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Yang-Baxter sigma models from 4D Chern-Simons theory

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0. Introduction

Our interest here

Construct a unified way to describe the 2D integrable models

Why is this issue so important? (My personal point of view)

In the study of integrable systems, integrable models are discovered suddenly and when a certain amount of them have been obtained, beautiful universal structures behind them are extracted such as Yang-Baxter equation.

Even now, new integrable models are being discovered one after another. But we did not know a method to describe everything from the traditional integrable models to the latest new types of models in a unified manner.

If this is compared to the study of elementary particle physics, the discovery of an integrable model corresponds to that of a new particle, and its unified theory corresponds to finding a unified model of elementary particles (though this theory would be replaced by a larger new theory, subsequently,,,).

The candidate of the unified theory

4D Chern-Simons (CS) theory

[Costello-Yamazaki, 1908.02289]

$$S[A] = \frac{i}{4\pi} \int_{\mathcal{M}\times\mathbb{C}P^1} \omega \wedge CS(A)$$

c.f. Costello-Yamazaki-Witten, 1709.09993, 1802.01579

A takes a value in Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of a semi-simple Lie group $G^{\mathbb{C}}$

$\mathcal{M}$: a 2D surface with the coordinates $(\tau,\ \sigma)$. z is a coordinate of $\mathbb{C}P^1$.

$$CS(A) \equiv \left\langle A, dA + \frac{2}{3} A \wedge A \right\rangle$$: Chern-Simons 3-form

$$\omega \equiv \varphi(z) dz$$: a meromorphic 1-form

This 1-form is closely related to the integrable structure of 2D integrable sigma model (ISM) to be derived.

The recipe to derive 2D ISMs from 4D CS

1. Prepare a meromorphic 1-form.

   The structure of poles and zeros determines the resulting 2D ISM.

2. Take a boundary condition for the gauge field A .

   Possible boundary conditions are governed by the equation of motion.

3. Reduce 4D CS to a 2D system by following a procedure.

   There are some reduction methods. Take one of them as you like.

As a result, we see that the resulting 2D system is classically integrable because the associated Lax pair can be constructed along this way.

The content of my talk

Explain how to derive 2D ISMs from 4D CS by taking a reduction method developed by Delduc-Lacroix-Magro-Vicedo (DLMV)

[Delduc-Lacroix-Magro-Vicedo, 1909.13824]

1. A reduction method by DLMV

2. Concrete examples: 2D principal chiral model
   Yang-Baxter sigma models
   A brief summary of my related works
   [Fukushima-Sakamoto-KY]

3. Summary and discussion

1. A reduction method by DLMV

A reduction method by DLMV

[Delduc-Lacroix-Magro-Vicedo, 1909.13824]

Our starting point:

$$S[A] = \frac{i}{4\pi} \int_{\mathcal{M}\times\mathbb{C}P^1} \omega \wedge CS(A) \ , \qquad CS(A) \equiv \left\langle A, dA + \frac{2}{3} A \wedge A \right\rangle$$

$\omega \equiv \varphi(z) dz$: a meromorphic 1-form

This action has an extra gauge symmetry:

$$A \mapsto A + \chi\, dz$$

Hence the z-component can always be gauged away:

$$A = A_\sigma\, d\sigma + A_\tau\, d\tau + A_{\bar{z}}\, d\bar{z}$$

Equations of motion:

$$\omega \wedge F(A) = 0 \qquad \text{(bulk eom)}$$

$$d\omega \wedge \langle A, \delta A \rangle = 0 \qquad \text{(boundary eom)}$$

(bulk eom) → Species of 2D ISM

(boundary eom) → Integrable twists

NOTE 1 : If φ is smooth, the boundary eom is trivially satisfied.

But now $d\omega = \partial_{\bar{z}}\varphi(z)\, d\bar{z} \wedge dz$ i.e., $\partial_{\bar{z}}\frac{1}{z} = 2\pi\delta(z, \bar{z})$

and hence a delta function may appear if φ has a pole.

NOTE 2 : From the bulk eom, the zeros of φ are also important because a derivative of *A* may be a distribution, i.e., $x\,\delta(x) = 0$.

Let us introduce the following notation:

$\mathfrak{p}$: set of poles of φ	$\mathfrak{z}$: set of zeros of φ

NOTE3: The boundary eom has the support only on $\mathcal{M} \times \mathfrak{p} \subset \mathcal{M} \times \mathbb{C}P^1$.

