Quiver Quantum Toroidal Algebra & Crystal Representations

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based on 2108.07104 & 2109.02045 (with Noshita)
See also 2101.03953 (with Harada, Matsuo, Noshita)

Introduction : q-deformation

- Deform algebras by adding parameters [Drinfeld 1987]
- In the limit $q \to 1$, the original algebra is recovered
- Hopf algebra structure & a larger symmetry.

q-Virasoro has paremeters $q; t; p$ [Shiraishi et al. 1995]

$$T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n};$$

$$[T_n; T_m] = -\sum_{\ell=1}^{\infty} f_\ell (T_{n-\ell} T_{m+\ell} - T_{m-\ell} T_{n+\ell}) - \frac{(1-q)(1-t^{-1})}{1-p} (p^n - p^{-n}) \delta_{m+n,0}$$

- There are two invariances

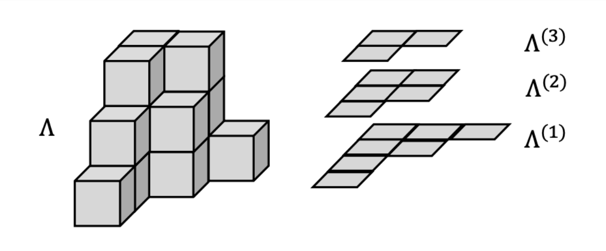
$$T_n \to T_{-n}; \quad (q; t) \to (q^{-1}; t^{-1})$$

- In the limit $q = e^{\hbar}; \hbar \to 0$, $T(z)$ becomes

$$T(z) = 2 + \beta^2 \left(L(z) + \frac{(1-\beta)^2}{4\beta} \right) \hbar^2 + O(\hbar^4)$$

Introduction : Affine Yangian & $\hbar$-deformation

- Affine Yangian of $\mathfrak{gl}_1$ [Tsymbaliuk 2014]
- Its $\hbar$-deformation is quantum toroidal $\mathfrak{gl}_1$ [Ding-Iohara 1997]
- Three representations [Feigin et al. 2012]
  - Vector representation (1d Young diagrams)
  - Fock representation (2d Young diagrams)
  - MacMahon representation (3d Young diagrams)



Introduction : Generalization of quantum toroidal $\mathfrak{gl}_1$

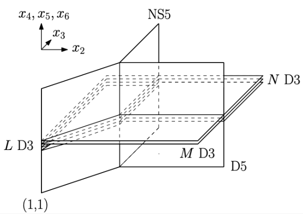
- Affine Yangian of $\mathfrak{gl}_1$ Quiver Yangian [Li-Yamazaki 2020]
- Quantum Toroidal $\mathfrak{gl}_1$ QQTA [Noshita-AW 2108.07104]

- Representations
  - 1d Young diagrams !
    1d crystals
  - 2d Young diagrams !
    2d crystals
    [Nishinaka-Yamaguchi-Yoshida 2013]
  - 3d Young diagrams !
    3d crystals [Ooguri-Yamazaki 2008]
- We found the relations between
  1d & 2d representations

[Ooguri-Yamazaki 2008]

[Noshita-AW 2109.02045]

Introduction : tϭ-deformation of corner VOA



[Gaiotto-Rapcak 2017]

Virasoro algebra	$\subset$	W_N algebra	$\subset$	corner VOA $Y_{L,M,N}$ [Gaiotto-Rapcak 2017]
↓ q-deformation		↓		↓
q-Virasoro algebra [Shiraishi et al. 1995]	$\subset$	q-W_N algebra [Awata et al. 1995]	$\subset$	q-corner VOA q-$Y_{L,M,N}$

We constructed it!! [2101.03953]

Today's talk

1. Introduction   Finished

2. ($t\sigma$-) affine Yangian of $\mathfrak{gl}_1$ & Young diagrams representations
    - Affine Yangian of $\mathfrak{gl}_1$
    - $t\sigma$-deformation
    - Representations

