Combinatorial algorithms for some multiflow problems and related network designs

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We address

I. Mincost node-demand multiflow problem

II. Maximum node-capacitated multiflow problem

Our result

First combinatorial polytime algo for I

First combinatorial strongly-polytime algo for II

Feature

Build on Discrete Convex Analysis beyond $\mathbf{Z}^n$

Application

Approx. of terminal backup & node-multiflow cut

I. Mincost node-demand multiflow problem

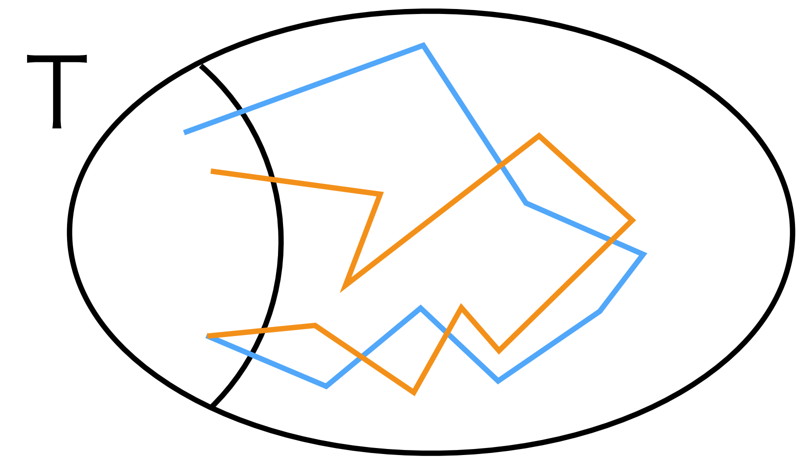
N = (V,E,c,a,T): undirected network

c: E → Z+: edge-capacity

a: E → Z+: edge-cost

T: terminal set (⊂ V)

r: T → Z+: demand



Def: Multiflow <=> f: { T-paths } → R+ s.t.

$$f(e) := \Sigma \{ f(P) \mid P: e \in P\} \leq c(e) \quad (e \in E)$$

feasible <=> (s,T-s)-flow in f ≥ r(s) (∀s in T)

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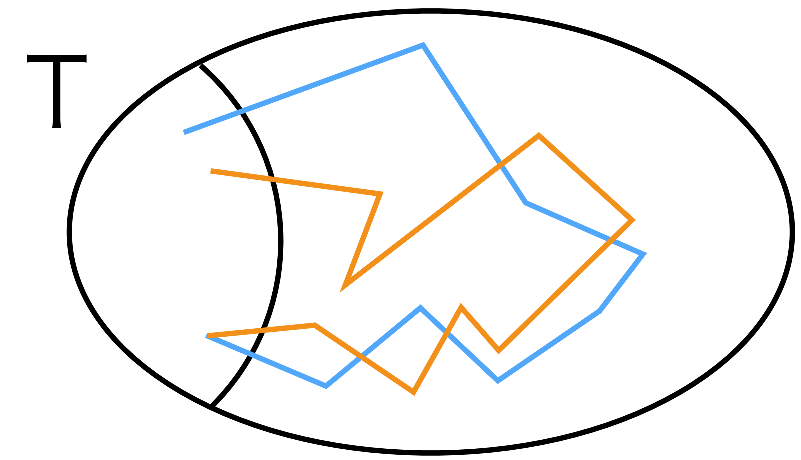

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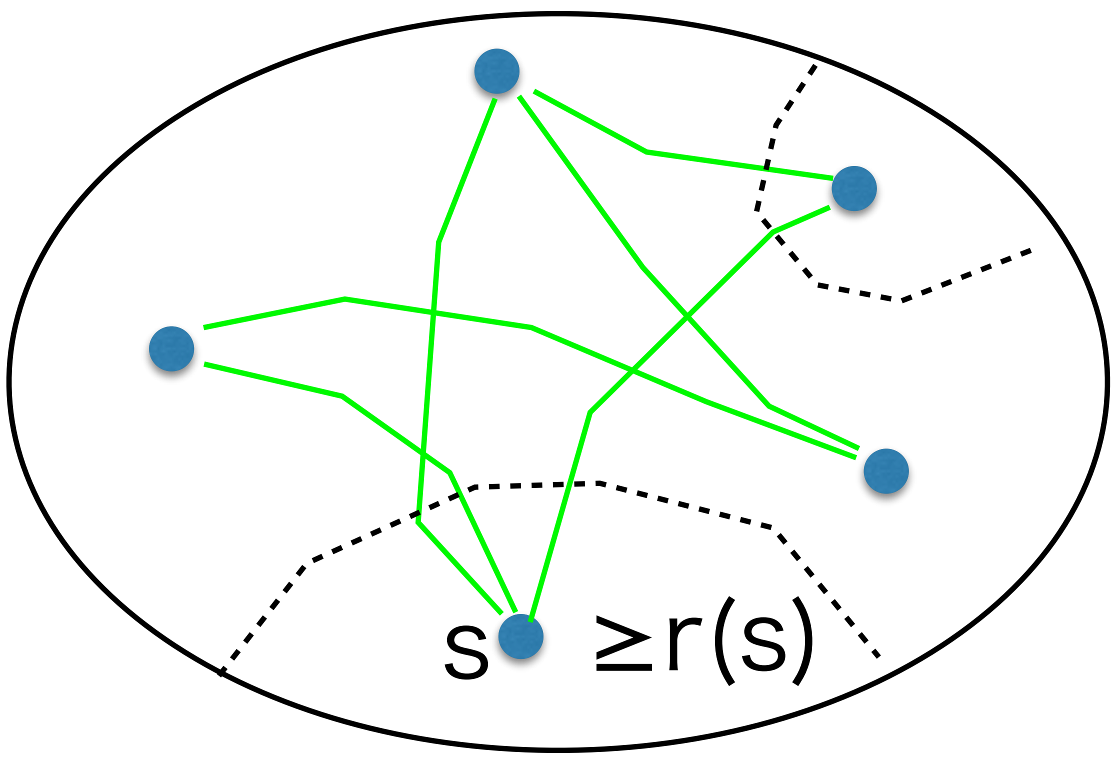
> Find a feasible multiflow f
> of minimum total cost Σ a(e)f(e)

Introduced by Fukunaga (2014)

as LP-relaxation of a class of network designs



Min. $\Sigma\, a(e)\, f(e)$

s.t. f: multiflow

$(s, T\text{-}s)$-flow in $f \geq r(s)$

$(s \in T)$

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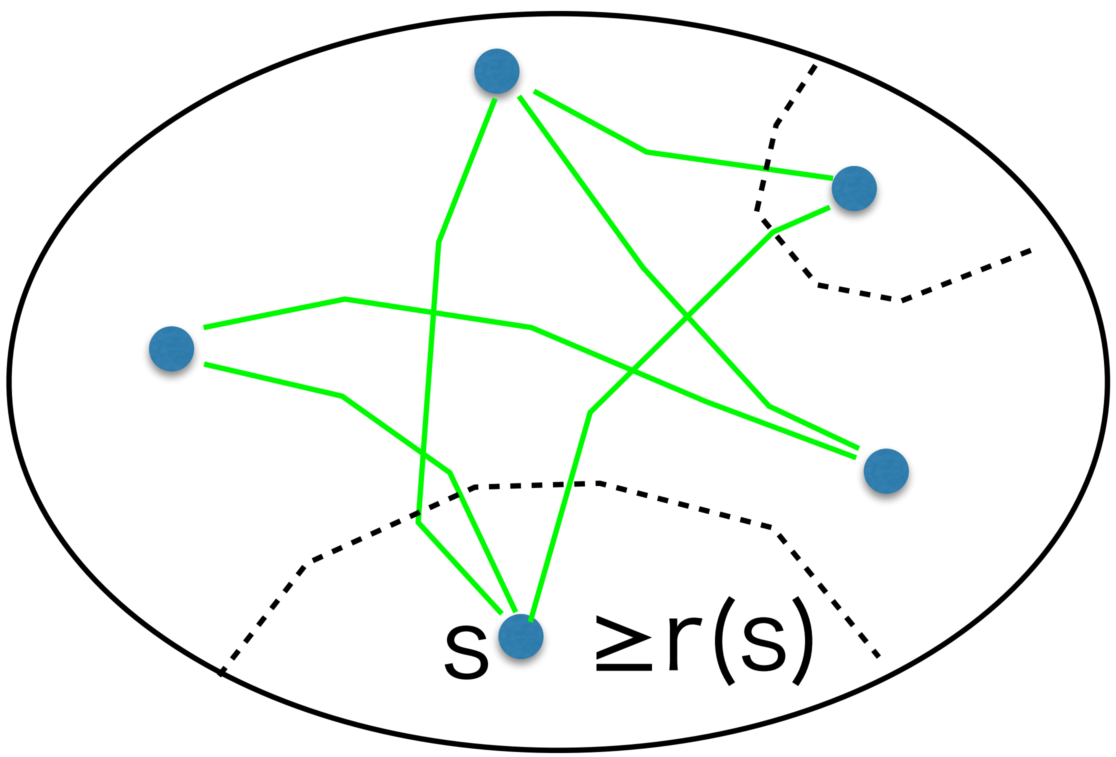

Min. Σ a(e) f(e)
s.t. f: multiflow
(s,T-s)-flow in f ≥ r(s)
(s ∈ T)

x(e) = f(e)

Min. Σ a(e) x(e)
s.t. Σ {x(e) | e ∈ ∂X} ≥ r(s) (s ∈ T, (s,T-s)-cut X)
0 ≤ x(e) ≤ c(e) (e ∈ E)

