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Chaotic string in a near pp-wave limit

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[S. Kushiro and K. Y., arXiv:2209.05171]

0. Introduction

Introduction

Integrability in AdS/CFT

has played an important role in checking conjectured relations.

A typical example of AdS/CFT

Type IIB superstring on $AdS_5 x S^5$ ⟺ N=4 super Yang-Mills (large *N*)

Classical string solutions.		Composite operators.
Spinning string solutions		$\mathrm{Tr}(\phi^{i_1}\cdots\phi^{i_l})$
Energy	$E = \Delta$	Scaling dimensions

One can check the relation even in non-BPS regions.

For the SYM side, the dilatation operator is represented by an integrable spin chain Hamiltonian. [Minahan-Zarembo, hep-th/0212208, citation 1410, 2023/10/15]

At 2-loop level, for the SU(2) subsector,

$$H = \epsilon_0 \sum_{i=1}^{L} \mathbb{1}_{i,i+1} + \lambda_1 \sum_{i=1}^{L} \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \lambda_2 \sum_{i=1}^{L} \mathbf{S}_i \cdot \mathbf{S}_{i+2}$$

$$\epsilon_0 = 1 - \frac{3\lambda}{16\pi^2}, \qquad \lambda_1 = -4 + \frac{\lambda}{\pi^2}, \qquad \lambda_2 = -\frac{\lambda}{4\pi^2}$$

λ: 't Hooft coupling

$\mathbf{S}_i = (\sigma_i^1, \sigma_i^2, \sigma_i^3)$, where σ_i^a $(a = 1, 2, 3)$ are the standard Pauli matrices.

It has been believed in that this system is quantum mechanically integrable.

However, the energy-level spacings of this system are well approximated by the Wigner-Dyson distribution at strong coupling, which indicates non-integrablity.

Chaotic spin chains in AdS/CFT [T. McLoughlin and A. Spiering, 2202.12075]

It is beyond the validity region of the perturbative expansion. But still it is suggestive.

Motivated by the work by McLoughin and Spiering [2202.12075], we will revisit the string dynamics in a near pp-wave limit of AdS_5xS^5.

Actually, in the previous work with Y. Asano, D. Kawai, H. Kyono [1505.07583], we have studied it. But we could not find any chaos.

However, the ansatz used in 1505.07583 did not include string winding numbers.

Our claim here

Chaotic string motions appear when string winding numbers are included.

[S. Kushiro and K. Y., arXiv:2209.05171]

In fact, the AdS_5xS^5 string and pp-wave string are exactly solvable.

However, a finite truncation of the AdS_5xS^5 string may lead to chaos!

In fact, the similar thing happens in the case of the Toda lattice.

Connected Harmonic Oscillators	Henon-Heiles model (1964)	Toda lattice model (1967)
2nd order potential	3rd order potential	Exp potential
Exactly solvable	Chaotic	Exactly solvable

PP-wave string	PP-wave string + interactions	AdS_5xS^5 string
2nd order potential	4th order potential	Full theory
Exactly solvable	Chaotic **(our result)**	Exactly solvable

← Penrose limit

The content of my talk

1. Chaos from a truncation of Toda lattice
2. String action in a near pp-wave limit
3. Reduction, Poincare sections and Lyapunov exponents
4. Summary and Discussion

1. Chaos from a truncation of Toda lattice

3-particle periodic Toda chain

Let us start from the 3-particle periodic Toda chain:

$$H = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + e^{-(Q_2-Q_1)} + e^{-(Q_3-Q_2)} + e^{-(Q_1-Q_3)} - 3\,.$$

Hamilton's equations are given by

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = P_1\,, \qquad \frac{\mathrm{d}Q_2}{\mathrm{d}t} = P_2\,, \qquad \frac{\mathrm{d}Q_3}{\mathrm{d}t} = P_3\,,$$
$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = e^{Q_3-Q_1} - e^{Q_1-Q_2}\,, \quad \frac{\mathrm{d}P_2}{\mathrm{d}t} = e^{Q_1-Q_2} - e^{Q_2-Q_3}\,, \quad \frac{\mathrm{d}P_3}{\mathrm{d}t} = e^{Q_2-Q_3} - e^{Q_3-Q_1}\,.$$

This system is classically integrable in the sense of Liouville.

