Computing the nc-rank via discrete convex optimization on CAT(0) spaces

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Edmonds Problem Edmonds 1967

Can we compute the rank of

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

in polynomial time ?

x_i : variables, A_i: $n \times n$ matrices over field $\mathbb{K}$

A: matrix over $\mathbb{K}[x_1, x_2, \ldots, x_m] \subset \mathbb{K}(x_1, x_2, \ldots, x_m)$

- RP, but P ? (for large field)
- Related to fundamental problems in diverse areas
  ～ combinatorial optimization, rigidity theory, TCS,…

Non-commutative Edmonds Problem

Ivanyos-Qiao-Subrahmanyam 2017

Can we compute the rank ***(nc-rank)*** of

$$A = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m$$

in polynomial time ?

x_i : noncommutative variables, A_i: matrices over field $\mathbb{K}$

A: matrix over free ring $\mathbb{K}\langle x_1, x_2, \ldots, x_m \rangle$

$\cap$

free skew field $\mathbb{K}(\langle x_1, x_2, \ldots, x_m \rangle)$

- Broaden the literature:
  Noncommutative algebra, Invariant theory, …

Null cone membership for left-right action
$GL_n \times GL_n \ni (S, T) \mapsto (SA_k T)_{k=1,2,\ldots,m}$

nc-rank in **P**

- Garg-Gurvits-Oliveira-Wigderson 2019 : $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

  Gurvits' operator scaling $\approx$ alternating minimization for a *geodesically convex* function on symmetric space GL_n/O_n

  nc-nonsingularity $\Leftrightarrow$ $\inf \left\{ \log \frac{\det \sum A_i X A_i^T}{\det X} : X \succ 0 \right\} > -\infty$

  → *Noncommutative optimization* (Bürgisser et al.)

- Ivanyos-Qiao-Subrahmanyam 2018: $\mathbb{K}$ arbitrary

  Wong sequence --- vector-space analogue of augmenting path

  Matrix completion by cyclic division algebra

Our contribution [Hamada-H 2020]

A different polytime algorithm to compute nc-rank

Features / techniques:

- Submodular optimization on modular lattice

$$f(p) + f(q) \geq f(p \wedge q) + f(p \vee q)$$

- Geodesically convex optimization on CAT(0) space

non-manifold

For nc-rank over $\mathbb{Q}$ (to avoid bit-complexity issue):

- reduction to nc-rank over $\mathrm{GF}(p)$ by p-adic valuation

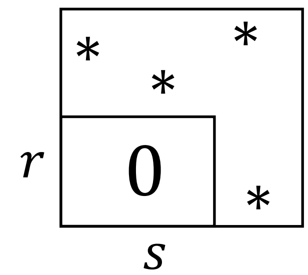
→ discrete convex optimization

on Euclidean building for $GL_n(\mathbb{Q})$

Submodularity in nc-rank

Thm (Fortin-Reutenauer 2004)

$$\text{nc-rank} \sum_k A_k x_k = 2n - \text{Max. } r + s$$

s.t. $PA_kQ =$ 

$(\forall k)$

$P, Q \in \mathrm{GL}_n(\mathbb{K})$

$$\text{Max. } \dim X + \dim Y \quad \text{s.t. } A_k(X, Y) = \{0\} \ (\forall k)$$

$X, Y \subseteq \mathbb{K}^n$ vector subspaces

where $A_k(x, y) \coloneqq x^T A_k y$

Max. $\dim X + \dim Y$ s.t. $A_k(X,Y) = 0\ (\forall k)$

$X, Y \subseteq \mathbb{K}^n$ vector subspaces

$2n+1$

$\cong$

Min. $-\dim X - \dim Y + C \sum_k \operatorname{rank} A_k|_{X \times Y}$

s.t. $(X,Y) \in \mathcal{L} \times \mathcal{L}$

$\mathcal{L}$: the modular lattice of all vector subspaces of $\mathbb{K}^n$

partial order reversed

Lem: The objective function is submodular in $\mathcal{L} \times \mathcal{L}^*$

$$f(X,Y) + f(X',Y') \geq f(X+X', Y \cap Y') + f(X \cap X', Y + Y')$$

$\wedge = (+, \cap),\ \vee = (\cap, +)$

Our approach to solve SFM for nc-rank

$$\text{Min.}\quad -\dim X - \dim Y + C \textstyle\sum_i \operatorname{rank} A_i|_{X\times Y}$$