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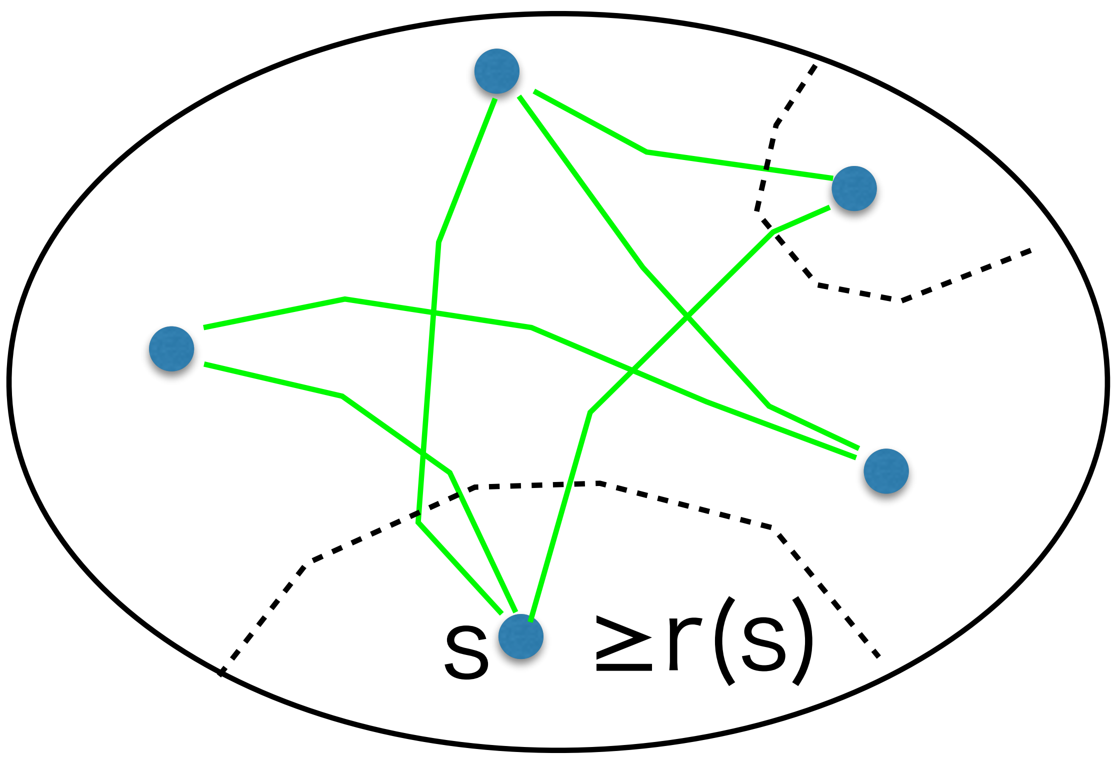

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$(s, T-s)$-flow in $f \geq r(s)$

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Lovasz-Cherkassky ↑ ↓ $x(e) = f(e)$

Min. $\Sigma\, a(e)\, x(e)$

s.t. $\Sigma \{x(e) \mid e \in \partial X\} \geq r(s) \quad (s \in T, (s, T-s)\text{-cut } X)$

$$0 \leq x(e) \leq c(e) \quad (e \in E)$$

Terminal backup problem

very special class of skew-supermodular covering

Min. Σ a(e) x(e)

s.t. Σ {x(e) | e $\in \partial$X} $\geq$ r(s) (s $\in$ T, (s,T-s)-cut X)

x(e) in {0,1,2,⋯,c(e)} (e $\in$ E)

Anshelevich-Karagiozova 11: r = 1 (—> c = 1) P

Bernath-Kobayashi-Matsuoka 13: c = ∞ P

P or NP-hard ??

Terminal backup problem

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Fukunaga 14:

half-integrality of LP-relax. (= Multiflow-relax.)

half-integral opt. —> 2-approx. —> 4/3-approx.

naive

obtained by ellipsoid method

Result

O(n log (n AC) MF(kn, km)) time algorithm to find half-integral opt. of

Min. Σ a(e) f(e)

s.t. f: multiflow

(s,T-s)-flow in f $\geq$ r(s) (s $\in$ T)

n = |V|, m = |E|, k = |T|, A= max a(e), C =Σ c(e)

—> combinatorial implementation of 4/3-approx. of terminal backup

Remark

Our multiflow problem generalizes
mincost maximum free multiflow problem (Karzanov 79)

Karzanov 94: strongly polynomial time algorithm
··· ellipsoid

Goldberg, Karzanov 97:
combinatorial weakly polynomial time algorithms
··· O(??)

Open problem:
combinatorial strongly polynomial time algorithm ?

Idea

Discrete Convex Analysis (Murota)

~ theory of "convex" optimization over $\mathbf{Z}^n$

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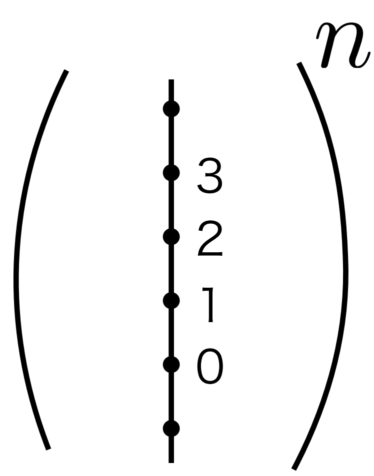
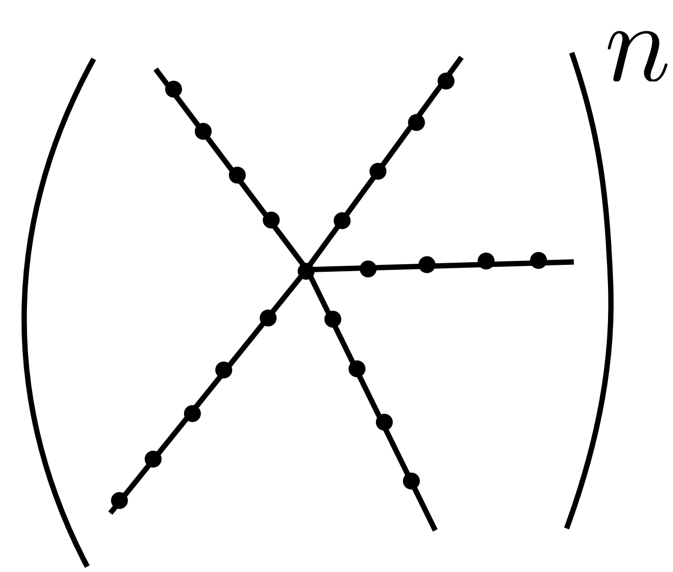
Discrete Convex Analysis (Murota)

~ theory of “convex” optimization over $\mathbf{Z}^n$ $\simeq$



Dual of our multiflow problem

~ “convex” optimization over



Adapting DCA algorithm & technique

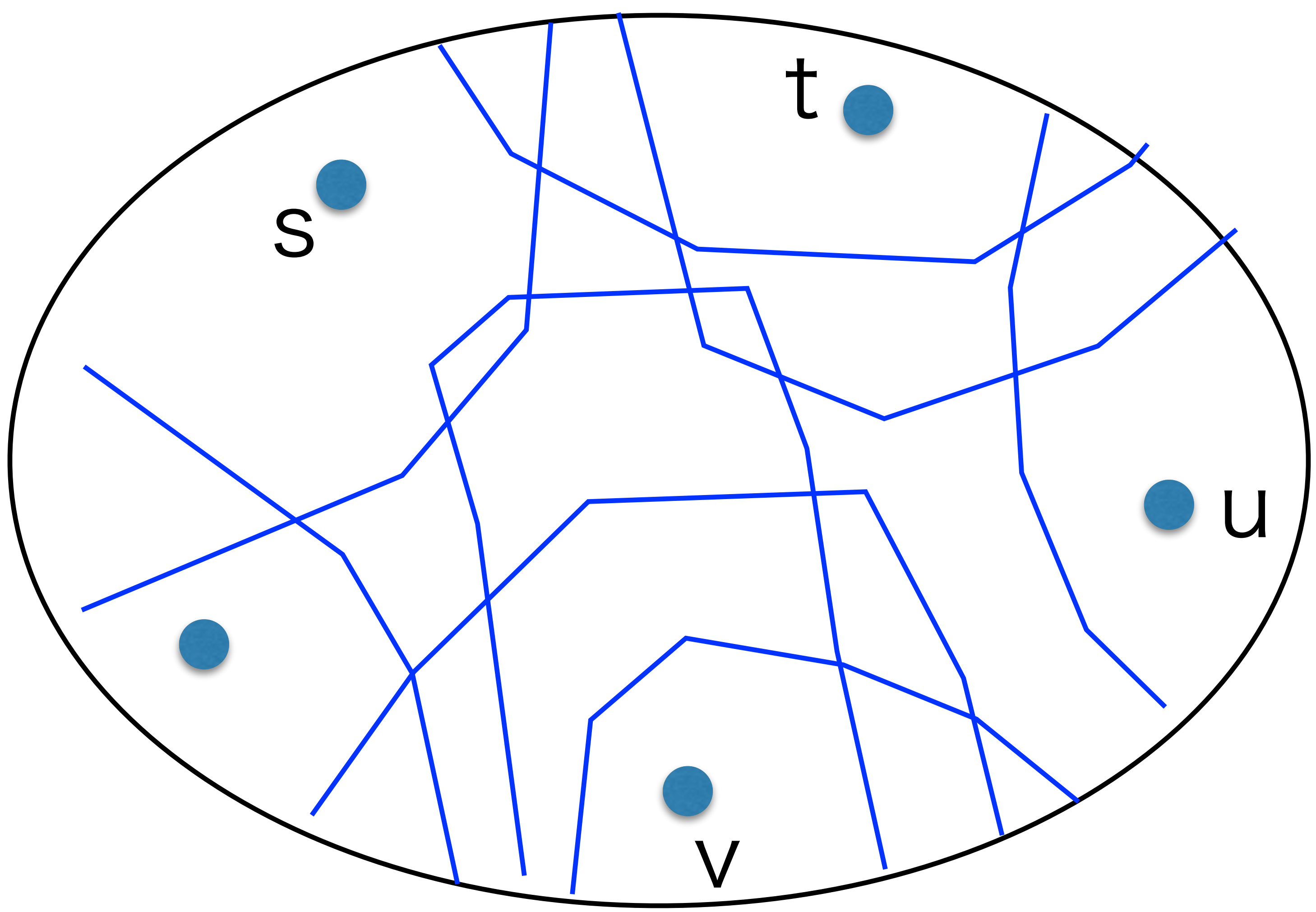
(steepest descent algo. & proximity scaling)