3 independent conserved charges:

$$I_1 \equiv P_1 + P_2 + P_3\,.$$

$$I_2 \equiv P_1P_2 + P_2P_3 + P_3P_1 - e^{Q_1-Q_2} - e^{Q_2-Q_3} - e^{Q_3-Q_1}$$

$$I_3 = P_1P_2P_3 - P_1e^{Q_2-Q_3} - P_2e^{Q_3-Q_1} - P_3e^{Q_1-Q_2}$$

Assume that Q_i's are small and expand the exponential potentials.

Then the resulting Hamiltonian at the third order is given by

$$H = H_0 + H_1 \,,$$
$$H_0 \equiv \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \frac{1}{2}\{(Q_1 - Q_2)^2 + (Q_2 - Q_3)^2 + (Q_3 - Q_1)^2\} \,,$$
$$H_1 \equiv \frac{1}{6}\{(Q_1 - Q_2)^3 + (Q_2 - Q_3)^3 + (Q_3 - Q_1)^3\} \,,$$

It is convenient to perform a rotation as follows:

$$Q_i = \sum_{j=1}^{3} A_{ij}\,\zeta_j \,, \qquad P_i = \sum_{j=1}^{3} A_{ij}\,\eta_j \,, \qquad A \equiv \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} . \qquad (A^T A = I)$$

Then the resulting Hamiltonian is

$$H = \frac{1}{2}(\eta_1^2 + \eta_2^2 + \eta_3^2 + 3\zeta_1^2 + 3\zeta_2^2) + \frac{3}{2\sqrt{2}}\left(\zeta_2\zeta_1^2 - \frac{1}{3}\zeta_2^3\right)$$

Since the total momentum $P_1 + P_2 + P_3 = \sqrt{3}\,\eta_3$ is conserved.

So we can drop off η_3 .

Then Hamilton's equations are given by

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} = \eta_1\,, \quad \frac{\mathrm{d}\zeta_2}{\mathrm{d}t} = \eta_2\,, \quad \frac{\mathrm{d}\eta_1}{\mathrm{d}t} = -3\zeta_1 - \frac{3}{\sqrt{2}}\zeta_1\zeta_2\,, \quad \frac{\mathrm{d}\eta_2}{\mathrm{d}t} = -3\zeta_2 - \frac{3}{2\sqrt{2}}\left(\zeta_1^2 - \zeta_2^2\right)$$

After rescaling the variables like

$$q_1 = \frac{1}{2\sqrt{2}}\,\zeta_1\,, \quad q_2 = \frac{1}{2\sqrt{2}}\,\zeta_2\,, \quad p_1 = \frac{1}{2\sqrt{6}}\,\eta_1\,, \quad p_2 = \frac{1}{2\sqrt{6}}\,\eta_2\,, \quad \tau = \sqrt{3}\,t\,,$$

Hamilton's equations are rewritten as

$$\frac{\mathrm{d}q_1}{\mathrm{d}\tau} = p_1\,, \quad \frac{\mathrm{d}q_2}{\mathrm{d}\tau} = p_2\,, \quad \frac{\mathrm{d}p_1}{\mathrm{d}\tau} = -q_1 - 2q_1q_2\,, \quad \frac{\mathrm{d}p_2}{\mathrm{d}\tau} = -q_2 - q_1^2 + q_2^2\,,$$