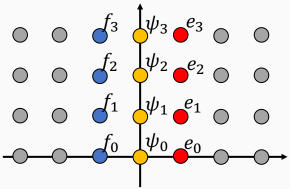
3. ($t\sigma$-) quiver Yangian & crystal representations
    - Quiver diagrams and quiver Yangian
    - $t\sigma$-deformation
    - Representations by crystals
    - Construction of 2d crystal reps from 1d

($t\sigma$-) affine Yangian of $\mathfrak{gl}_1$ & Young diagrams representations

Affine Yangian of $\mathfrak{gl}_1$

- Generators are [Tsymbaliuk 2014]

$$e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}}, \quad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}, \quad \psi(z) = 1 + \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}}$$



- Their relations depend on h_i symmetrically

$$h_1 + h_2 + h_3 = 0, \quad e(z)e(w) \sim \varphi(z-w)e(w)e(z)$$

$$\varphi(z) = \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$

- Affine Yangian of $\mathfrak{gl}_1 \simeq \mathcal{W}_{1+\infty}$

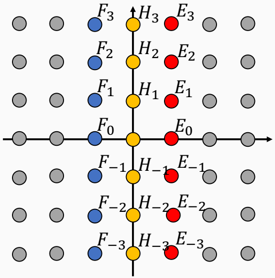
quantum toroidal $\mathfrak{gl}_1$

- q-deformation of affine Yangian of $\mathfrak{gl}_1$
- Generators are

$$e(z) = \sum_{j \in \mathbb{Z}} e_j z^{-j},$$

$$f(z) = \sum_{j \in \mathbb{Z}} f_j z^{-j},$$

$$\psi^{\pm}(z) = \psi_0^{\pm 1} \exp\left(\pm \sum_{r=1}^{\infty} a_{\pm r} z^{\mp r} \right)$$



- Their relations depend on q_1, q_2, q_3 symmetrically

$$q_1 q_2 q_3 = 1, \quad e(z)e(w) \sim \varphi(z,w)\, e(w)e(z)$$

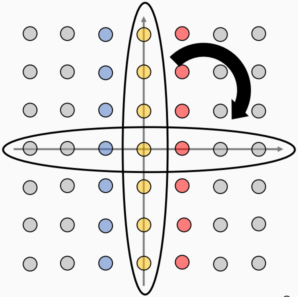
$$\varphi(z,w) = \prod_{i=1}^{3} \frac{(q_i^{1/2} z - q_i^{-1/2} w)}{(q_i^{-1/2} z - q_i^{1/2} w)}$$

Properties of quantum toroidal gl_1

- In the limit $tc\ !\ 1$, it becomes affine Yangian of $\mathbf{gl}_1$
- The modes are $j > 0$ in the degenerate case, while $j\ 2\ \mathbb{Z}$ in tc-deformed case
- Hopf algebra structure(unit, counit, product, coproduct, antipode)

- Coproduct is important here

$$
\begin{aligned}
\_(\ ) &= \_(\ ) \quad 1 + É \quad (G_{1\_}\ ) \quad \_(G_{1\_}\ ) \\
\{(\ ) &= \{(G_{2\_}\ ) \quad É^{+}(G_{2\_}\ ) + 1 \quad \{(\ ) \\
É^{+}(\ ) &= É^{+}(\ ) \quad É^{+}(G_1{}^{1}{}_{\_}\ ) \\
É\ (\ ) &= É\ \ (!_2{}^{1}{}_{\_}\ ) \quad É\ (\ )
\end{aligned}
$$



- Triality : exchange of $tc_1; tc_2; tc_3$
- Miki duality : $\hat{h}Í(2;\mathbb{Z})$

Representations of quantum toroidal $\mathfrak{gl}_1$

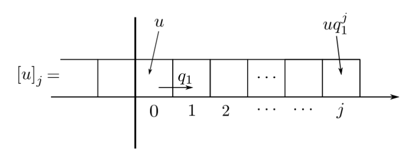
- There are 3 vertical representations
  - Vector representation : 1d Young diagrams
  - Fock representation : 2d Young diagrams
  - MacMahon representation : 3d Young diagrams