LP-dual (~~ cut packing)

Max. $$\sum_{s \in T,\, X:(s,T\setminus s)\text{-cut}} r(s)\lambda(X) - \sum_{ij \in E} c(ij) \max\{0, \sum_{X} \lambda(X)[\partial X](ij) - a(ij)\}$$

characteristic vector

s.t. $$\lambda(X) \geq 0 \quad (s \in T,\ X : (s, T \setminus s)\text{-cut})$$



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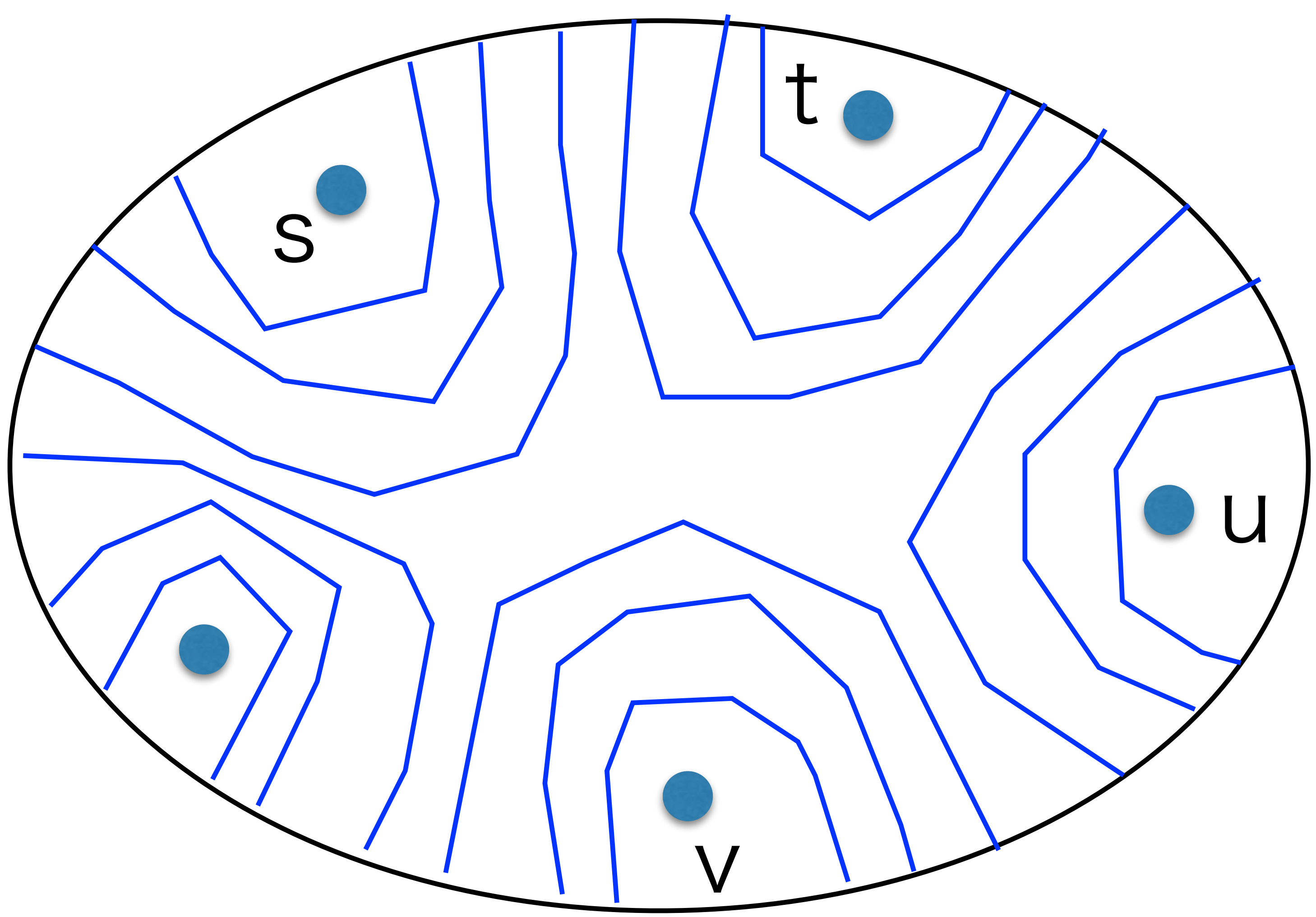

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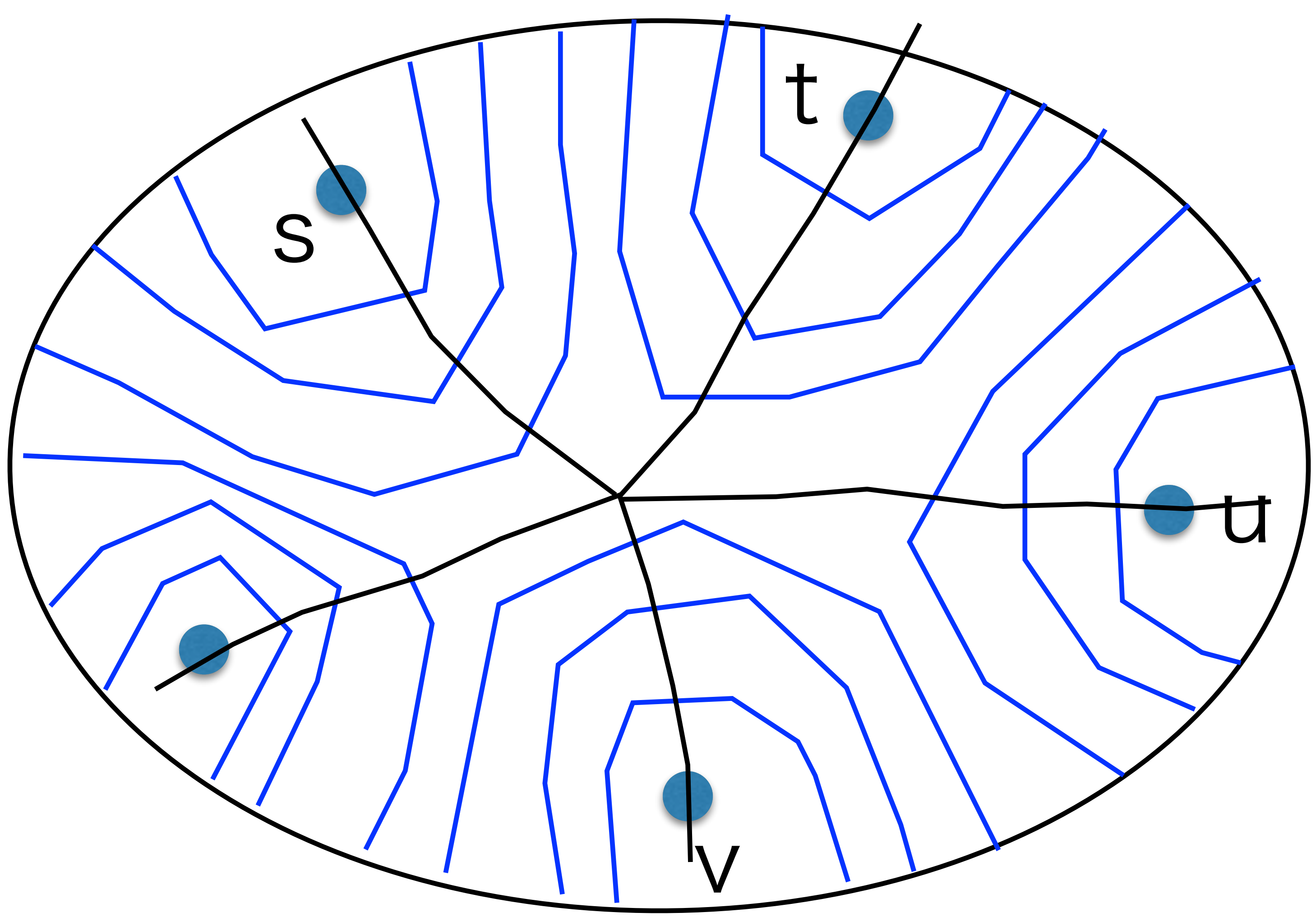
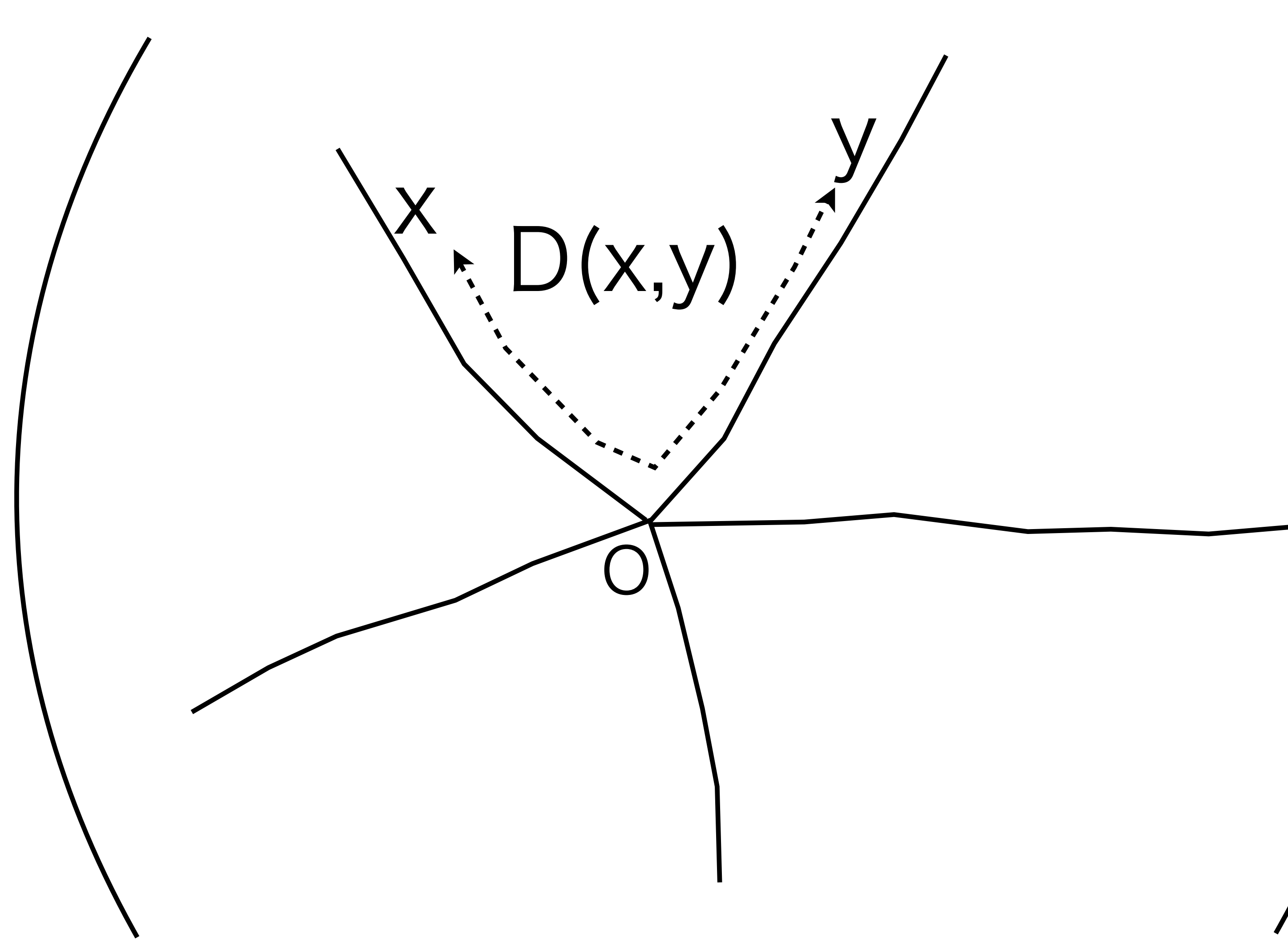