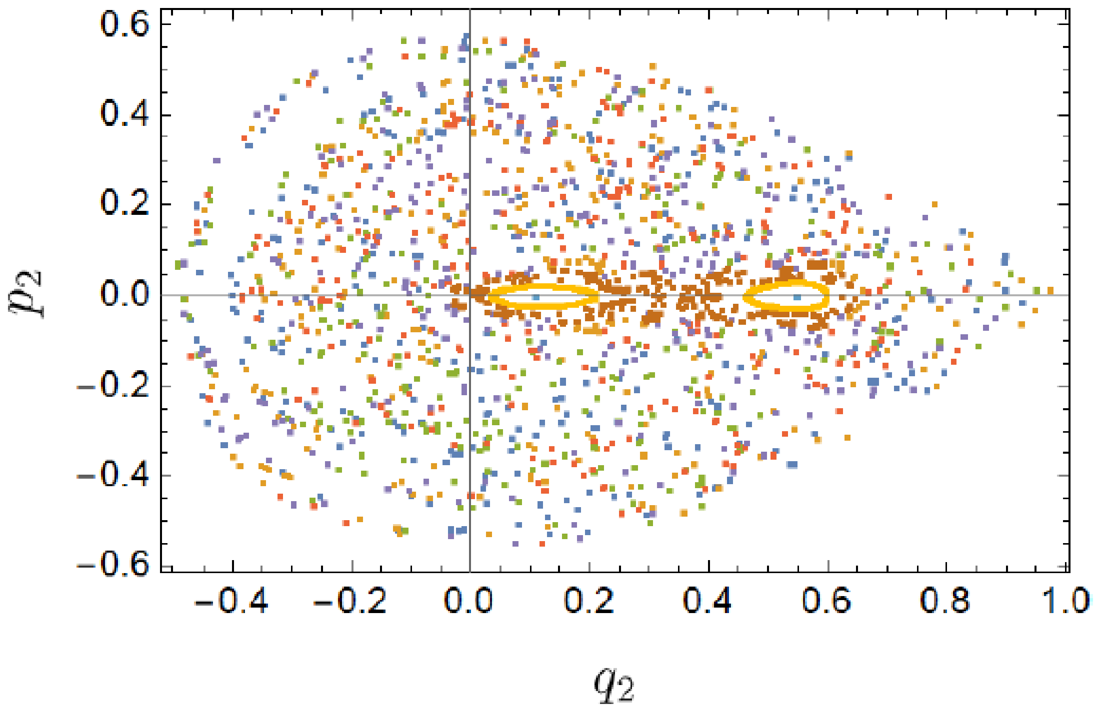
These equations are provided from the following Hamiltonian:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(q_1^2 + q_2^2\right) + q_2q_1^2 - \frac{1}{3}q_2^3\,.$$

This is nothing but the Henon-Heiles model. This is the well-known chaotic system!

A Poincare section for the Henon-Heiles model

E=1/6, q_1=0, p_1 > 0



Each of different colors corresponds to each of different initial conditions.

2. String action

in a near pp-wave limit

A near pp-wave limit of AdS_5xS^5

The metric of AdS_5xS^5

$$ds^2 = R^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\Omega_3'^2)$$

By introducing new coordinates $\tilde{z}$ and $\tilde{y}$ defined as

$$\cosh\rho \equiv \frac{1+\tilde{z}^2/4}{1-\tilde{z}^2/4}\,, \qquad \cos\theta \equiv \frac{1-\tilde{y}^2/4}{1+\tilde{y}^2/4}$$

the resulting metric is

$$ds^2 = R^2\left(-\left(\frac{1+\tilde{z}^2/4}{1-\tilde{z}^2/4}\right)dt^2 + \left(\frac{1-\tilde{y}^2/4}{1+\tilde{y}^2/4}\right)d\phi^2 + \frac{d\tilde{z}^2+\tilde{z}^2 d\Omega_3^2}{(1-\tilde{z}^2/4)^2} + \frac{d\tilde{y}^2+\tilde{y}^2 d\Omega_3'^2}{(1+\tilde{y}^2/4)^2}\right)$$

The light-cone coordinates:

$$\tilde{x}^+ = t\,, \qquad \tilde{x}^- = -t + \phi\,.$$

Penrose limit

After rescaling the coordinates as

$$\tilde{x}^+ \longrightarrow x^+ , \qquad \tilde{x}^- \longrightarrow \frac{x^-}{R^2} , \qquad \tilde{z} \longrightarrow \frac{z}{R} , \qquad \tilde{y} \longrightarrow \frac{y}{R} ,$$

take the $R \to \infty$ limit.

The resulting metric (at the order of $1/R^2$)

$$ds^2 = ds_0^2 + \frac{1}{R^2} ds_2^2 + \mathcal{O}\left(\frac{1}{R^4}\right) ,$$

$$ds_0^2 \equiv 2dx^+ dx^- - (z^2 + y^2)(dx^+)^2 + dz^2 + z^2 d\Omega_3^2 + dy^2 + y^2 d\Omega_3'^2 ,$$

$$ds_2^2 \equiv -2y^2 dx^+ dx^- + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2} z^2 (dz^2 + z^2 d\Omega_3^2) - \frac{1}{2} y^2 (dy^2 + y^2 d\Omega_3'^2) .$$

The leading part is the maximally supersymmetric pp-wave background.