$$\text{s.t.}\ (X, Y) \in \mathcal{M} = \mathcal{L} \times \mathcal{L}^*$$

- Continuous & convex relaxation to *CAT(0) space*

$\mathcal{M} \overset{\text{orthoscheme complex}}{\Longrightarrow} K(\mathcal{M})$, a CAT(0) space

$f \overset{\text{Lovász extension}}{\Longrightarrow} \bar{f}: K(\mathcal{M}) \to \mathbb{R}$, geodesically convex

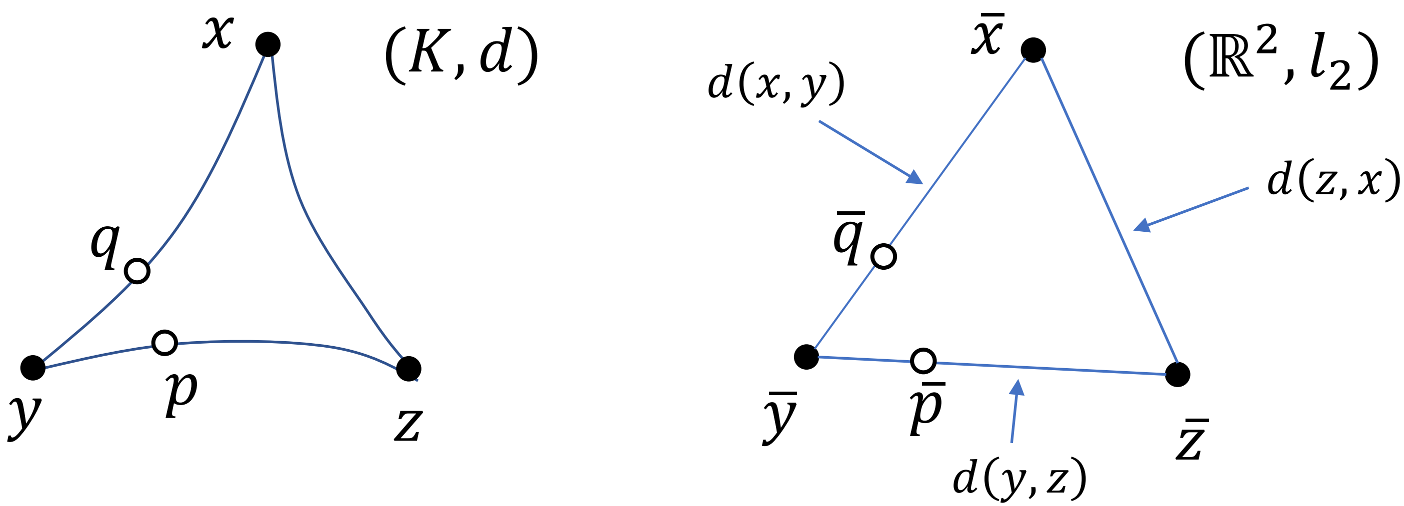
- Minimize $\bar{f}$ by an optimization algorithm on CAT(0) space

Inspired by the classical SFM [Lovász-Grötschel-Schrijver] :

- $f: \{0,1\}^n \to \mathbb{R}$, submodular $\Rightarrow$ $\bar{f}: [0,1]^n \to \mathbb{R}$, convex
- Minimize $\bar{f}$ by ellipsoid method

CAT(0) space (Gromov 1987)

A geodesic metric space (K, d) s.t. every triangle is "slimmer"