Max. $$\sum_{s \in T} r(s) D(x_s, O) - \sum_{ij \in E} c(ij) \max\{0, D(x_i, x_j) - a(ij)\}$$

s.t.

$(x_1, x_2, \ldots, x_n) \in$ 

^{n}

x_s in s-th branch

(s in T)

V={1,2,⋯,n}

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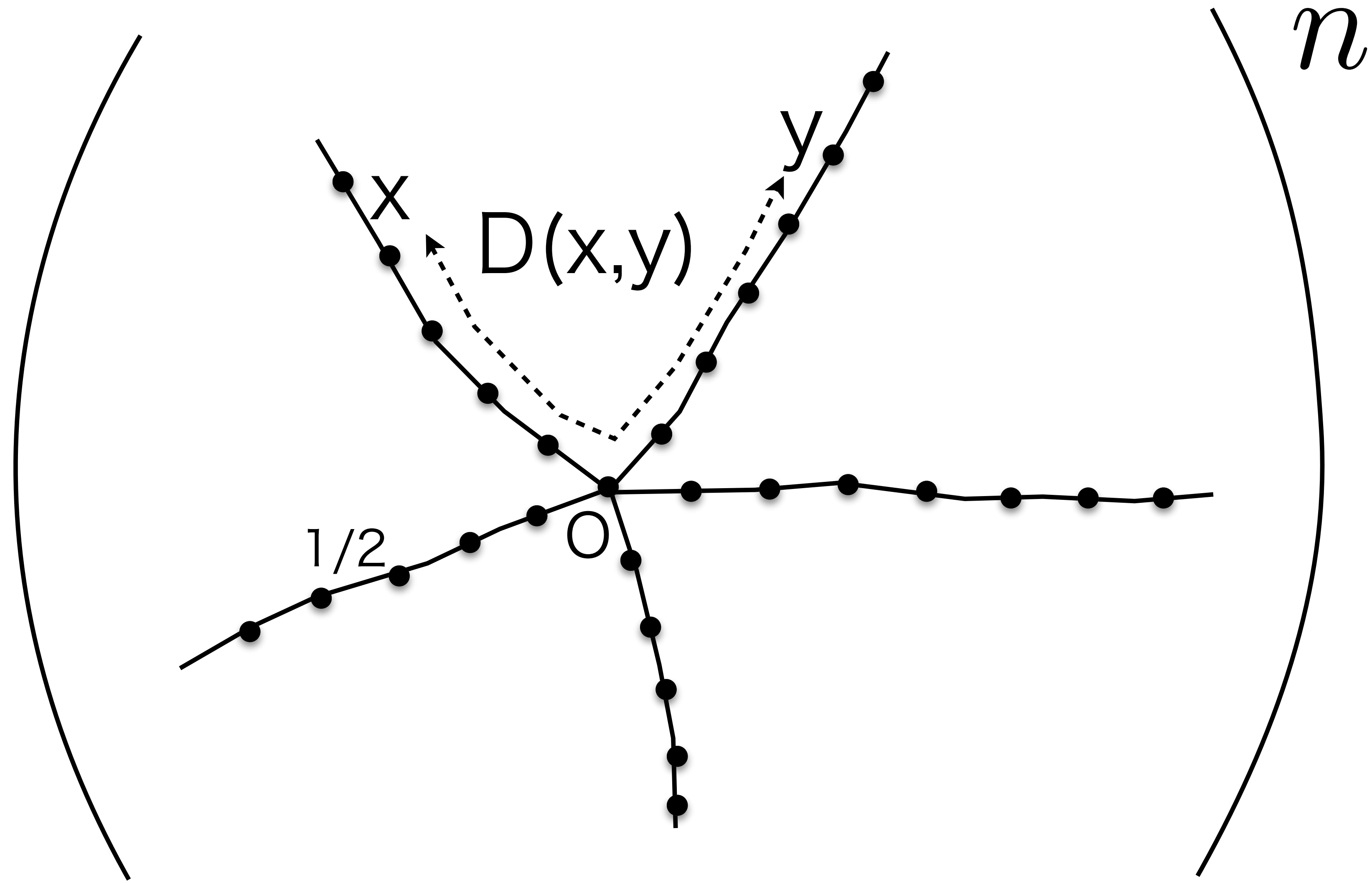

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half-integrality

“discretely concave”

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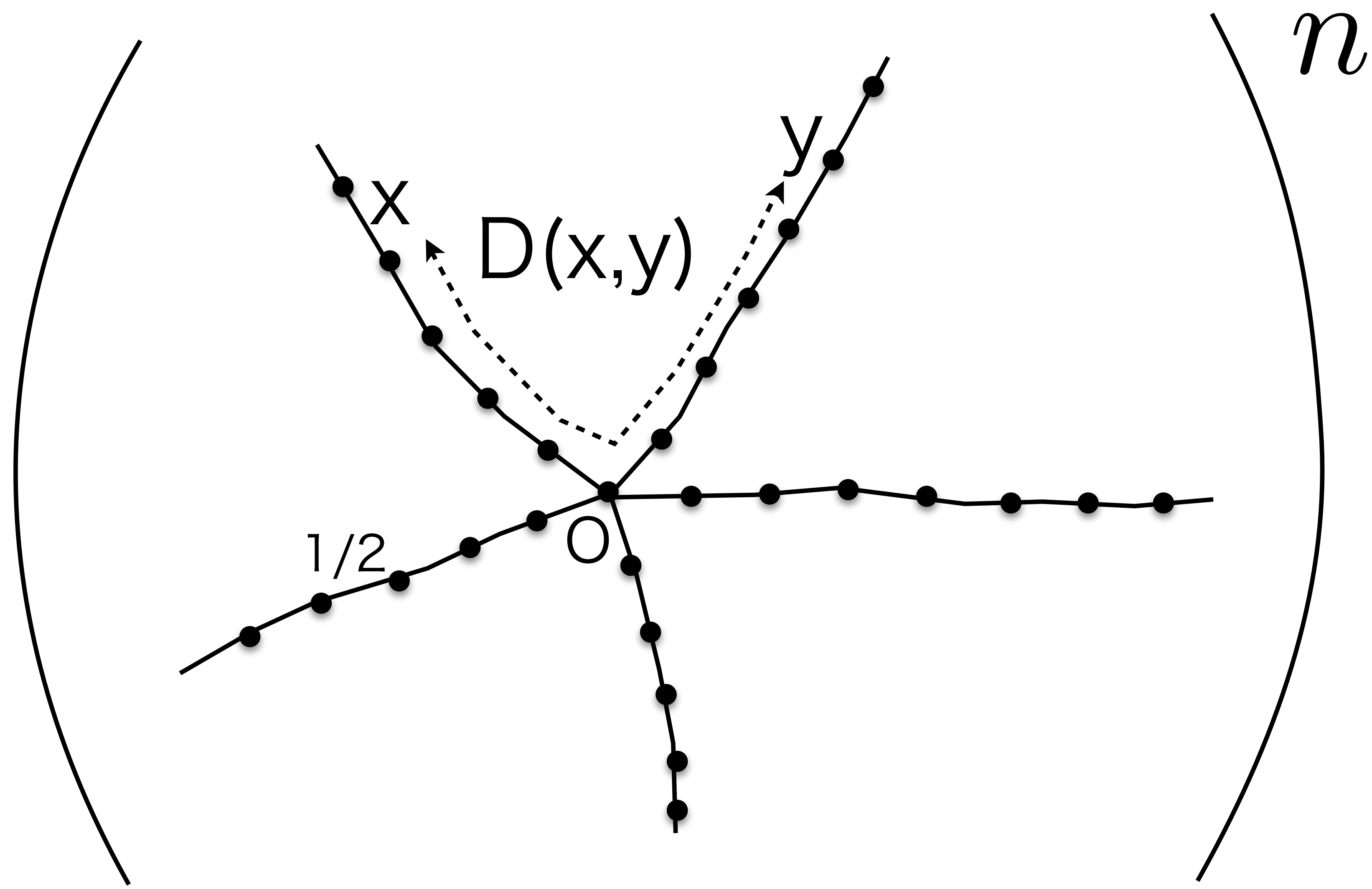