[Blau-Figueroa-O'Farill-Hall-Papadopoulos, hep-th/0110242, 0201081]

String action

$$S = \int d\tau d\sigma \mathcal{L} = \frac{1}{2}\int d\tau d\sigma \sqrt{-\det(h_{ab})} h^{ab}\partial_a x^\mu \partial_b x^\nu g_{\mu\nu}$$

The vanishing stress tensor:

$$T_{ab} = \partial_a x^\mu \partial_b x^\nu g_{\mu\nu} - \frac{1}{2} h_{ab} h^{cd} \partial_c x^\mu \partial_d x^\nu g_{\mu\nu} = 0$$

By using the canonical momentum

$$p_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\tau x^\mu)} = h^{\tau a}\partial_a x^\nu g_{\mu\nu}\,, \qquad \dot{x}^\mu = \frac{1}{h^{\tau\tau}} g^{\mu\nu} p_\nu - \frac{h^{\tau\sigma}}{h^{\tau\tau}} x'^\mu$$

the vanishing stress tensor is written as

$$p_\mu p_\nu g^{\mu\nu} + x'^\mu x'^\nu g_{\mu\nu} = 0\,,$$

$$p_\mu x'^\mu = 0\,.$$

We will work in the light-cone gauge as usual:

Light-cone gauge:

$$x^+ = \tau \,, \qquad p_- = \text{const} \,.$$

Light-cone Hamiltonian:

$$\mathcal{H}_{\rm lc} \equiv -p_+$$

This L.C. Hamltonian can be expressed in terms of the transverse variables:

$$\mathcal{H}_{\rm lc} = -\frac{p_- g^{+-}}{g^{++}} - \frac{1}{g^{++}} \sqrt{{p_-}^2 g - g^{++} \left(g_{--} \left(\frac{p_I x'^I}{{p_-}^2} \right)^2 + p_I p_J g^{IJ} + x'^I x'^J g_{IJ} \right)} \,,$$

$$g \equiv (g^{+-})^2 - g^{++} g^{--}$$

By substituting the expanded metric

$$g_{\mu\nu} = (g_0)_{\mu\nu} + \frac{1}{R^2}(g_2)_{\mu\nu} + \mathcal{O}\left(\frac{1}{R^4}\right), \qquad g^{\mu\nu} = (g'_0)^{\mu\nu} + \frac{1}{R^2}(g'_2)^{\mu\nu} + \mathcal{O}\left(\frac{1}{R^4}\right),$$

where $g'_0 = g_0^{-1}$ and $g'_2 = -g_0^{-1} g_2 g_0^{-1}$.

into the L.C. Hamiltonian, we obtain that

$$\mathcal{H}_{\rm lc} = \mathcal{H}_0 + \frac{1}{R^2}\mathcal{H}_{\rm int} + \mathcal{O}\left(\frac{1}{R^4}\right),$$

$$\mathcal{H}_0 = \frac{1}{2}\left(p_I p_J (g'_0)^{IJ} + x'^I x'^J (g_0)_{IJ} + y^2 + z^2\right),$$

$$\mathcal{H}_{\rm int} = \frac{1}{8}\Big(\left(y^2+z^2\right)^2 - \left(p_I p_J (g'_0)^{IJ} + x'^I x'^J (g_0)_{IJ}\right)^2\Big) + \frac{1}{2}\left(p_I x'^I\right)$$
$$+ \frac{1}{4}\left(z^2 - y^2\right)\left(p_I p_J (g'_0)^{IJ} + x'^I x'^J (g_0)_{IJ}\right) + \frac{1}{2}\left(p_I p_J (g'_2)^{IJ} + x'^I x'^J (g_2)_{IJ}\right).$$

Here we have set $p_- = 1$.

This expanded form was originally obtained by J. Schwarz et. al.

[Callan-Lee-McLoughlin-Schwarz-Swanson-Wu, hep-th/0307032]

3. Reduction, Poincare sections and Lyapunov exponents

Let us consider how to reduce the L.C. Hamiltonian so as to describe chaotic dynamics.