CAT(0)-inequality: $d(p, q) \leq \|\bar{p} - \bar{q}\|_2$

Ex: $\mathbb{R}^n$, hyperbolic space, symmetric space of noncompact type

tree, Euclidean building, …

Modeling & Algorithm & Optimization on CAT(0) spaces

Fact: CAT(0) space is uniquely geodesic

→ Convexity is naturally defined

- The space of phylogenetic trees (BHV-tree space) [Billera et al 2001]
- Configuration spaces of robots [Abram &Ghrist 2004]
- Polynomial-time geodesic computation

  for BHV-tree space [Owen 2011]

  for CAT(0) cubical complex [Hayashi 2021]
- Proximal point algorithm for convex optimization [Bačák 2014]
- Combinatorially-defined CAT(0) spaces:

  median graph → CAT(0) cubical complex [Chepoi 2000]

  poset / lattice → orthoscheme complex [Brady-McCammond 2012]

Orthoscheme Complex Brady-McCammond 2012

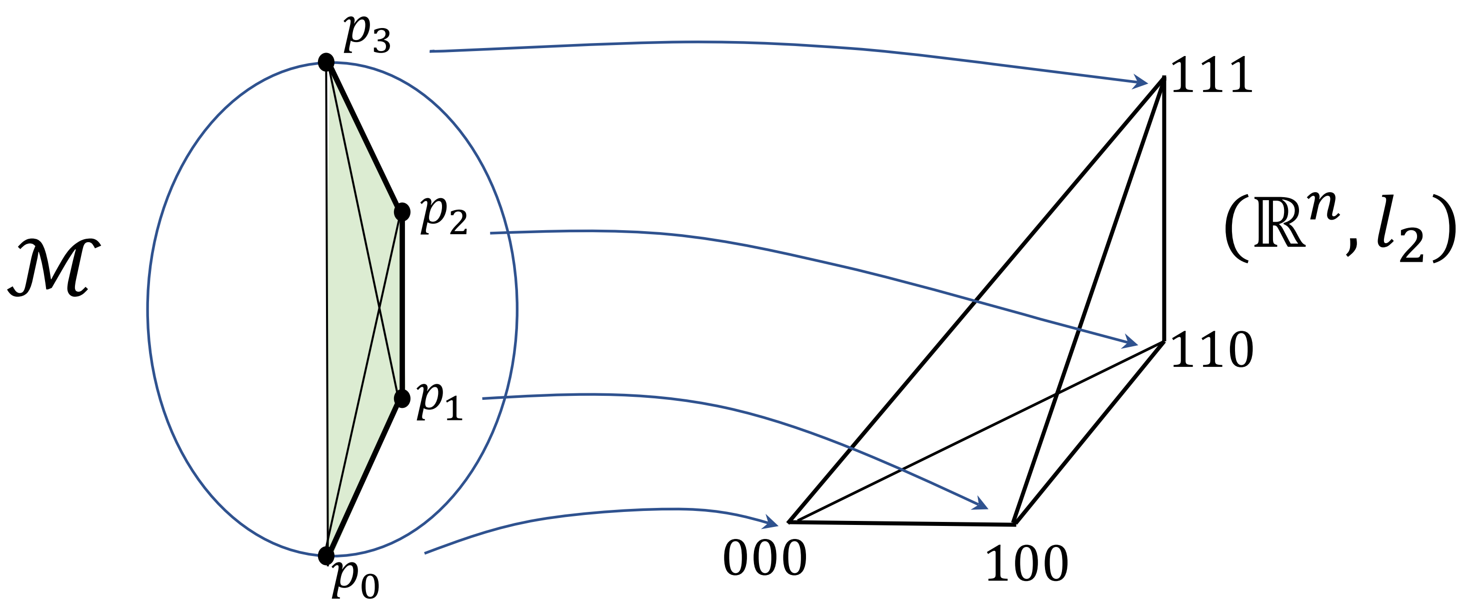
$\mathcal{M}$: graded poset

formal convex combination $\sum_i \mu_i = 1, \mu_i \geq 0$

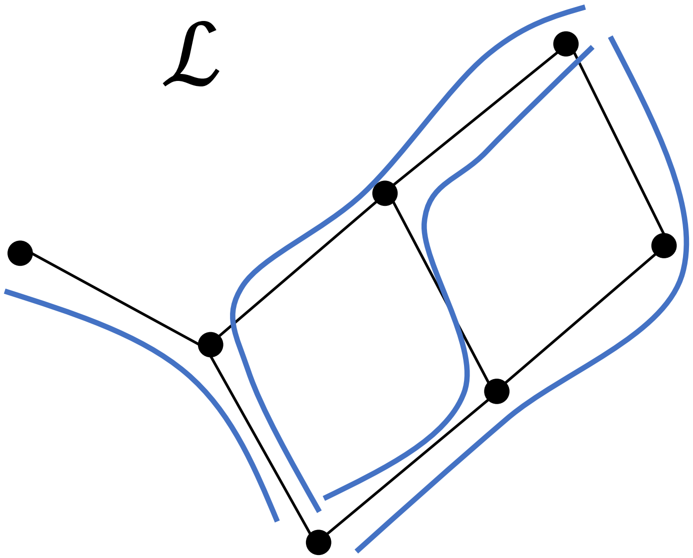
$K(\mathcal{M}) \coloneqq \{ \ \sum_i \mu_i p_i \ | \ p_0 \prec p_1 \prec \cdots \prec p_n, \ \ p_i \in \mathcal{M} \ \}$ = the order complex of $\mathcal{M}$

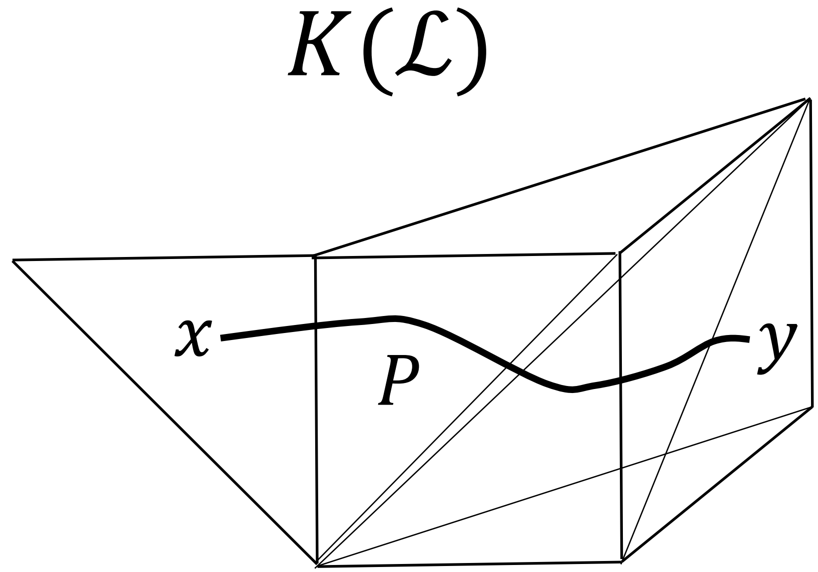
metrized by filling *orthoscheme* to each simplex:

$$\sum_i \mu_i p_i \longmapsto \sum_i \mu_i (\overbrace{1, \dots, 1}^{i}, 0, \dots, 0)$$



Orthoscheme = conv $\{(0,0,\dots,0), (1,0,\dots,0), (1,1,0,\dots,0), \dots, (1,1,1,\dots,1)\}$





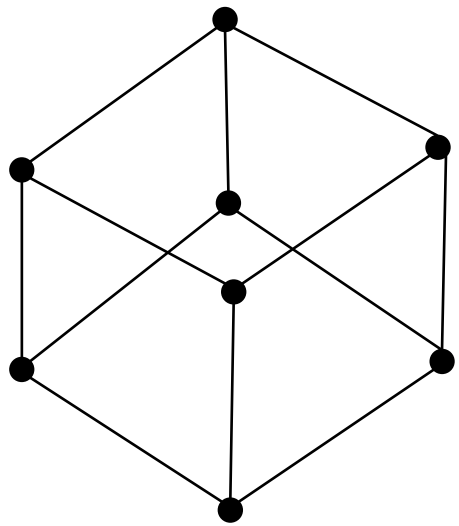
The length $d(P)$ of a path $P: [0,1] \to K(\mathcal{L})$ is well-defined

