Discrete Convexity and Polynomial Solvability in Minimum 0-Extension Problems

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Combinatorial Geometries
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   - Discrete convex analysis (Murota 1996 $\sim$)
   - Valued CSP & fractional polymorphism (Thapper-Živný 2012)

Multifacility location problem (70's $\sim$)

Γ: undirected graph
d_Γ: path-metric on V_Γ
$y_1, y_2, \ldots, y_k$: existing facilities on Γ

Locate new facilities $x_1, x_2, \ldots, x_n$ on Γ such that
the mutual *communication cost* between facilities

$$\underbrace{\sum_{i,j} b_{ij} d_\Gamma(x_i, y_j)}_{\text{cost between new and old}} + \underbrace{\sum_{i<j} c_{ij} d_\Gamma(x_i, x_j)}_{\text{cost between new facilities}}$$

is minimum.

Recent applications: clustering, learning theory, computer vision, ...
c.f. *Metric Labeling* (Kleinberg-Tardos 1998)

$V_\Gamma = \{v_1, v_2, \ldots, v_k\}$
Input: number n of variables, $b_{ij}, c_{ij} \geq 0$

$$\textbf{MultiFac}[\Gamma]: \quad \textit{Min.} \quad \sum_{i,j} b_{ij} d_\Gamma(x_i, v_j) + \sum_{i<j} c_{ij} d_\Gamma(x_i, x_j)$$
$$\textit{s.t.} \quad (x_1, x_2, \ldots, x_n) \in V_\Gamma \times V_\Gamma \times \cdots \times V_\Gamma$$

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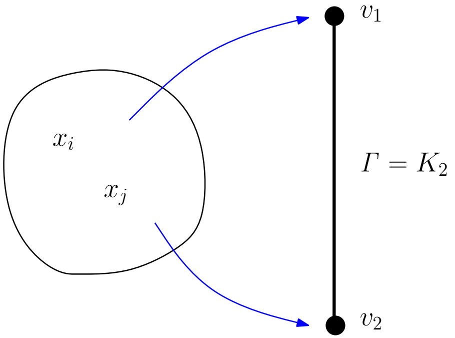
Ex. $\Gamma = K_2$



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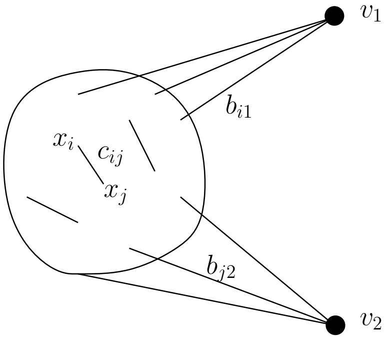

MultiFac$[K_2] \simeq$ Minimum (v_1, v_2)-cut

Tractability Question (Chepoi 1996, Karzanov 1998, 2004)

$\mathit{\Gamma} = K_2 \Leftrightarrow$ minimum cut $\Rightarrow$ P (Ford-Fulkerson 1956)

$\mathit{\Gamma} = K_{n \geq 3} \Leftrightarrow$ multi-termical cut $\Rightarrow$ NP-hard

(Dahlhaus, Johnson, Papadimitriou, Seymour, Yannakakis 1994)

Question

What is $\mathit{\Gamma}$ for which **MultiFac**$[\mathit{\Gamma}]$ *is in P ?*

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What is Γ for which **MultiFac**$[\Gamma]$ *is in P ?*

Known tractable class:

- Trees (Picard-Ratliff 1978)
- Median graphs (Chepoi 1996)
- Frames (Karzanov 1998)

Theorem (Picard-Ratliff 1978)

If Γ is a tree, then **MultiFac**$[\Gamma]$ is in P.

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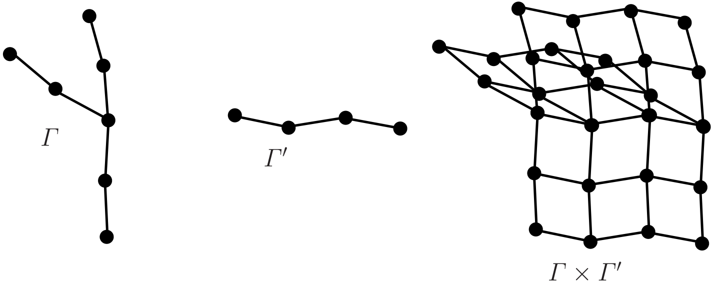
Obs. $\Gamma \in$ P and $\Gamma' \in$ P $\Rightarrow \Gamma \times \Gamma' \in$ P

