verheul 93

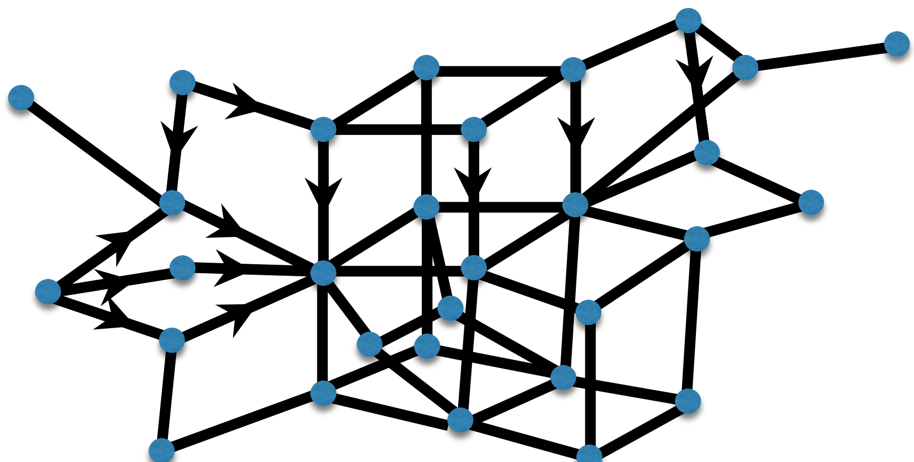


- Define submodular function on modular semilattice
- Define L-convex function on oriented modular graph
  := submodular in each “neighborhood modular semilattice”
- Definition is not easy

L-convex function on orient**ed** modular graph & submodular function on modular semilattice [H.16, H.18]

Oriented modular graph = gluing of modular lattices
& modular semilattices

Bandelt-Van de Vel-Verheul 93



- Define submodular function on modular semilattice
- Define L-convex function on oriented modular graph
  := submodular in each “neighborhood modular semilattice”
- Definition is not easy

- Local-to-global optimality condition → Steepest Descent Algorithm
- Local optimality ← SFM on neighborhood modular semilattice
- VCSP tractability of SFM on modular semilattice

  Use a criterion of Kolmogorov-Thapper-Živný 2015 on VCSP
- # iteration of SDA = d_{Γ^Δ}(opt, initial), Γ^Δ: *thickening* of Γ

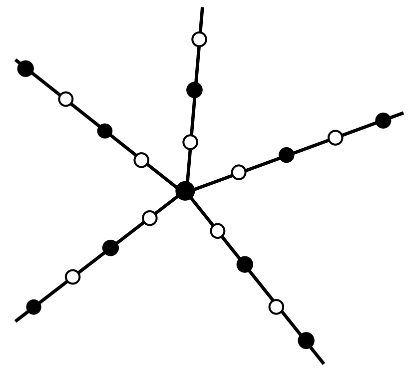
To 0-Ext$[\Gamma]$:

- $(x, y) \mapsto d_\Gamma(x, y)$ is L-convex on $\vec{\Gamma} \times \vec{\Gamma}$
- 0-Ext$[\Gamma]$ is L-convex minimization on $\vec{\Gamma} \times \vec{\Gamma} \times \cdots \times \vec{\Gamma}$

L-convex function minimization on om-graphs often arise as the dual of related multiflow & network designs problems

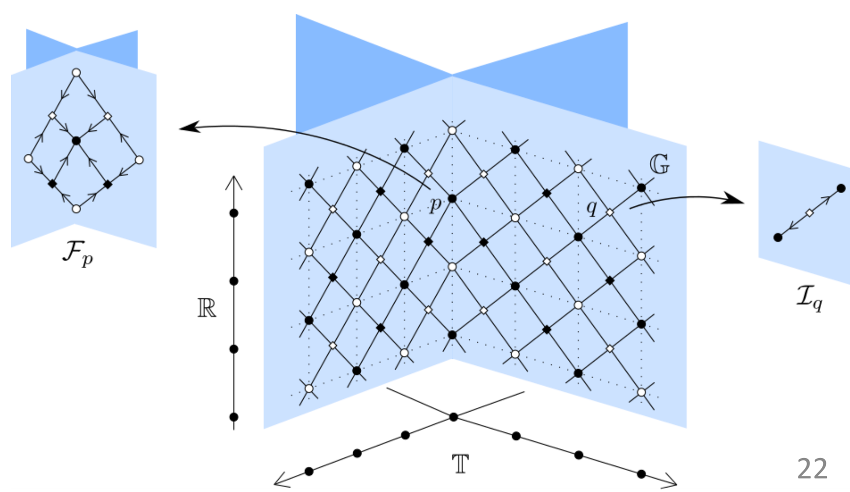
- Mincost free multiflow & terminal backup [H. 15]

≈ L-convex minimization of the product of



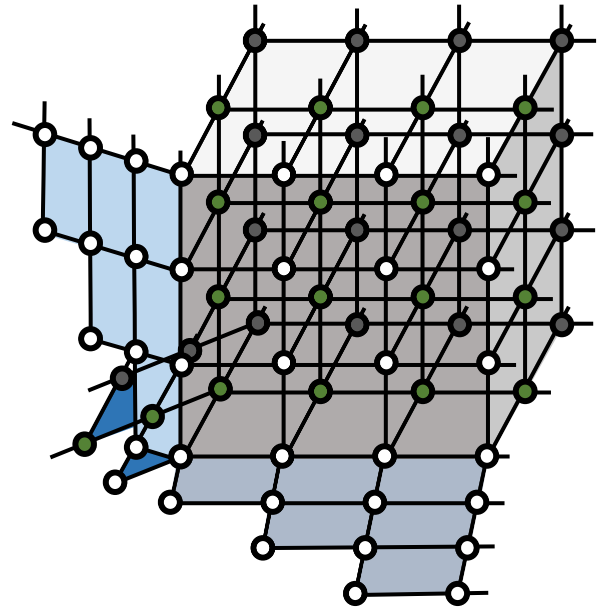
- (Mincost) node-capacitated free multiflow [H. 18, H-Ikeda 20]