Suppose that the string is sitting at a point in S^3 in AdS_5 hence the $d\Omega_3^2$ part is omitted.

It is helpful to parametrize the S^3 part in S^5 like

$$d\Omega_3'^2 = d\eta^2 + \sin^2\eta\, d\xi_1^2 + \cos^2\eta\, d\xi_2^2\,.$$

Note that p_{ξ_1} and p_{ξ_2} are conserved and hence we will set $p_{\xi_1} = p_{\xi_2} = 0$ below.

To study the chaotic dynamics, we will reduce the system from 2D field theory to

a mechanical system with a winding string ansatz

$$z = 0\,, \quad p_z = 0\,, \quad y = y(\tau)\,, \quad p_y = p_y(\tau)\,,$$
$$\xi_1 = a_1\sigma\,, \quad \xi_2 = a_2\sigma \quad \eta = \eta(\tau)\,, \quad p_\eta = p_\eta(\tau)\,.$$

Here a_i $(i = 1,2)$ are integers due to the periodicity of ξ_i .

In the previous work [1505.07583], a1 = a2 = 0.

The Hamilton equations of the reduced system:

$$\dot{y} = p_y - \frac{1}{2R^2}\left[P_y^3 + \frac{P_y P_\eta^2}{y^2} + y^2\underline{\left(a_1^2 \sin^2\eta + a_2^2\cos^2\eta\right)}\right],$$

$$\dot{\eta} = \frac{p_\eta}{y^2} - \frac{1}{2R^2}\left[\frac{p_\eta^3}{y^4} + \frac{p_y^2 p_\eta}{y^2} + \underline{\left(a_1^2 \sin^2\eta + a_2^2\cos^2\eta\right)}\right],$$

$$\dot{p_y} = \frac{p_\eta^2}{y^3} - y\left(\underline{a_1^2 \sin^2\eta + a_2^2\cos^2\eta} + 1\right)$$
$$+ \frac{1}{2R^2}\left[-5y^3 - 2yp_y^2 - \frac{p_\eta^4}{y^5} - \frac{p_y^2 p_\eta^2}{y^3} - \frac{p_y^4}{4y} + y^3\left(\underline{a_1^2 \sin^2\eta + a_2^2\cos^2\eta} + \frac{p_y^2 + 4y^2}{2y^2}\right)^2\right],$$

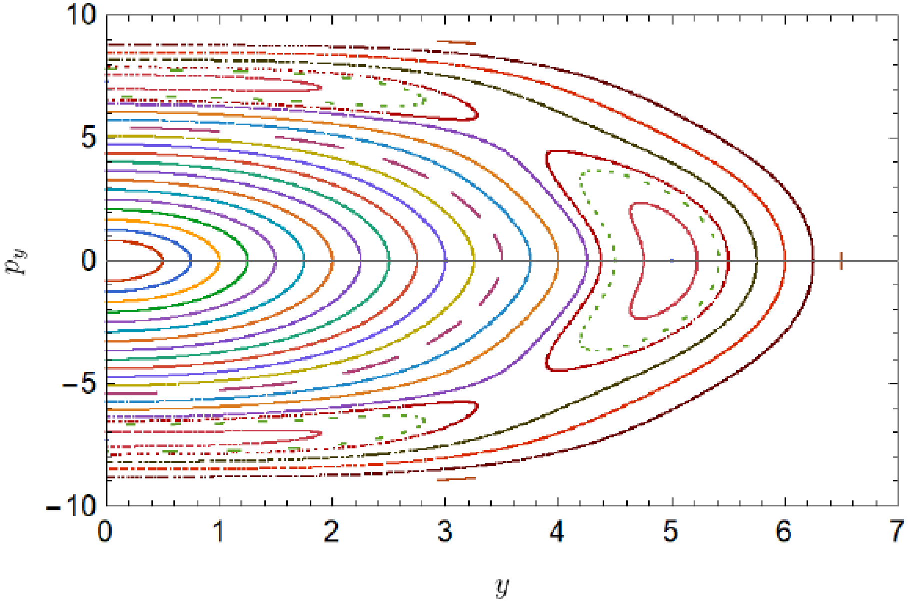
$$\dot{p_\eta} = \underline{\frac{1}{2}\left(a_2^2 - a_1^2\right) y^2 \sin 2\eta}$$
$$- \frac{1}{8R^2}\underline{\left(a_2^2 - a_1^2\right)\sin 2\eta\left[\left(a_2^2 - a_1^2\right)\right.} y^4 \cos 2\eta + \left(a_1^2 + a_2^2 + 4\right)y^4 + 2p_\eta^2 + 2y^2 p_y^2\left.\right].$$

Compute the Poincare sections and Lyapunov spectrum.