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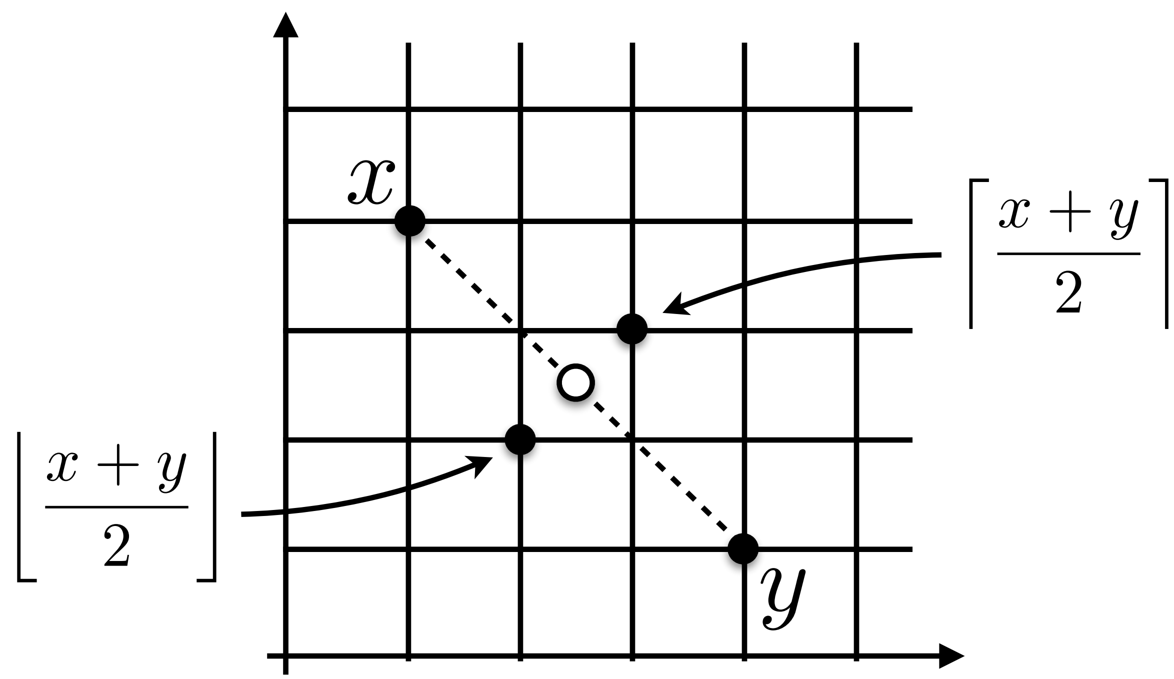
L$^\natural$-convex function: (Murota, Murota-Fujishige, Favati-Tardella)

$$g(x) + g(y) \geq g\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + g\left(\left\lceil \frac{x+y}{2} \right\rceil\right) \quad (x, y \in \mathbf{Z}^n)$$



- Optimality check —> Submodular Func. Min.
- Steepest Descent Algorithm by successive SFMs
- l_∞-bound of # iterations of SDA (Kolmogorov-Shioura 04)
- Proximity & domain scaling technique

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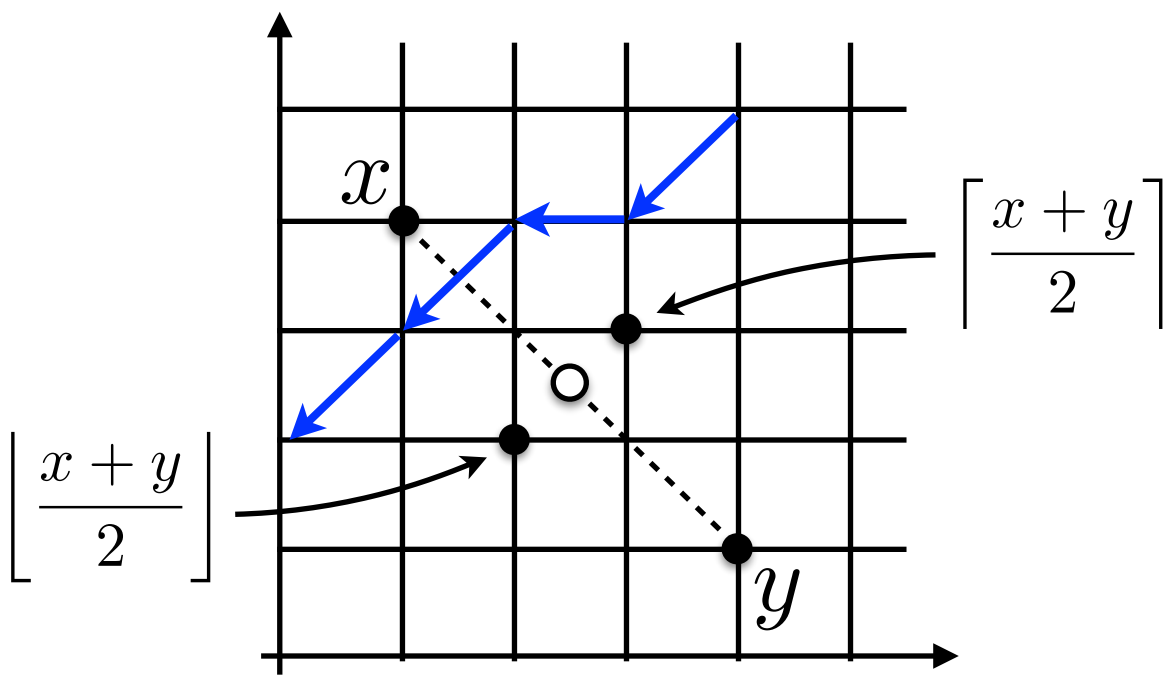

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(negative of) our dual objective satisfies

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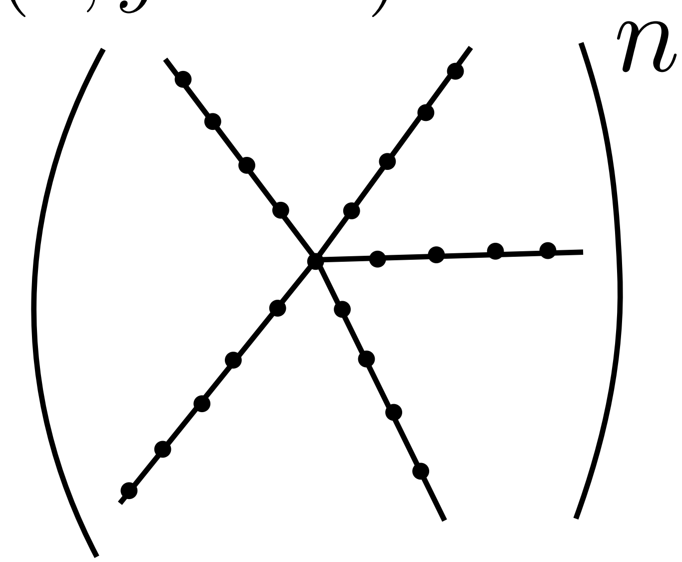
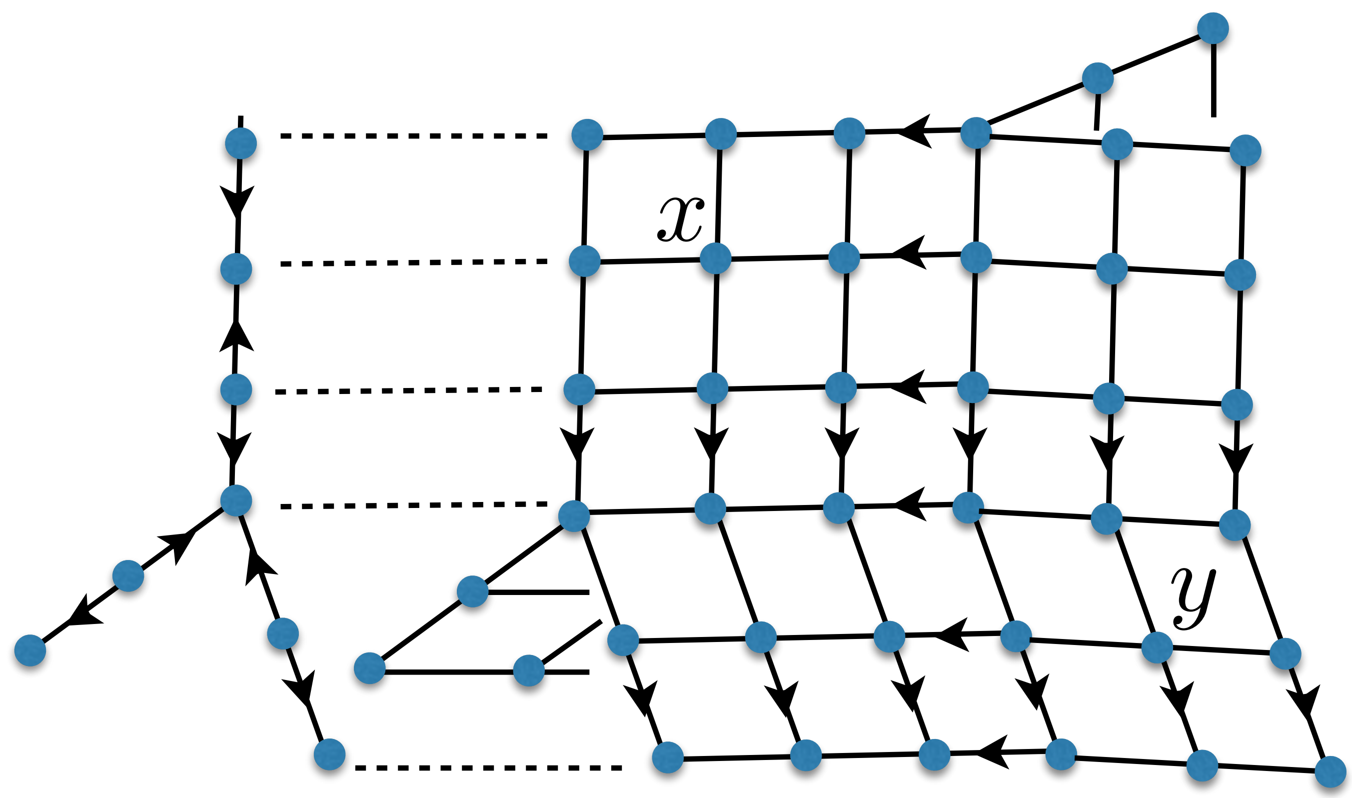