- Vector representation

$$\psi^{\pm}(z)[u]_j = \left[\psi_{[u]_j}(z)\right]_{\pm}[u]_j,$$

$$E(z)[u]_j = \mathcal{E}(q_1^{j+1}u/z)[u]_{j+1},$$

$$F(z)[u]_{j+1} = \mathcal{F}(q_1^{j+1}u/z)[u]_j$$



- $E(z)$ adds a box ,and $F(z)$ removes a box.

$$[E(z), F(w)] = \delta(w/z)\psi^+(z) - \delta(z/w)\psi^-(w)$$

Tensor product of two vector representations

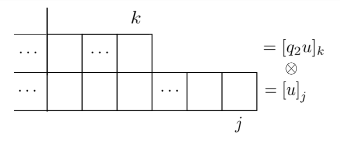
- Tensor product of two vector reps becomes Fock rep by a two-row Young diagram.
- Generators are defined by coproduct as

  _(_) = _(_) 1 + É (_) _(_);

  {(_) = {(_) É⁺(_) + 1 {(_);

  É (_) = É (_) É (_)

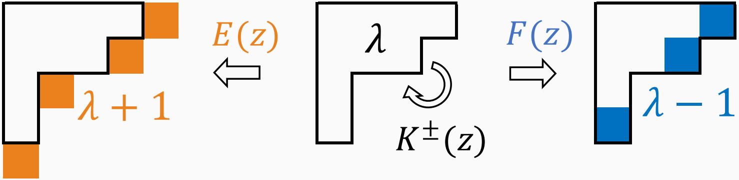
- _(_) adds a box, and {(_) removes a box. If the condition of Young diagram breaks, the coefficient becomes zero.



- Furthermore, tensor product of infinite vector reps becomes Fock rep by a general Young diagram.

$$|\lambda\rangle = \bigotimes_{i=1}^{\infty} [u q_2^{i-1}]_{\lambda_i - 1}$$

- Generators are defined by $\acute{Y} - 1$ coproducts.
- $\Delta^{(\acute{Y}-1)}E(z)$ adds a box, and $\Delta^{(\acute{Y}-1)}F(z)$ removes a box.
- If the condition of Young diagram breaks, the coefficient becomes zero.



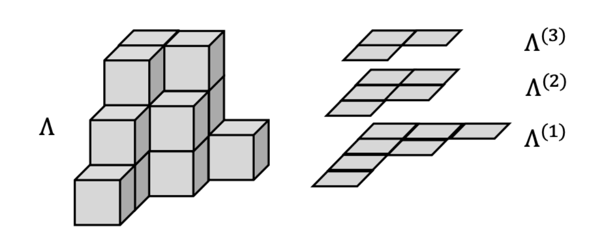
MacMahon representation

- Tensor product of vector rep (1d) ! Fock rep (2d)
- Similarly,
  tensor product of Fock rep (2d) ! MacMahon rep (3d)

$$= (\ {}^{(1)}; \ {}^{(2)}; \ {}^{(3)}; \ \ ) \qquad {}^{(i)} : \text{Young diagram}$$

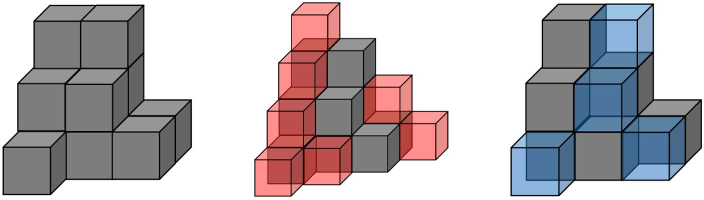
- is called Plane Partition, satisfying

1 2 3



MacMahon representation

- Similarly to the 1d or 2d case, É ( ) acts as eigenvalue、 _( ) adds a box to the Plane Partition, and {( ) removes a box.
- If the condition of Plane Partition breaks, the coefficient becomes zero.