Indeed, it can be rewritten as

$$\sum_{x\in\mathfrak{p}} \sum_{p\geq 0} (\mathrm{res}_x \xi_x^p \,\omega)\, \epsilon^{ij} \frac{1}{p!} \partial_{\xi_x}^p \, \langle A_i, \delta A_j \rangle|_{\mathcal{M}\times\{x\}} = 0$$

Here the local holomorphic coordinates ξ_x are defined as

$$\xi_x \equiv z - x \quad (x \in \mathfrak{p}\backslash\{\infty\})\,, \quad \xi_\infty \equiv 1/z$$

Lax form

Let us perform a formal gauge transformation:

$$A = -d\hat{g}\hat{g}^{-1} + \hat{g}\,\mathcal{L}\,\hat{g}^{-1} \qquad \text{a smooth function } \hat{g} \;:\; \mathcal{M} \times \mathbb{C}P^1 \to G^{\mathbb{C}}$$

Then the $\bar{z}$-component of $\mathcal{L}$ can be removed as $\mathcal{L}_{\bar{z}} = 0$ (e.g., temporal gauge)

Then the Lax form is given by

$$\mathcal{L} \equiv \mathcal{L}_\sigma \, d\sigma + \mathcal{L}_\tau \, d\tau$$

(to be identified with Lax of 2D ISM)

The bulk eom leads to

$$\partial_\tau \mathcal{L}_\sigma - \partial_\sigma \mathcal{L}_\tau + [\mathcal{L}_\tau, \mathcal{L}_\sigma] = 0$$

→ Flatness condition

$$\omega \wedge \partial_{\bar{z}} \mathcal{L} = 0$$

NOTE: the set of zeros of φ is that of poles of $\mathcal{L}$

For simplicity, we assume below that φ has

at most first-order zero & at most double poles

The ansatz for Lax form

$$\mathcal{L} = \sum_{i \in \mathfrak{z}} V^i(\tau, \sigma) \xi_i^{-1} d\sigma^i + U_\sigma(\tau, \sigma) d\sigma + U_\tau(\tau, \sigma) d\tau$$

Here $V^i(\tau, \sigma) \quad (i = +, -)$, $U_\tau(\tau, \sigma)$, $U_\sigma(\tau, \sigma)$ are smooth functions.

$$\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$$

These functions are unknown functions at this moment and to be determined from a boundary condition for the gauge field.

Later, we will see how to do it for 2D principal chiral model concretely.

The original 4D CS can be rewritten as

$$S[A] = -\frac{i}{4\pi}\int_{\mathcal{M}\times\mathbb{C}P^1} \omega \wedge d\langle \hat{g}^{-1} d\hat{g}, \mathcal{L}\rangle - \frac{i}{4\pi}\int_{\mathcal{M}\times\mathbb{C}P^1} \omega \wedge I_{\rm WZ}[\hat{g}]$$

$$I_{\rm WZ}[u] \equiv \frac{1}{3}\langle u^{-1}du, u^{-1}du \wedge u^{-1}du\rangle$$

To reduce this 4D action to a 2D theory, let us suppose

the archipelago conditions:

There exist open disks $V_x,\ U_x$ for each $x \in \mathfrak{p}$ such that $\{x\} \subset V_x \subset U_x$ and

i) $U_x \cap U_x = \phi$ If $x \neq y$ for all $x,\ y \in \mathfrak{p}$

ii) $\hat{g} = 1$ outside $\mathcal{M} \times \cup_{x\in\mathfrak{p}} U_x$

iii) $\hat{g}|_{\mathcal{M}\times U_x}$ depends only on $\tau,\ \sigma$ and the radial coordinate $|\xi_x|$