(Huber, Kolmogorov 12)

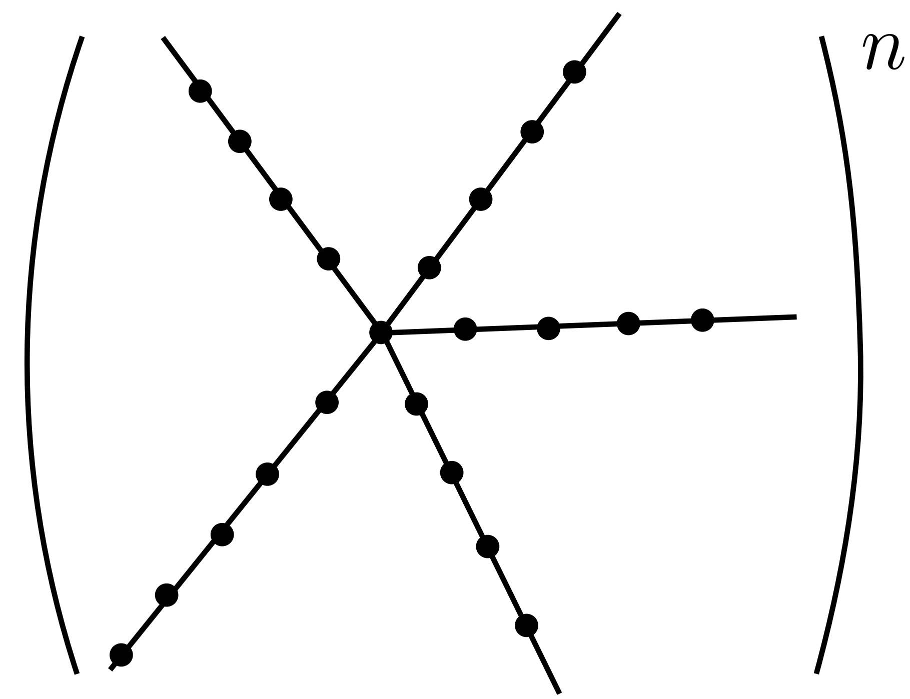
- Optimality check —> k-Submodular Func. Min.
- Steepest Descent Algorithm by successive k-SFMs
- l_∞-bound of # iterations of SDA
- Proximity & domain scaling technique

- k-SFM in oracle model ~~ P or NP-hard ??
- k-SFM in VCSP model ~~ P (Thapper, Zivny 12)
- Our special case ~~ O(MF(kn,km))

(Iwata,Wahlstrom,Yoshida 14)

SDA + scaling

—> dual opt x* in O(n log (n AC) MF(kn, km)) time

min g(x) s.t. x ∈ 

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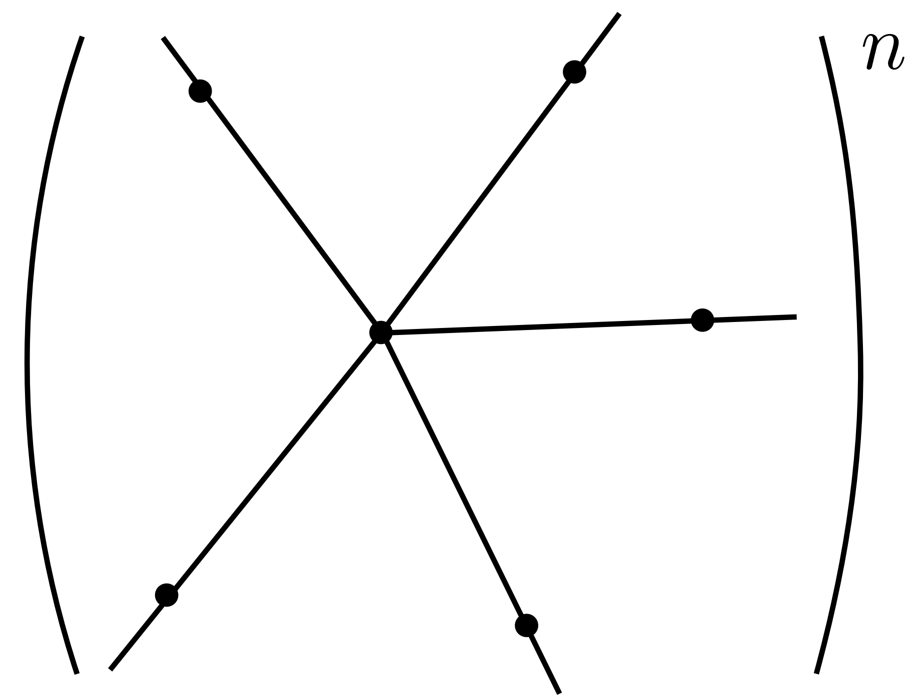

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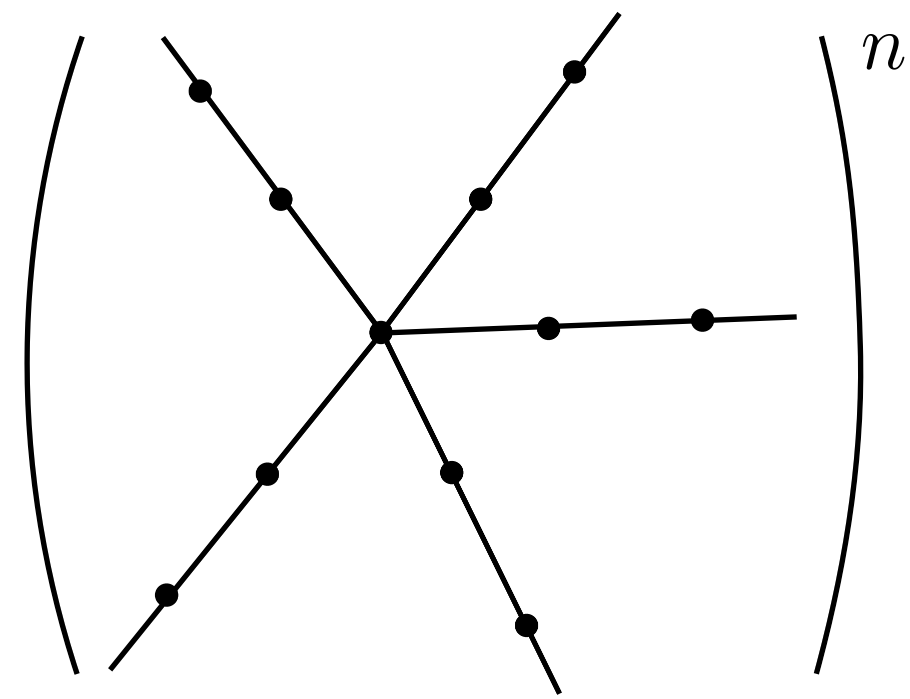
min g(x) s.t. x ∈ 

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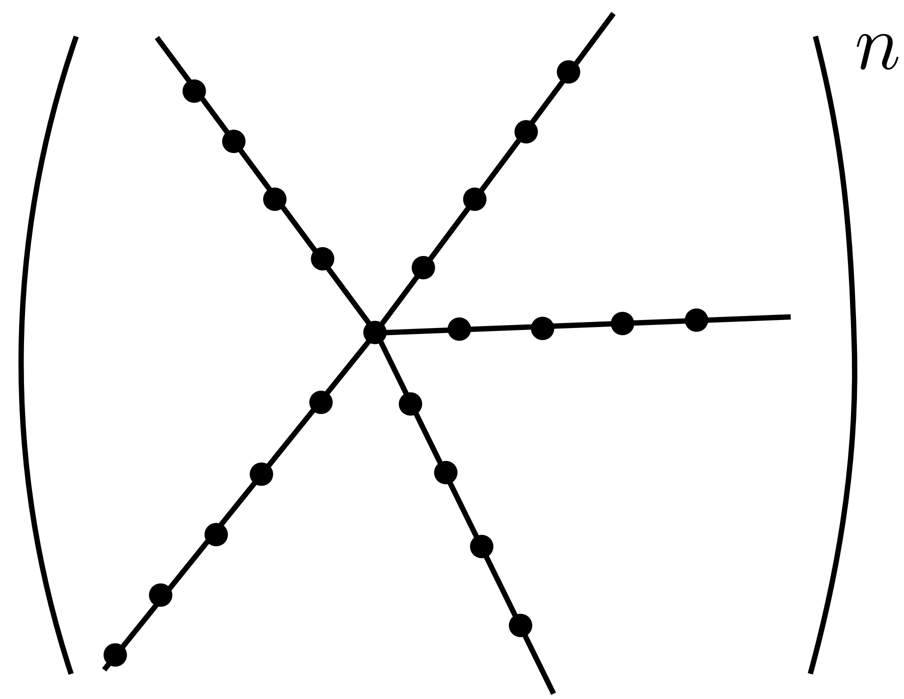