$\because d_{\Gamma\times\Gamma'}((x,x'),(y,y')) = d_\Gamma(x,y) + d_{\Gamma'}(x',y')$

$\rightarrow$ **MultiFac**$[\Gamma \times \Gamma']$ = **MultiFac**$[\Gamma]$ + **MultiFac**$[\Gamma']$



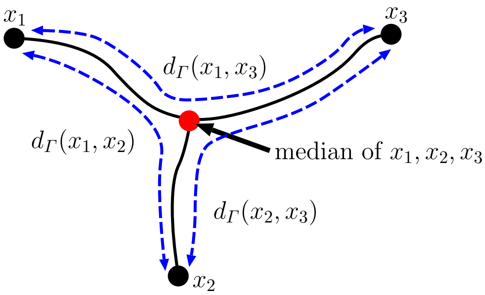
Cor. cube, grid graph, product of trees $\in$ P

Median graph

- **Median** of $x_1, x_2, x_3 \Leftrightarrow y \in V_\Gamma$ satisfying

$$d_\Gamma(x_i, x_j) = d_\Gamma(x_i, y) + d_\Gamma(y, x_j) \quad (1 \leq i < j \leq 3)$$

- **Median graph** $\Leftrightarrow \forall$ triple has a *unique* median.



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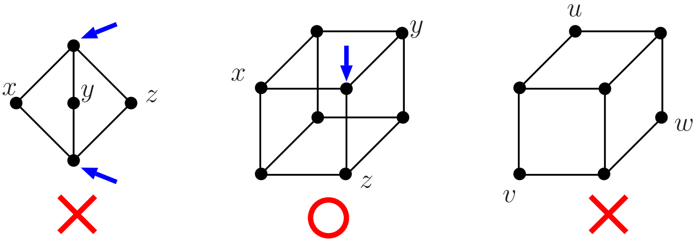

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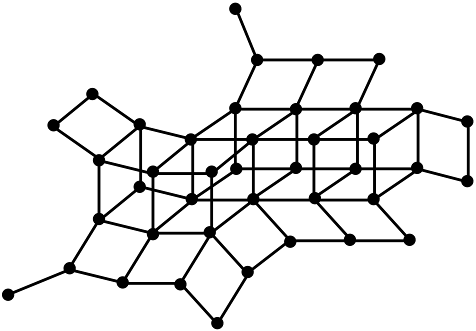

c.f. median graph $\simeq$ graph of CAT(0) cube complex (Chepoi 2000)

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- **Median graph** $\Leftrightarrow \forall$ triple has a *unique* median.

Theorem (Chepoi 1996)

If Γ is a median graph, then **MultiFac**$[\Gamma]$ is in P.

$\because$ use of minimum cut

0-extension formulation (Karzanov 1998)

(V, μ): finite metric space

Def: *extension* of (V, μ)
$\Leftrightarrow$ metric space (X, d) s.t. $X \supseteq V$ and $d|_V = \mu$

Def: 0-*extension* of (V, μ)
$\Leftrightarrow$ extension (X, d) s.t. $\forall x \in X, \exists s \in V : d(x, s) = 0$.
$\Leftrightarrow$ $d = \mu \circ \rho$ for some retraction $\rho : X \to V$.

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Obs. **MultfiFac**$[\Gamma] \simeq$

$$\textbf{0-Ext}[\Gamma]: \quad Min. \quad \sum_{x,y} c_{xy} d(x, y)$$
$$s.t. \quad (X, d) \text{ is a 0-extension of } (V_\Gamma, d_\Gamma)$$

$\because (\{v_1, v_2, \ldots, v_k\}, d_\Gamma) \to (\{v_1, v_2, \ldots, v_k, x_1, x_2, \ldots, x_n\}, d)$

Metric-LP relaxation (Karzanov 98)

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Metric-LP relaxation (Karzanov 98)

$$\begin{aligned} \textbf{Ext}[\Gamma]: \qquad \text{Min.} \quad & \sum_{xy} c_{xy} d(x,y) \\ \text{s.t.} \quad & d(x,x) = 0 \\ & d(x,y) = d(y,x) \geq 0 \\ & d(x,y) + d(y,z) \geq d(x,z) \\ & d|_{V_\Gamma} = d_\Gamma \end{aligned}$$