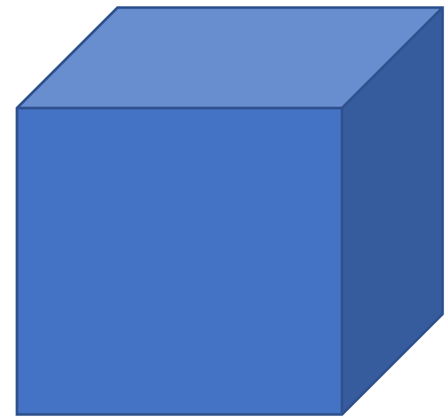
$$d(x, y) := \inf\{d(P) \mid P: (x, y)\text{–path}\}$$

$\to (K(\mathcal{L}), d)$ is a geodesic metric space

Question [Brady-McCammond 2012]:
Which is the class of posets $\mathcal{L}$ having CAT(0) $K(\mathcal{L})$?

Boolean Lattice





$\mathcal{L} = \{0,1\}^n$

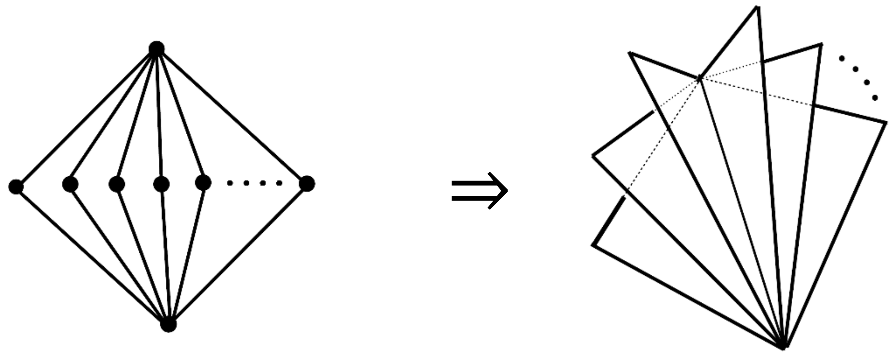
$K(\mathcal{L}) \simeq [0,1]^n$

$\rightarrow$ CAT(0)

$\mathcal{L}$: distributive lattice → $K(\mathcal{L})$: order polytope, CAT(0)

Thm [Haettel-Kielak-Schwer 2015, Chalopin-Chepoi-H-Osajda 2020]

$\mathcal{M}$: modular lattice $\Rightarrow K(\mathcal{M})$ is CAT(0)



$\mathcal{M}$: modular lattice of subspaces of $\mathbb{K}^2$

$K(\mathcal{M})$

Thm [H.2018] $\mathcal{M}$: modular lattice

$f: \mathcal{M} \to \mathbb{R}$ is submodular $\Leftrightarrow$ $\bar{f}: K(\mathcal{M}) \to \mathbb{R}$ is convex

$$\bar{f}(x) := \sum_i \mu_i f(p_i) \quad \left(x = \sum_i \mu_i p_i \, ; \, p_0 \prec p_1 \prec \cdots \prec p_n, \right)$$

Classical Lovász extension: $f: \{0,1\}^n \to \mathbb{R}$ is submodular $\Leftrightarrow \bar{f}: [0,1]^n \to \mathbb{R}$ is convex

$$\text{Min.}\quad -\dim X - \dim Y + C\textstyle\sum_i \operatorname{rank} A_i|_{X\times Y}$$

$$\text{s.t.}\quad (X,Y) \in \mathcal{L}\times\mathcal{L}^*$$

$\Downarrow$ Lovász extension

$$\text{Min.}\quad -\overline{\dim}(x) - \overline{\dim}\,(y) + C\textstyle\sum_i \overline{\operatorname{rank} A_i}(x,y)$$

$$\text{s.t.}\quad (x,y) \in K(\mathcal{L}\times\mathcal{L}^*) = K(\mathcal{L})\times K(\mathcal{L}^*)$$

→ convex optimization on CAT(0) space $K(\mathcal{L}\times\mathcal{L}^*)$

but we have no powerful method such as ellipsoid method

Feature: the objective is a sum of convex functions

→ splitting proximal point algorithm

Splitting Proximal Point Algorithm Bačák 2014

(K, d): complete CAT(0) space, $f_1, f_2, \dots, f_m$: convex func on K

Goal: Minimize $f = \sum_{i=1}^{m} f_i$

SPPA: Iterate

$$z^{k+1} \longleftarrow \operatorname{argmin}_{z \in K} \; f_{k \bmod m}(z) + \frac{1}{\lambda_k} d(z, z^k)^2$$