(a) Y_Λ (b) $CC(Y_\Lambda)$ (c) $CV(Y_\Lambda)$

($t\sigma$-) quiver Yangian & crystal representations

Quiver data

- Quiver Yangian is defined by a quiver data and generalizes affine Yangian of $\mathfrak{gl}_1$ [Li-Yamazaki 2020]
- Quiver data = Quiver diagram $(\check{E}_0; \check{E}_1)$ + Loops $\check{E}_2$.
- (b) corresponds to the affine Yangian of $\mathfrak{gl}_{2j1}$

$$\check{E}_0 = f1;2;3g; \ \check{E}_1 = f1 \ ! \ 1; \ 1 \ ! \ 2; \ 1 \ ! \ 3; \ 2 \ ! \ 1; \ 2 \ ! \ 3; \ 3 \ ! \ 1; \ 3 \ ! \ 2g;$$

$$\check{E}_2 = f1 \ ! \ 1 \ ! \ 3 \ ! \ 1;1 \ ! \ 2 \ ! \ 3 \ ! \ 2 \ ! \ 1;1 \ ! \ 1 \ ! \ 2 \ ! \ 1;1 \ ! \ 3 \ ! \ 2 \ ! \ 3 \ ! \ 1g$$

From toric diagram to quiver diagram

- We focus on symmetric quivers constructed from toric diagram without compact 4-cycles.
- Draw the red arrows perpendicular to the toric diagram on the torus.(brane configuration)
- White region $\$$ vertex, their connection $\$$ arrows
- Loops on brane configuration $\$$ $\check{E}_2$

Definition of quiver Yangian

- We define a set of generators $\bar{\dot{a}}; \grave{\imath};$ for each vertex $ħ$.

$$\bar{\dot{a}}^{(ħ)}(\_) = \sum_{ɧ=0}^{\infty} \frac{\bar{\dot{a}}^{(ħ)}_{ɧ}}{\;^{ɧ+1}}; \quad {}^{(ħ)}(\_) = \sum_{ɧ=-\infty}^{\infty} \frac{{}^{(ħ)}_{ɧ}}{\;^{ɧ+1}}; \quad \grave{\imath}^{(ħ)}(\_) = \sum_{ɧ=0}^{\infty} \frac{\grave{\imath}^{(ħ)}_{ɧ}}{\;^{ɧ+1}};$$

- Bond facters are defined by $\check{E}_1$.

$$,ħ)ε(ᵕ) \quad \frac{\prod_{ᶻ2\{ε\,!\,ħ\}}(ᵕ + d_ᶻ)}{\prod_{ᶻ2\{ħ\,!\,ε\}}(ᵕ \quad d_ᶻ)}$$

$$\bar{\dot{a}}^{(ħ)}(\_)\bar{\dot{a}}^{(ε)}(″) \quad (\;\; 1)^{|ħ||ε|}\,{,ε)ħ}(\_ \quad ″)\bar{\dot{a}}^{(ε)}(″)\bar{\dot{a}}^{(ħ)}(\_)$$

Quiver Quantum Toroidal Algebra (QQTA)

- We proposed the ℓ-deformation of quiver Yangian called QQTA [Noshita-AW 2108.07104]
- Similar to the quiver Yangian, we use quiver data $(\check{E}_0; \check{E}_1; \check{E}_2)$ to define QQTA.
- Parameters are replaced into $d_2 \;!\; t_{62} = \bar{a}^{d_2}$
- We define a set of generators $\_; \{; \acute{E}$ for each vertex $\mathring{0}$.

$$\_\mathring{0}(\_) = \sum_{j \in \mathbb{Z}} \_\mathring{0}; j\_ \; {}^{j}; \quad \{\mathring{0}(\_) = \sum_{j \in \mathbb{Z}} \{\mathring{0}; j\_ \; {}^{j}; \quad \acute{E}_{\mathring{0}} (\_) = \acute{E}_{\mathring{0}}^{\;1} \, 0^{2} q \left(\sum_{w=1}^{1} a_{\mathring{0}; \, w\_} \; {}^{w} \right) :$$