For simplicity, we will take the values: a1 = 1, a2 = 2, *R* =5.0 below.

Poincare sections

Condition: $\eta = \pi/2,\ p_\eta > 0$

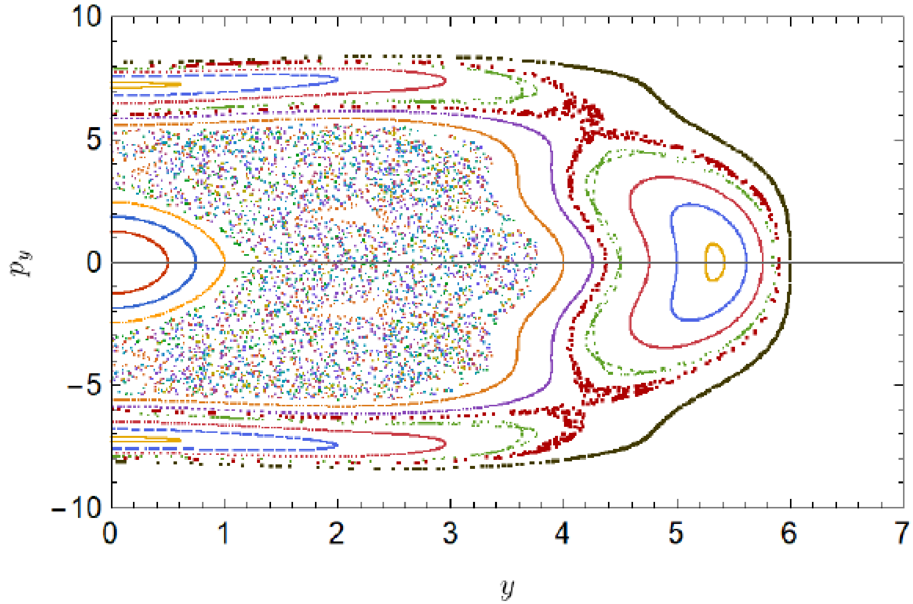


(a) Poincaré section for $E = 5.0$

There are only the KAM tori and no chaos.

Poincare sections

Condition: $\eta = \pi/2,\ p_\eta > 0$

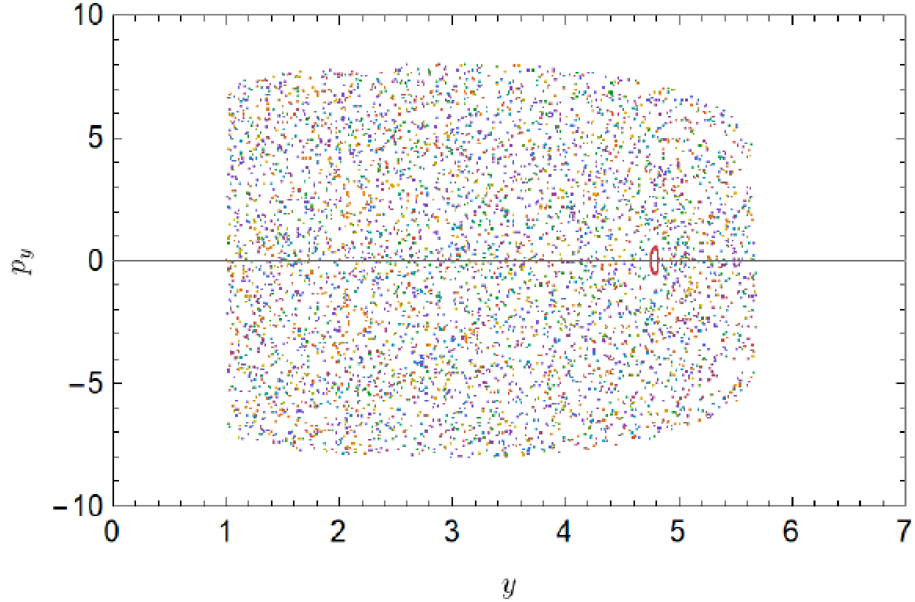


(b) Poincaré section for $E = 11.0$

There are some KAM tori and chaotic motions also appear.