≈ L-convex minimization of the product of

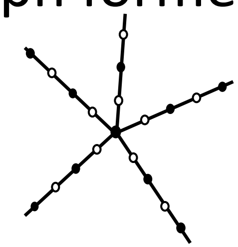


- Node-capacitated terminal backup [H-Ikeda 20]

≈ L-convex minimization of the product of



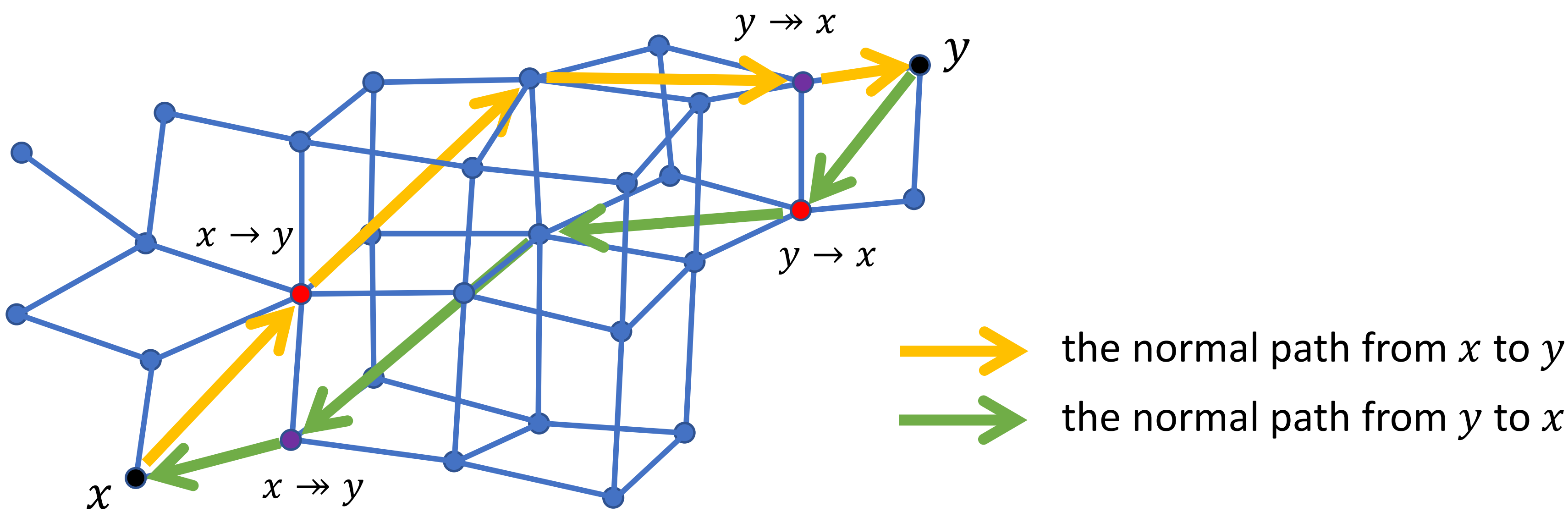
= median graph formed by subtrees of



For these problems,
SDA framework provides efficient combinatorial algorithms,
where such algorithms were previously not known.

N-convexity [H-Ikeda 20]

$\sim$ another useful discrete convexity concept



Median graph
=
Normal cube-path for CAT(0) cubical complex: Niblo-Reeves 1998

Def: $f: V(G) \to \mathbb{R} \cup \{\infty\}$: N-convex $\Leftrightarrow$

$$f(x) + f(y) \geq f(x \to y) + f(y \to x)$$

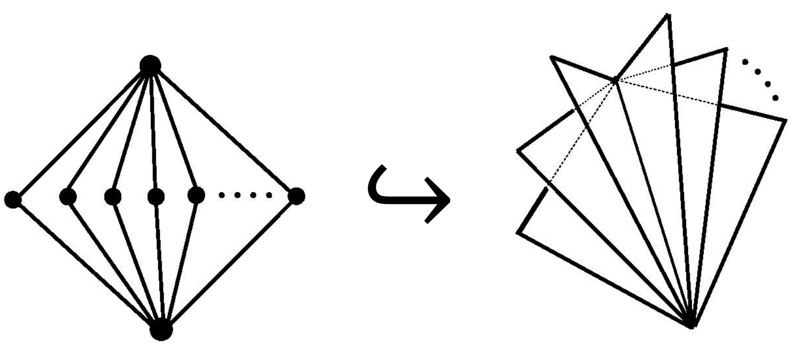
$$f(x) + f(y) \geq f(x \twoheadrightarrow y) + f(y \twoheadrightarrow x)$$

Summary

- MGT provides several tractability characterizations for (distance-related) combinatorial optimization problems, and ground structures for defining discrete convex functions, which lead to efficient algorithms.

- Convexity of NPC space $\overset{?}{\rightarrow}$ tractability in (discrete) optimization

- Future work: Develop a framework to solve discrete problems via continuous/convex relaxation to NPC spaces



Application: noncommutative-rank (Hamada-H 21)

Thank you for your attention !