Rem: $\textbf{Ext}[\Gamma]$ is polysize LP $\rightarrow$ P

Metric-LP relaxation (Karzanov 98)

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Rem: $\textbf{Ext}[\Gamma]$ is polysize LP $\to$ P

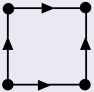
Q. *What is Γ for which* $\textbf{Ext}[\Gamma]$ *is exact ?*

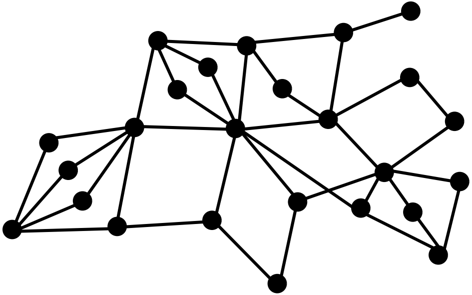
c.f. LP-dual of multicommodity flow (Karzanov 1998, H. 2009 $\sim$)

Frame = graph for which **Ext**[Γ] is exact

Γ: **frame** $\Leftrightarrow$

- bipartite
- no isometric cycle of length > 4
- *orientable* $\Leftrightarrow \exists$ orientation: $\forall$ 4-cycle





Rem: frame is obtained by *gluing* $K_{2,m}$ and K_2 (in a certain way)

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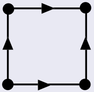

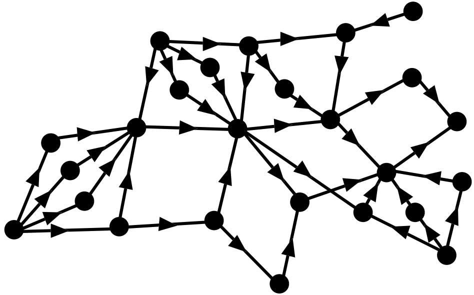

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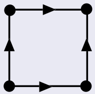
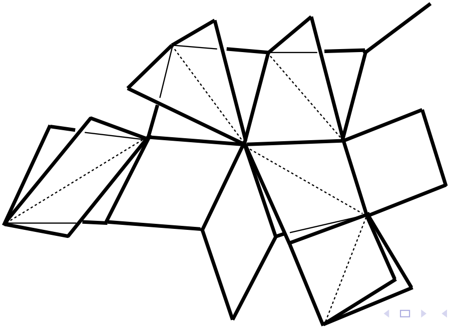


c.f. frame $\simeq$ CAT(0)-complex of *folders* (Chepoi 2000)



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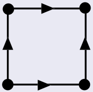


Theorem (Karzanov 1998)

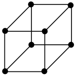
Γ is a frame if and only if **Ext**$[\Gamma]$ = **0-Ext**$[\Gamma]$.

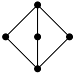
Corollary (Karzanov 1998)

If Γ is a frame, then **MultiFac**$[\Gamma]$ (= **0-Ext**$[\Gamma]$) is in P.

Rem. $\{\text{frames}\}$ is not closed under product

Rem. frame $\neq$ median graph

 $\in$ $\{\text{median graphs}\}$, $\notin$ $\{\text{frames}\}$

 $\notin$ $\{\text{median graphs}\}$, $\in$ $\{\text{frames}\}$

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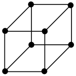
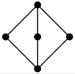
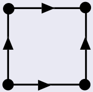
 $\in$ {median graphs}, $\notin$ {frames}

 $\notin$ {median graphs}, $\in$ {frames}

- Γ: *modular* $\Leftrightarrow$ $\forall$ triple has a median.
- Γ: *orientable* $\Leftrightarrow$ $\exists$ orientation: $\forall$ 4-cycle 

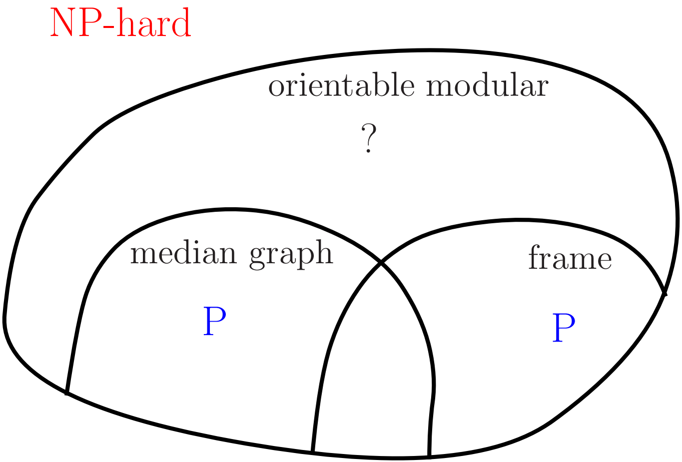
Rem. frame = orientable *hereditary* modular graph (c.f. Bandelt 85)