Poincare sections

Condition: $\eta = \pi/2,\ p_\eta > 0$

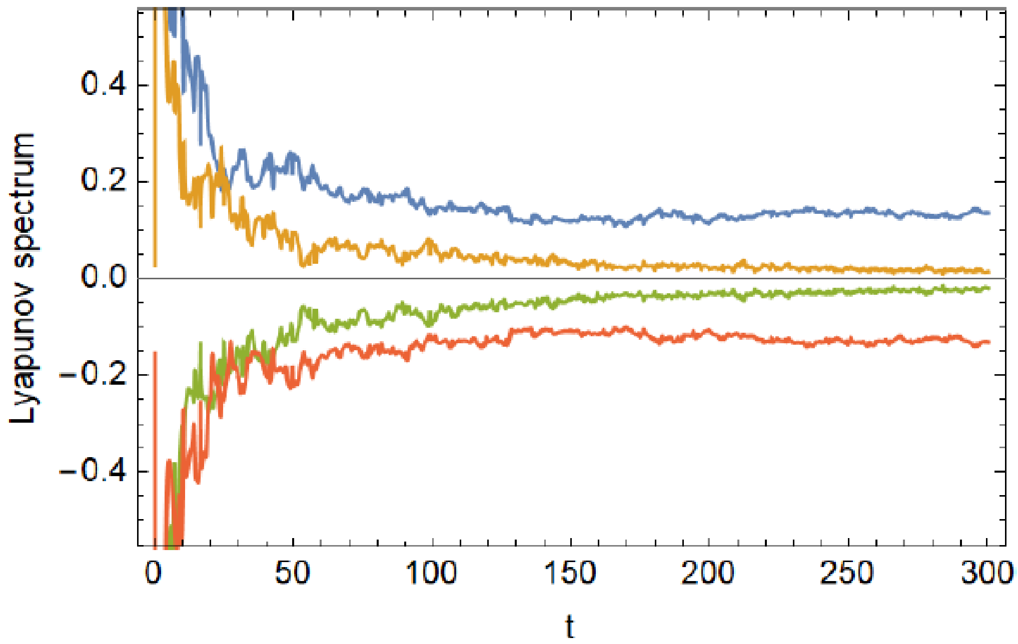


(c) Poincaré section for $E = 13.0$

There are a few tiny KAM tori and almost of the orbits are chaotic.

Lyapunov spectrum

Condition: y = 2.0, η = $\pi/2$, p_y = 1.0



(d) Lyapunov spectrum for $E = 13.0$

A Lyapunov exponent is non-zero.

4. Summary and Discussion

Summary and Discussion

Summary

We found chaos in a near pp-wave limit. [S. Kushiro and K. Y., arXiv:2209.05171]

String winding numbers are crucial for this chaotic behavior.

c.f., There was no chaos without string winding numbers

[A. Asano, D. Kawai, H. Kyono and K.Y.,1505.07583]

Discussion

Physical implications of this chaos?

What is the SYM side counterpart?

Chaos from truncations of other integrable theories?

Find snakes sneaking in integrable theories

Thank you for your attention!

Backup

The Hamiltonian of the reduced system:

$$
\begin{aligned}
\mathcal{H}_{\rm lc} &= \mathcal{H}_0 + \frac{1}{R^2}\mathcal{H}_{\rm int}\,, \\
\mathcal{H}_0 &= \frac{1}{2}\left(p_y^2 + \frac{p_\eta^2}{y^2} + y^2 + y^2(a_1^2\sin^2\eta + a_2^2\cos^2\eta)\right), \\
\mathcal{H}_{\rm int} &= y^2 p_y^2 + p_\eta^2 + y^4\big(1 - a_1^2\sin^2\eta - a_2^2\cos^2\eta\big) \\
&\quad - \frac{\big(y^2 p_y^2 + p_\eta^2 + y^4\big(1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta\big)\big)^2}{2y^4}\,.
\end{aligned}
$$