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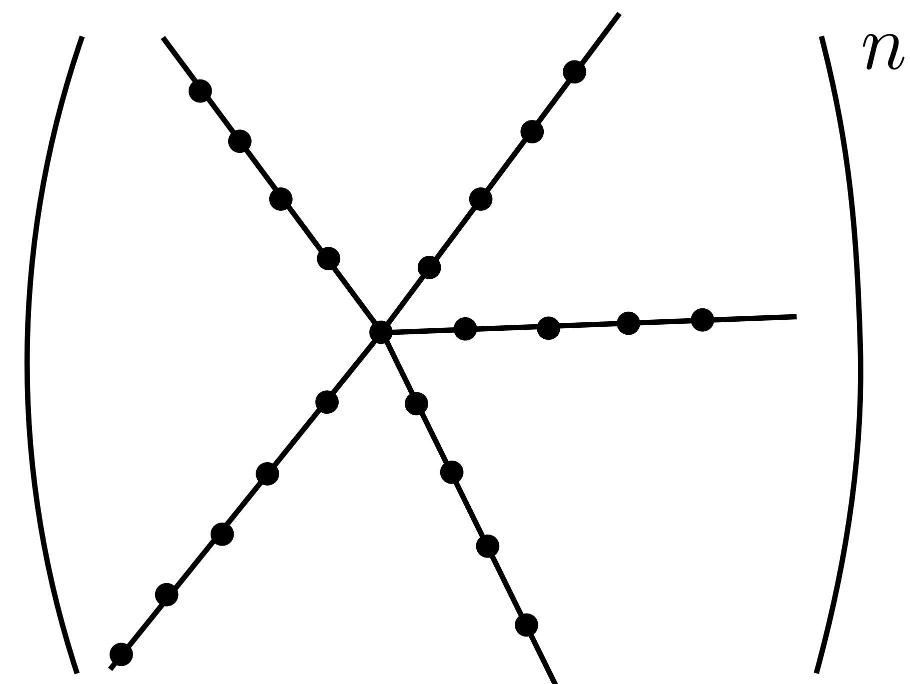
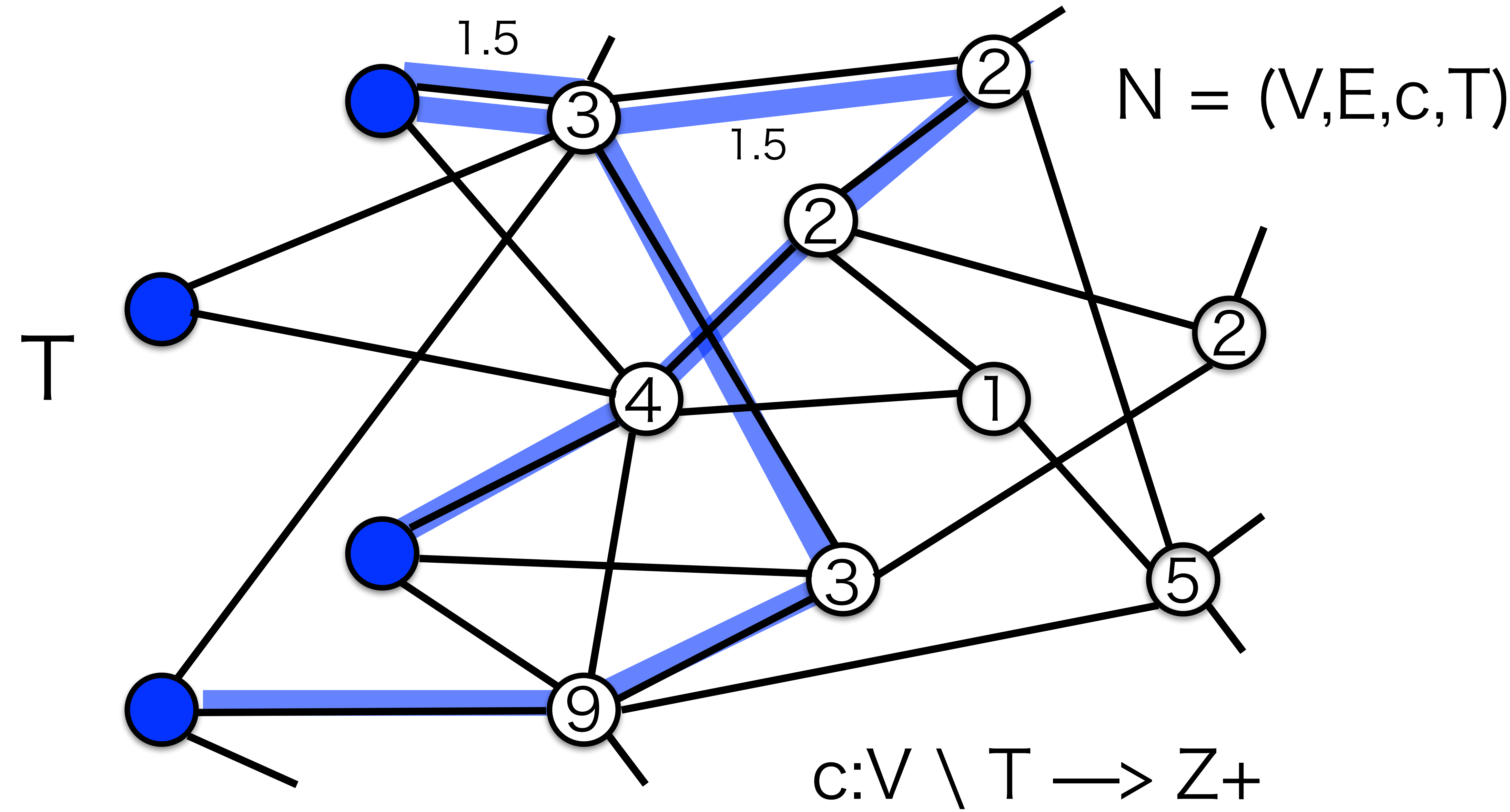


x* —> primal opt f*

by complementary slackness with x*

one circulation problem

II. Maximum node-capacitated free multiflow problem



Max. Σ f(P) s.t. f: multiflow

Garg, Vazirani, Yannakakis 04

- dual of LP-relaxation of node-multiway cut
- dual half-integrality —> 2-approximation

Pap 07,08

Mader's openly-disjoint T-path thm

- primal half-integrality
- strong polynomial time solvability (ellipsoid)

Babenko, Karzanov 08

- combinatorial O(MF(n, m, C) n^2 (log n)^2 log C) time

Result

O(m (log k) MSF(n, m,1)) time algorithm
to find half-integral primal & dual opt.

MSF(n,m,h): max. submodular flow on network of
n nodes, m edges,
time complexity h of computing exchange capacity

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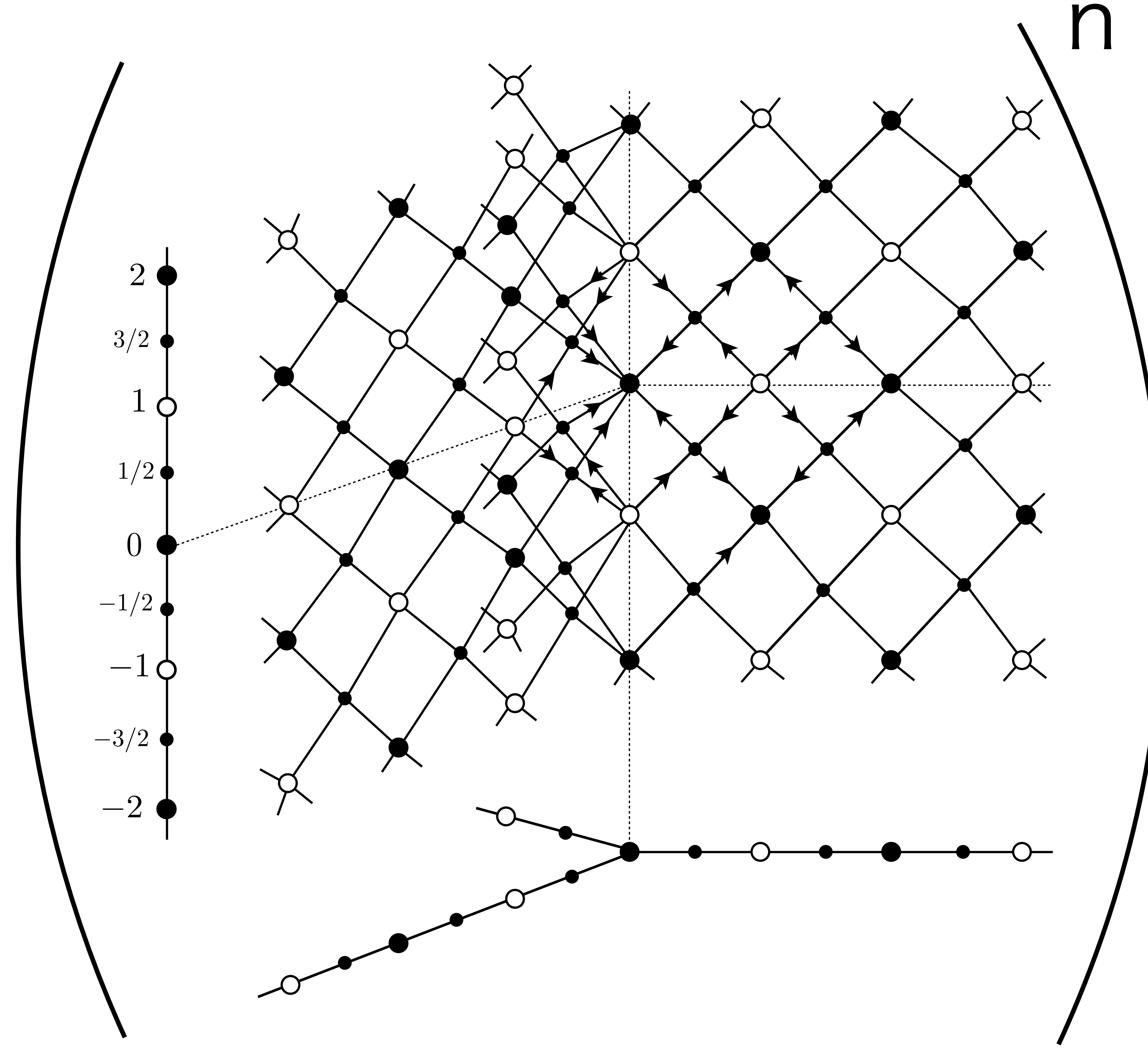
MSF(n,m,h): max. submodular flow on network of
n nodes, m edges,
time complexity h of computing exchange capacity