Thm (Ohta-Pálfia 2015)

f_i: L-Lipschitz, f: ε-strongly convex, $\lambda_k := \frac{1}{2\varepsilon(k+1)}$

$$\Rightarrow \qquad f(z^k) - \min f \le \frac{\mathrm{poly}(L, m, \varepsilon^{-1}, \mathrm{diam}\, K)}{\sqrt{k}}$$

$$\text{Min.} \quad -\overline{\dim}(x) - \overline{\dim}\,(y) + C\sum_i \overline{\operatorname{rank} A_i}(x,y)$$

+ Perturbation term for ε-strong convexity

$$\text{s.t.} \quad (x,y) \in K(\mathcal{L}\times\mathcal{L}^*) = K(\mathcal{L}) \times K(\mathcal{L}^*)$$

- Apply SPPA with:

  $f_0(x,y) \coloneqq -\overline{\dim}(x) - \overline{\dim}(y)$

  $f_i(x,y) \coloneqq C\,\overline{\operatorname{rank} A_i}(x,y) \quad (i = 1,2,\dots,m)$

- After $k = \operatorname{poly}(n,m)$ iterations, we have $f\left(x^k, y^k\right) - \mathrm{opt} < 1/2$

- By integrality of f, an optimum (X^*, Y^*) exists

  in the support of $\left(x^k, y^k\right) = \sum_i \mu_i (X_i, Y_i)$

Main technical contribution

Thm: SPAA is implementable in polytime, i.e.,

$$\text{Min. } f_i(x,y) + \frac{1}{\lambda} d((x,y),(x_0,y_0))^2$$

$$\text{s.t. } (x,y) \in K(\mathcal{L} \times \mathcal{L}^*)$$

where $f_i(x,y) = -\overline{\dim}(x) - \overline{\dim}(y)$ or

$f_i(x,y) = C\,\overline{\operatorname{rank} A_i}(x,y)$

is solvable in polytime.

Proof: Lattice theoretic / building theoretic argument

- An optimum exists in an *apartment* = the subcomplex of a Boolean sublattice including the supports of $x_0, y_0, x_0^{\perp_{A_i}}, y_0^{\perp_{A_i}}$
- In the apartment $\approx [0,1]^{2n}$, the problem is an easy convex quadratic program

Good / bad points

- Conceptually simple, described uniformly in an arbitrary field
- Applications to other problems ?

- Polytime, but very slow:

  ← Less use of characteristics of the objective function

  ← Undevelopment of optimization theory in CAT(0) spaces

- Bit complexity problem for $\mathbb{K} = \mathbb{Q}$:

  ← Polynomial number of vector-space operations $X \cap X', X + X'$

  can cause exponential bit explosion of bases

Reduction of nc-singularity over $\mathbb{Q}$ to that over $\mathbb{F}_p$ by p-adic valuation v_p & discrete convex optimization on Euclidean building

$$\begin{aligned} \text{Min.} \quad & v_p \det P + v_p \det Q \\ \text{s.t.} \quad & v_p(PA_kQ)_{ij} \geq 0 \quad (\forall ij, k), \\ & P, Q \in GL_n(\mathbb{Q}) \end{aligned}$$

- Nc-singular $\Leftrightarrow$ optimal value $= -\infty$

$$z = \underbrace{1011\,\overbrace{000}^{v_2(z)}}_{bitsize}$$

- opt $> -\infty \Rightarrow$ opt $> -\text{poly (input)}$ ← $v_p(|z|) \leq$ bit-size of z for $0 \neq z \in \mathbb{Z}$
- *L-convex function* minimization on Euclidean building for $GL_n(\mathbb{Q})$
- Descent algorithm [H.2019], where each step is nc-rank computation over $\mathbb{F}_p$