Ex. Hasse diagram of modular lattice

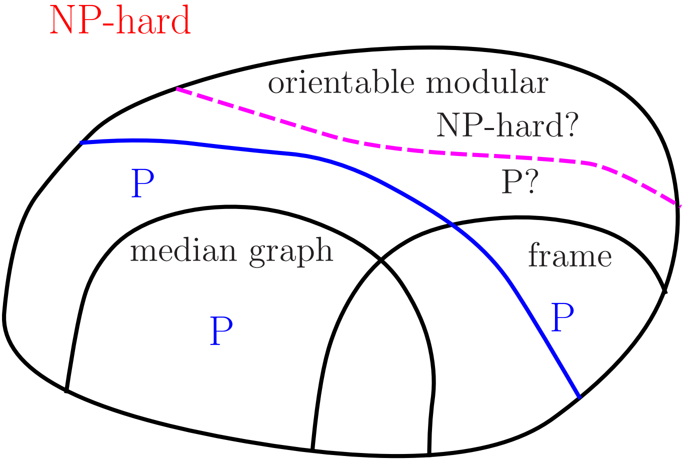
Hardness result

Theorem (Karzanov 1998)

If Γ is not modular or not orientable, **MultiFac**$[\Gamma]$ is NP-hard.



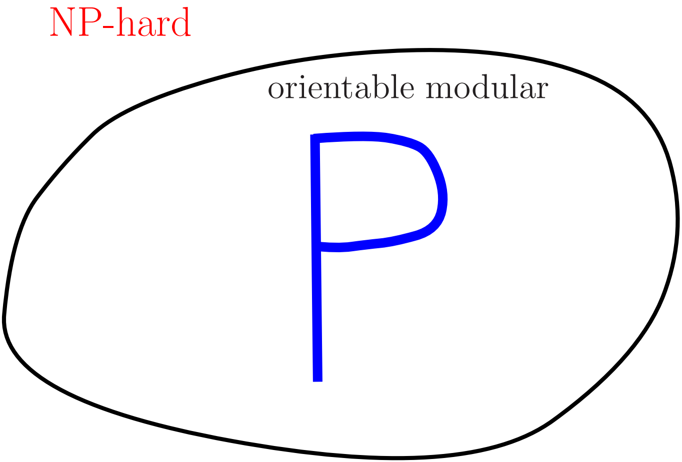
Partial result & conjectures (Karzanov 2004)



Main result

Theorem (H. 2012; SODA'13)

If Γ is orientable modular, then **MultiFac**$[\Gamma]$ is in P.



Proof idea and outline

$$\textbf{MultiFac}[\Gamma]: \quad \textit{Min.} \quad \sum_{i,j} b_{ij} d_\Gamma(x_i, v_j) + \sum_{i<j} c_{ij} d_\Gamma(x_i, v_j)$$
$$\text{s.t.} \quad (x_1, x_2, \ldots, x_n) \in V_\Gamma \times V_\Gamma \times \cdots \times V_\Gamma$$

Basic idea $\sim$ *formulate* **MultiFac** *as a 'convex' optimization over a simplicial complex assoc													itated with orientable modular graph, based on a philosophy of Discrete Convex Analysis.*

A theory of 'convex' functions on $\mathbf{Z}^n = \mathbf{Z} \times \mathbf{Z} \times \cdots \times \mathbf{Z}$ *for well-solvable combinatorial optimization problems including network flows, matroids, and submodular functions.*

Obs. If $\Gamma =$ path, then $\Gamma \times \Gamma \times \cdots \times \Gamma \simeq$ box B in $\mathbf{Z}^n$,

$$\begin{aligned} \textit{Min.} \quad & \sum_{i,j} b_{ij} d_\Gamma(x_i, v_j) + \sum_{i<j} c_{ij} d_\Gamma(x_i, x_j) \\ \textit{s.t.} \quad & (x_1, x_2, \ldots, x_n) \in V_\Gamma \times V_\Gamma \times \cdots \times V_\Gamma \end{aligned}$$