Fujishige, Zhang 92: O(n^3 h)
—> O(m n^3 log k)
—> 2-approx. of node-multiway cut in O(m n^3 log k) time

c.f. Madan-Chekuri 15: (2+ε)-approx. in $\tilde{O}$(mn/ε^2) time

Sketch

Dual objective is again “L-convex” on



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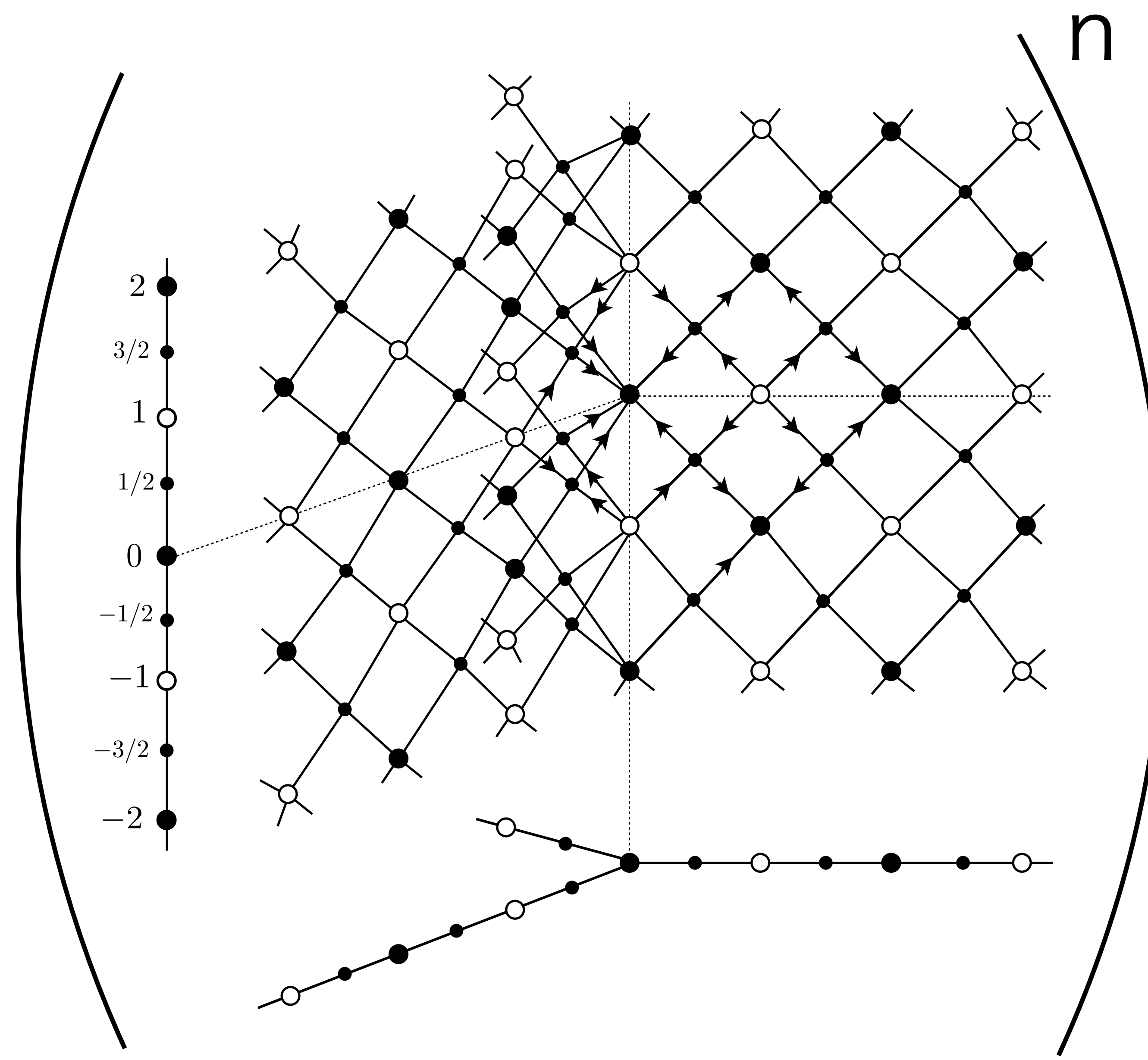

- Steepest descent algo.
- l_∞-bound
- Steepest direction

  <— Max. subflow
- Dual opt —> primal opt

  subflow feasibility

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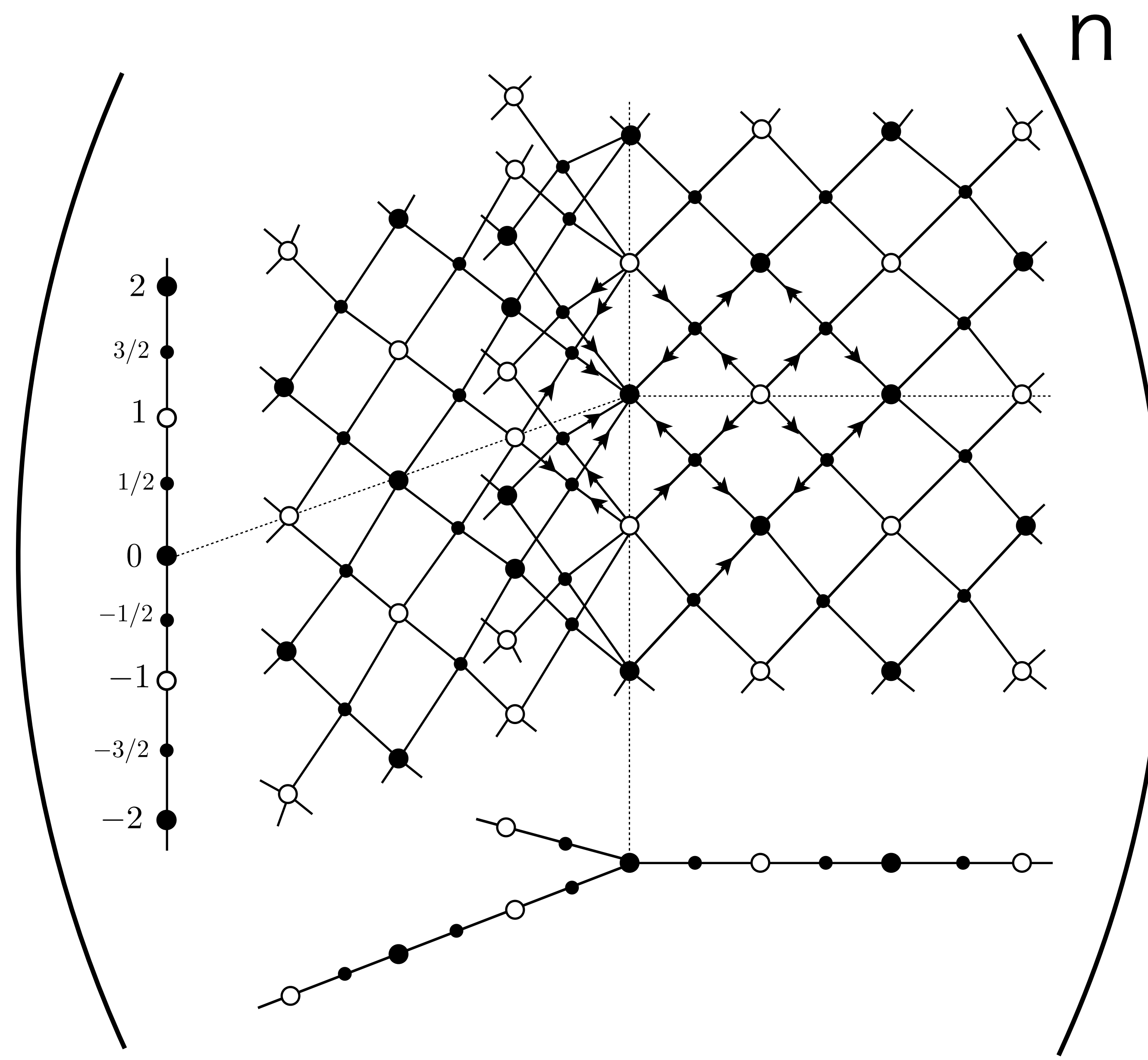

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  <— Max. subflow
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  subflow feasibility

perturbation

bisubflow—>subflow

- H. Hirai: L-extendable functions and a proximity scaling algorithm for minimum cost multiflow problem, Discrete Optimization, 18 (2015), 1-37.
- H. Hirai: A dual descent algorithm for node-capacitated multiflow problems and its applications, 2015, arXiv:1508.07065.

Thank you for your attention !

http://www.misojiro.t.u-tokyo.ac.jp/~hirai/