- Bond factors are defined by $\check{E}_1$

$$, \mathring{0}) \, ?(\_; {}'') = \frac{\prod_{2 \in f? \,!\, \mathring{0} g} (t_{62}^{1/2} \_ \;\; t_{62}^{\;1/2''})}{\prod_{2 \in f \mathring{0} \,!\, ?g} (t_{62}^{\;1/2} \_ \;\; t_{62}^{1/2''})}$$

Properties

- In the limit $\to 0$, it reduces to quiver Yangian.

$$\frac{\prod_{I\in\{i\to j\}}(t\epsilon_I^{1/2} - t\epsilon_I^{-1/2}\,)}{\prod_{I\in\{j\to i\}}(t\epsilon_I^{-1/2} - t\epsilon_I^{1/2}\,)} \to \frac{\prod_{I\in\{i\to j\}}(u - v + h_I)}{\prod_{I\in\{j\to i\}}(u - v - h_I)};$$

- Hopf superalgebra structure, especially coproduct

$$\begin{aligned}
\Delta e_i(z) &= e_i(z) \otimes 1 + \psi_i^-(C_1 z) \otimes e_i(C_1 z);\\
\Delta f_i(z) &= f_i(C_2 z) \otimes \psi_i^+(C_2 z) + 1 \otimes f_i(z);\\
\Delta \psi_i^+(z) &= \psi_i^+(z) \otimes \psi_i^+(C_1^{-1} z);\\
\Delta \psi_i^-(z) &= \psi_i^-(C_2^{-1} z) \otimes \psi_i^-(z)
\end{aligned}$$

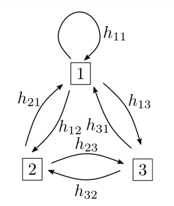
- Coproduct is important in the relation of different representations.

3d crystal

- 3d crystals are defined by the quiver data [Ooguri-Yamazaki 2008]

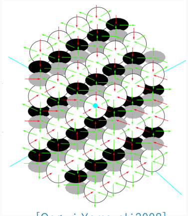
$$\check{E}_2 = \{1 \to 1 \to 3 \to 1; 1 \to 2 \to 3 \to 2 \to 1; 1 \to 1 \to 2 \to 1; 1 \to 3 \to 2 \to 3 \to 1\}$$

- vertex in quiver diagram $\to$ atom of the crystal
  arrow between vertices $\to$ bond between atoms
- $\check{E}_2$ identifies some path on crystal.



(b) Three vertices and one self-loop case

$+Q_2$



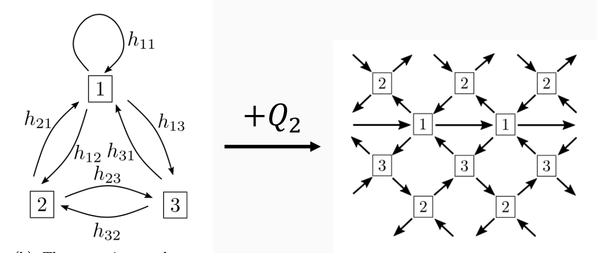
[Ooguri-Yamazaki 2008]

3d crystal

- quantum toroidal $\mathfrak{gl}_1$ acts to Plane Partitions.
- QQTA acts to general 3d crystals.
- Such crystals are defined by the quiver data

  [Ooguri-Yamazaki 2008]
- First, we obtain periodic quiver from $(\check{E}_0; \check{E}_1; \check{E}_2)$.