Application to related problems

(p, q)-scalability of $\{A_k\}$ by triangular matrices [Franks 2018]

$$\text{Max.} \quad \sum_i (p_i - p_{i+1}) \dim E_i \cap X + \sum_i (q_i - q_{i+1}) \dim F_i \cap Y$$

E_i, F_i:standard flags

$$\text{s.t.} \quad A_k(X, Y) = \{0\} \quad (\forall k)$$

X, Y: vector subspaces

Brascamp-Lieb separation for BL-data (B, p) [Garg et al. 2018]

$$\text{Max.} \quad \dim X + \sum_i p_i \dim Y_i$$

$$\text{s.t.} \quad B_i(X, Y_i) = \{0\} \quad (\forall i)$$

X, Y_i: vector subspaces

Our approach is applicable, but it only gives pseudo-polynomial complexity

I don’t know whether p-adic reduction can work

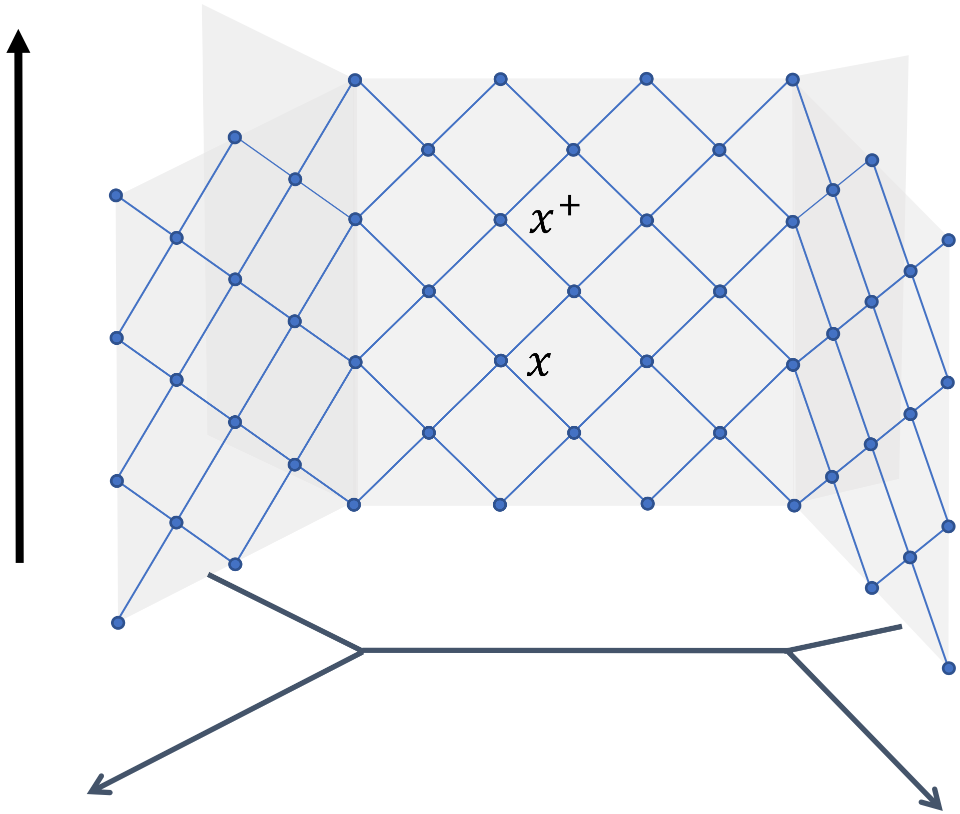
Reference

M. Hamada and H. Hirai: Computing the nc-rank via discrete convex optimization on CAT(0) spaces, *SIAM Journal on Applied Geometry and Algebra (SIAGA),* to appear.

Papers and slides are available at

http://www.misojiro.t.u-tokyo.ac.jp/~hirai/

Thank you for your attention !



L-convex function on Euclidean building (of type A) [H. 2019]

$$f(x^+) = f(x) + \text{const}$$

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$$

extends the one in *Discrete Convex Analysis* on $\mathbb{Z}^n$ [Murota 1998]

→ *Discrete Convex Analysis beyond* $\mathbb{Z}^n$ [H. 2013～]