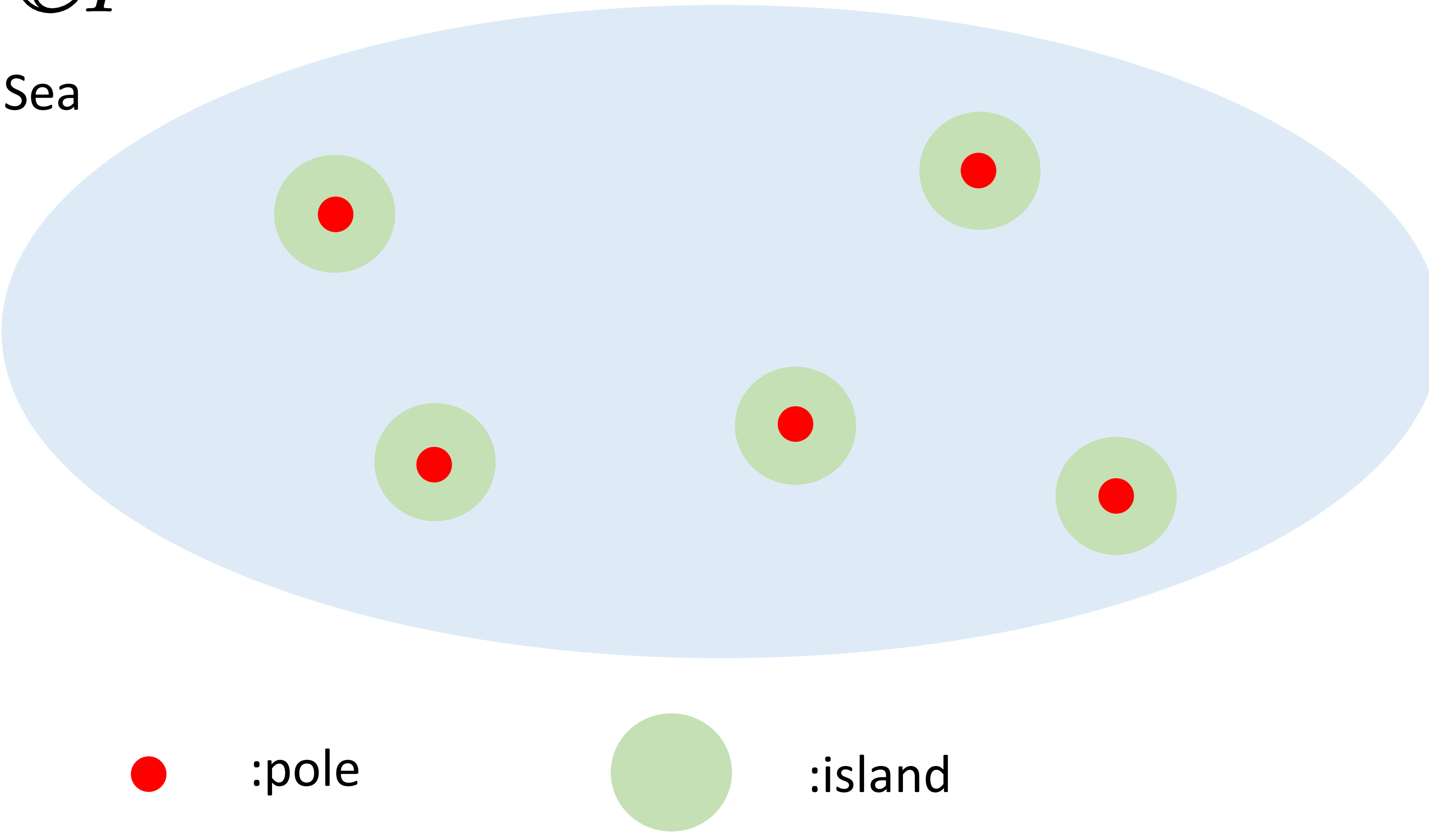
iv) $\hat{g}|_{\mathcal{M}\times V_x}$ depends only on $\tau,\ \sigma$, that is, $g_x \equiv \hat{g}|_{\mathcal{M}\times V_x} = \hat{g}|_{\mathcal{M}\times\{x\}}$

$\hat{g}$ depends only on τ, σ on the islands

Otherwise, $\hat{g} = 1$ (i.e., on the sea).

$\mathbb{C}P^1$

= Sea



:pole

:island

Master formula

$$S\left[\{g_x\}_{x\in\mathfrak{p}}\right] = \frac{1}{2}\sum_{x\in\mathfrak{p}}\int_{\mathcal{M}}\left\langle \mathrm{res}_x(\varphi\,\mathcal{L}),\ g_x^{-1}dg_x\right\rangle - \frac{1}{2}\sum_{x\in\mathfrak{p}}(\mathrm{res}_x\omega)\int_{\mathcal{M}\times[0,R_x]} I_{\mathrm{WZ}}[g_x]$$

Refined recipe to derive 2D ISM

1. Specify the form of ω
2. Take a boundary condition of A at the poles of ω
3. Fix the form of Lax form $\mathcal{L}$ with the above information.
4. Finally, evaluate the above master formula.