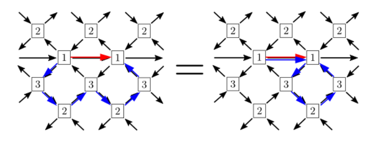
  $\check{E}_2 = f1 \ ! \ 1 \ ! \ 3 \ ! \ 1; 1 \ ! \ 2 \ ! \ 3 \ ! \ 2 \ ! \ 1; 1 \ ! \ 1 \ ! \ 2 \ ! \ 1; 1 \ ! \ 3 \ ! \ 2 \ ! \ 3 \ ! \ 1g$

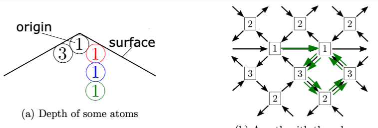


(b) Three vertices and one self-loop case

3d crystal

- a path in periodic quiver ′ an atom on 3d crystal
- However, it is not always one-to-one correspondence. Several paths may be identified to an atom (F-term relation)
- The depth of atom from the surface is the number of loops in the periodic quiver.



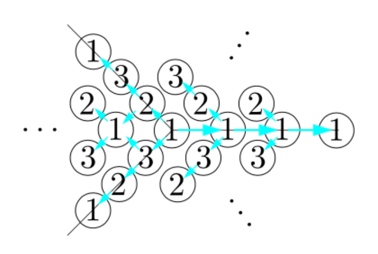


(a) Depth of some atoms

(b) A path with three loops

3d crystal

- By the identification of path and atom, we obtain 3d crystal.



[Ooguri-Yamazaki 2008]

- This is different from the Plane Partition of affine Yangian of $\mathfrak{gl}_1$.
- If we start with the quiver diagram of $\mathfrak{gl}_1$, we obtain the Plane Partition.

- quantum toroidal $\mathfrak{gl}_1$ acts to Plane Partitions.
- QQTA acts to general 3d crystals.
- $e_a(z)$ adds an atom of vertex a, and $f_a(z)$ removes an atom of vertex a.

$$\psi_a(z)|\Lambda\rangle = [\psi^{(a)}(z;u)]_{\Lambda}|\Lambda\rangle,$$

$$e_a(z)|\Lambda\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\Lambda)} E^{(a)}(\Lambda \to \Lambda + \boxed{a})\,\delta\left(\frac{z}{u\,h(\boxed{a})}\right)|\Lambda + \boxed{a}\rangle,$$

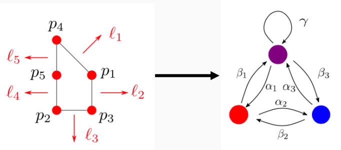
$$f_a(z)|\Lambda\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\Lambda)} F^{(a)}(\Lambda \to \Lambda - \boxed{a})\,\delta\left(\frac{z}{u\,h(\boxed{i})}\right)|\Lambda - \boxed{a}\rangle$$

Reduction to subcrystals

- Quantum toroidal $\mathfrak{gl}_1$ has not only Plane Partition rep, but also vector rep and Young diagram rep.
- QQTA also has representations by subcrystals.
- The surface of 3d crystal is 2d crystal, and the edge of 2d crystal is 1d crystal.

Reduction to 2d crystals

- A quiver diagrams of 3d crystal is constructed from a toric diagram.



- A quiver diagrams of 2d crystal is constructed from a toric diagram with a corner divisor [Nishinaka-Yamaguchi-Yoshida2013].
- Different corner divisor creates different 2d crystals. Left figure is the case £1, and right is £2.

Reduction to 1d crystals

- Furthermore, removing all arrows of two neighboring corner divisors gives 1d crystal.
- The below figure is the case $ɏ_1$, and arrows correponding to $£_1$ and $£_4$ are removed.

- 1d crystals and 2d crystals are called subcrystals.

1d crystal

- Subcrystals are the representations of ”shifted” QQTA.
- QQTA has 4 types generators $\_;\{;\acute{E}^+;\acute{E}$, and we slightly ”shift” $\acute{E}^+$ as $\acute{E}^+_{\hat{\jmath}}(\_) \; ! \; \_^{w_{\hat{\jmath}}}\acute{E}^+_{\hat{\jmath}}(\_)$.
- $(w_1; \quad ; w_{\check{E}_0})$ is parameters determined by the shape of crystal. Of course, different subcrystals give different $w_{\hat{\jmath}}$.
- This shift is essential to keep generated states on surface.