Discrete Convex Analysis (Murota 1996 $\sim$)

A theory of 'convex' functions on $\mathbf{Z}^n = \mathbf{Z} \times \mathbf{Z} \times \cdots \times \mathbf{Z}$ *for well-solvable combinatorial optimization problems including network flows, matroids, and submodular functions.*

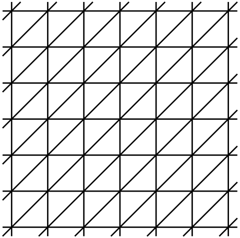
Obs. If $\Gamma =$ path, then $\Gamma \times \Gamma \times \cdots \times \Gamma \simeq$ box B in $\mathbf{Z}^n$,

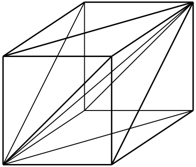
$$\begin{array}{ll} \textit{Min.} & \sum_{i,j} b_{ij}|x_i - v_j| + \sum_{i<j} c_{ij}|x_i - x_j| \\ \textit{s.t.} & (x_1, x_2, \ldots, x_n) \in B \subseteq \mathbf{Z}^n \end{array}$$

$\rightarrow$ *This is an L-convex function minimization in DCA*

L-convex functions (for combinatorial geometers)

Triangulation Δ of $\mathbf{Z}^n$ diced by hyperplanes of normal e_i, $e_i - e_j$





Def: A function g on $\mathbf{Z}^n$ is L-convex
$\Leftrightarrow \underbrace{\text{piecewise-linear interpolation of } g \text{ w.r.t. } \Delta}_{\textit{Lovász extension of } g}$ is convex in $\mathbf{R}^n$.

- L-convex function is minimized by tracing the 1-skeleton of Δ
- Descent direction is found by *Submodular Function Minimization*

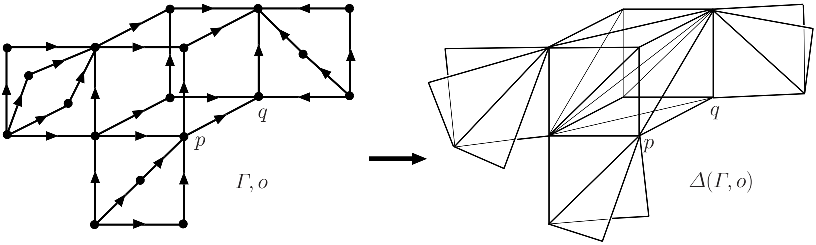
$f : \{0,1\}^n \to \mathbf{R}$ is submodular if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$

We develop an analogous theory for orientable modular graph Γ (with orientation o)

Fact: o.m.graph is obtained by *gluing geometric modular lattices*

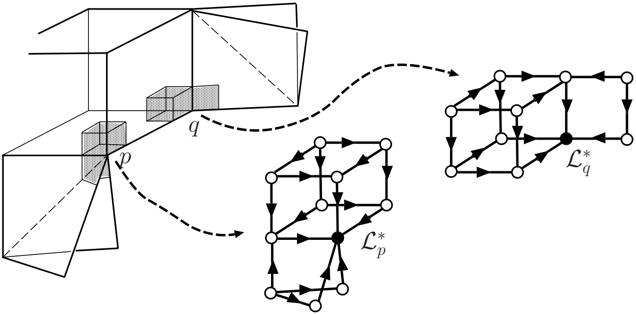
- Median graph $\leftrightarrow$ Boolean lattices
- Frame $\leftrightarrow$ diamonds (geometric modular lattices of rank 2)

Def: *Modular complex* $\Delta(\Gamma)$
$:=$ union of order complexes of these lattices



$g : V_\Gamma \to \mathbf{R} \Longrightarrow$ piecewise-linear interpolation $\bar{g} : \Delta(\Gamma) \to \mathbf{R}$

Neighborhood $\mathcal{L}_p^*$ of $p \in V_\Gamma$ in $\Delta(\Gamma)$
$\simeq$ *modular semilattice* (Bandelt, Van de Vel, Verheul 1993)



- define *submodular function* on modular semilattice [Appendix 1]
- $g : V_\Gamma \to \mathbf{R}$ is L-convex if $\bar{g}$ is submodular on $\mathcal{L}_p^*$ $(\forall p \in V_\Gamma)$

- L-convex function is minimized by tracing 1-skeleton of $\varDelta$.
- Descent direction is found by SFM on a modular semilattice.
- $d_{\varGamma}$ is L-convex on $\varGamma \times \varGamma$.
- **MultiFac**$[\varGamma]$ is L-convex minimization on $\varGamma \times \varGamma \times \cdots \times \varGamma$.