2d crystal from 1d crystal

- Quantum toroidal $\mathfrak{gl}_1$ has vector rep (1d) and Fock rep (2d). [Feigin et al. 2011]
  - Fock rep is constructed from the tensor product of vector rep.
  - The action of quantum toroidal algebra to Fock rep is the coproduct of that of vector rep.
- Also in the QQTA, the representations of 2d crystals are constructed from that of 1d crystals. [Noshita-AW 2109.02045]
- For example 2d crystal correponding to $\mathfrak{f}_1$ is constructed from the tensor product of 1d crystal $\mathfrak{y}_1$.

- 1d crystals stacked vertically become a 2d crystal.

1d crystal y_1

2d crystal ς_1

- Vacuum state of 2d crystal is defined by the tensor product of vacuum of 1d crystals $[\iota]\ _1$.

$$j;i\ =\ \ {}_{\mathring{0}=1}^{1}[(t\sigma_1 t\sigma_3)^{\mathring{0}\ \ 1}\iota]\ _1$$

- The generators acting to 2d crystal is defined by the coproduct of these to 1d crystal. $\_{\mathring{0}}(\_\ )$ adds an atom, and $\{_{\mathring{0}}(\_\ )$ removes an atom.

2d crystal from another 1d crystal

- In the previous slide, we construct 2d crystal from 1d crystal along with $ɏ_1$.
- We can construct the same 2d crystal from 1d crystal along with $ɏ_2$.
- This is the difference of slicing direction.

1d crystal $ɏ_2$

2d crystal $£_1$

Summary

quantum toroidal $\mathfrak{gl}_1$

vector rep (1d) ⟶ Fock rep (2d) ⟶ MacMahon rep (3d)

↓

(shifted) QQTA [Noshita-AW 2108.07104]

1d crystal rep ⟶ 2d crystal rep   3d crystal rep

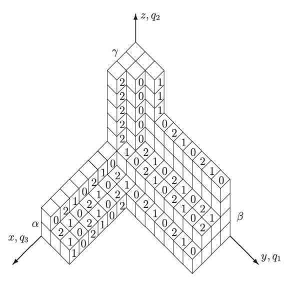
tensor product
[Noshita-AW 2109.02045]

Construction of 2d quiver

- 2d quiver is obtained from 3d quiver by remobing some arrows.
- Physically, it consists of D0-D2-D4 brane system.
  $ 3d quiver comes from D0-D2-D6 brane system.
- We need to determine the boundary in the brane configuration by finding a perfect matching in dimer model.

Related works

- Feigin-Jimbo-Miwa-Mukhin 1204.5378
  ”Representations of Quantum Toroidal $\mathfrak{gl}_{\mathfrak{n}}$”



- Galakhov-Li-Yamazaki 2108.10286
  ”Toroidal and Elliptic Quiver BPS Algebras and Beyond”

Parameters of quiver Yangian

- Parameter h_I exsits for each arrow $I \in Q_1$.
- Not all of these are independent, and the degree of freedom is 2 due to two kinds of constraints.

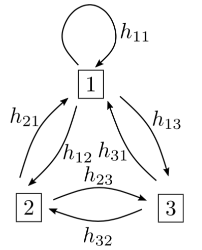
- Vertex constraints

$$\sum_{I \in Q_1(a)} \mathrm{sign}_a(I) h_I = 0$$

- Loop constraints

$$\sum_{I \in L} h_I = 0$$

L is an arbitrary loop in Q_2



(b) Three vertices and one self-loop case

$$Q_2 = \{1 \to 1 \to 3 \to 1; 1 \to 2 \to 3 \to 2 \to 1; 1 \to 1 \to 2 \to 1; 1 \to 3 \to 2 \to 3 \to 1\}$$