I don't know whether SFM in P. But ...

- L-convex function is minimized by tracing 1-skeleton of Δ.
- Descent direction is found by SFM on a modular semilattice.
- d_Γ is L-convex on $\Gamma \times \Gamma$.
- **MultiFac**$[\Gamma]$ is L-convex minimization on $\Gamma \times \Gamma \times \cdots \times \Gamma$.

I don't know whether SFM in P. But ...
Thapper-Živný fractional polymorphism criterion in Valued-CSP (FOCS'12) is applicable. [Appendix 2]

Minimizing a sum of submodular functions of *bounded arity* in P

Our problem is minimizing a sum of arity-2 submodular functions.

$\rightarrow$ **MultiFac**$[\Gamma]$ in P

Summary

Question

What is Γ for which **MultiFac**$[\Gamma]$ *is in P ?*

Answer

$$\textbf{MultiFac}[\Gamma] \in \begin{cases} \text{P} & \text{if } \Gamma \text{ is } \textbf{orientable modular} \\ \text{NP-hard} & \text{otherwise} \end{cases}$$

Many interesting aspects:

- Submodularity/L-convexity for multicommodity flows
  $\rightarrow$ multiflow analogue of [Max-flow $=$ Min-cut $\in$ SFM]
- Modular lattices, and modular semilattices
- Dichotomy theorem in Valued-CSP
- Optimization theory on CAT(0)-complex ??

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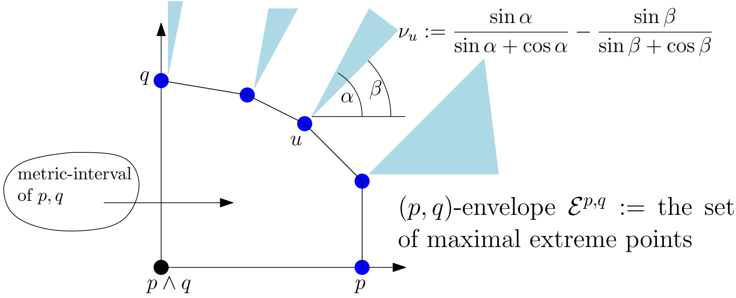
Thank you for your attention !

Appendix 1: Modular semilattice & submodular function

Meet-semilattice $\mathcal{L}$ is **modular** $\Leftrightarrow$

- every lower ideal is a modular lattice
- $u \vee v, v \vee w, w \vee u \in \mathcal{L} \Rightarrow u \vee v \vee w \in \mathcal{L}$

- *We can define a fractional join* $\sum\{\nu_u u \mid u \in \mathcal{E}^{p,q}\}$.



Def: f is submodular if $f(p) + f(q) \geq f(p \wedge q) + \sum_{u \in \mathcal{E}^{p,q}} \nu_u f(u)$

Appendix 2: Valued-CSP & fractional polymorphism

Λ: a set of functions f on D with arity K

$$\textbf{VCSP}[\Lambda]: \quad \text{Min.} \quad \sum_{f \in \Lambda; i_1, i_2, \cdots, i_K} c^f_{i_1, i_2, \ldots, i_K} f(x_{i_1}, x_{i_2}, \ldots, x_{i_K})$$
$$\text{s.t.} \quad (x_1, x_2, \ldots, x_N) \in D \times D \times \cdots \times D$$

$\simeq$ Minimization of a sum of functions with *bounded arity*

- Input size $= O(|\Lambda||D|^K)$
- **VCSP** has polysize IP formulation $\rightarrow$ Basic LP-relaxation

Theorem (Thapper-Živný, FOCS'12)

$\mathbf{VCSP}[\Lambda]$ = Basic LP $\in$ P
if Λ has a fractional polymorphism $\ni$ semilattice operation

Def: *fractional polymorphism* for Λ
$\Leftrightarrow$ probability ω on the *space of operations* $\{g : D \times D \to D\}$:

$$(f(p) + f(q))/2 \geq \sum_g \omega(g) f(g(p, q)) \quad (\forall f \in \Lambda, \forall (p, q)).$$