→ 2D ISM

2. Concrete Examples

1. Principal chiral model with Wess-Zumino (WZ) term

INPUT A meromorphic 1-form

$$\omega = \varphi(z)\, dz = K \frac{1 - z^2}{(z-k)^2}\, dz \qquad K,\ k\ :\ \text{real constants}$$

$z = k,\ \infty$ are double poles $\Rightarrow$ $\mathfrak{p} = \{k,\ \infty\}$

$z = \pm 1$ are zeros $\Rightarrow$ $\mathfrak{z} = \{+1, -1\}$

Boundary condition

The boundary condition of A at the poles of ω is

$$A_i|_k = 0, \quad A_i|_\infty = 0 \quad (i = \tau,\ \sigma)$$

By using the Archipelago condition, the group element $\hat{g}$ is restricted as

$$g_k = g(\tau, \sigma)\,, \quad \underline{g_\infty = 1}$$

due to the gauge symmetry

Then the boundary condition can be rewritten as

$$A|_k = -dg \cdot g^{-1} + g\mathcal{L}g^{-1} = 0$$

$$A|_\infty = \mathcal{L}|_\infty = 0$$

Due to the second condition, $U_\tau\,,\ U_\sigma$ in the Lax form should be zero.

Thus the Lax form is

$$\mathcal{L} = \frac{V^{+1}}{z-1}d\sigma^+ + \frac{V^{-1}}{z+1}d\sigma^-$$

Then, by substituting the Lax form into the first boundary condition, we obtain

$$V^{\pm 1} = (k \mp 1) j_\pm \, , \qquad j_\pm \equiv g^{-1} \partial_\pm g$$

Thus, the Lax form has been determined as

$$\mathcal{L} = \frac{k-1}{z-1} j_+ d\sigma^+ + \frac{k+1}{z+1} j_- d\sigma^-$$

Finally, by putting this Lax form into the master formula, 2D action is given by

$$S[g] = \frac{K}{2} \int_{\mathcal{M}} d\sigma \wedge d\tau \, \langle j_+, j_- \rangle + K\, k\, I_{\mathrm{WZ}}[g]$$

This is nothing but 2D principal chiral model with the WZ term.

2. Homogeneous Yang-Baxter sigma model

The 1-form is the same as the previous (but *k*=0 for simplicity)

$$\omega = \varphi(z)\,dz = K\frac{1-z^2}{z^2}\,dz \qquad K \ : \ \text{a real constant}$$

But the boundary condition of *A* at the poles of *ω* is replaced by

$$A_i|_0 = -R\,\partial_z A_i|_0\ , \qquad A_i|_\infty = 0 \quad (i=\tau,\ \sigma)$$

Here R is a linear operator from $\mathfrak{g} \to \mathfrak{g}$ satisfying

the homogeneous Yang-Baxter equation

$$[R(x),R(y)] - R([R(x),y]+[x,R(y)]) = 0$$

It is useful to introduce the notation: $R_g \equiv \mathrm{Ad}_{g^{-1}} \circ R \circ \mathrm{Ad}_g$

Lax form: $$\mathcal{L} = \frac{1}{z-1}\frac{-1}{1+R_g} j_+ \, d\sigma^+ + \frac{1}{z+1}\frac{1}{1-R_g} j_- \, d\sigma^-$$

2D action: $$S[g] = \frac{K}{2}\int_{\mathcal{M}} d\sigma \wedge d\tau \left\langle j_+, \frac{1}{1-R_g} j_- \right\rangle$$

Homogeneous Yang-Baxter sigma model

[Klimcik, hep-th/0210095, 0802.3518] [Delduc-Magro-Vicedo, 1308.3581] [Matsumoto-KY, 1501.03665]

My related works:

- Generalization of the DLMV method to symmetric coset case

[Fukushima-Sakamoto-KY, 2005.04950]

Introduction of the grading automorphism

In particular, we have derived homogeneous YB deformed $AdS_5 x S^5$ superstring from 4D CS

[Kawaguchi-Matsumoto-KY, 1401.4855]

c.f. λ-deformation of $AdS_5 x S^5$ superstring from 4D CS [Tian-He-Chen, 2007.00422]

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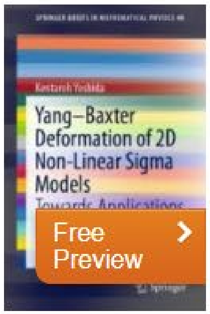


Yang–Baxter Deformation of 2D Non-Linear Sigma Models

Towards Applications to AdS/CFT

Authors: **Yoshida**, Kentaroh

Introduces a new method called Yang–Baxter deformation to perform integrable deformations systematically

» see more benefits

About this book

In mathematical physics, one of the fascinating issues is the study of integrable systems. In particular, non-perturbative techniques that have been developed have triggered significant insight for real physics. There are basically two notions of integrability: classical integrability and quantum integrability. In this book, the focus is on the former, classical integrability. When the system has a finite number of degrees of freedom, it has been well captured by the Arnold–Liouville theorem. However, when the number of degrees of freedom is infinite, as in classical field theories, the

» Show all

My other works

- Generalization to include order defects

[Fukushima-Sakamoto-KY, 2012.07370]
[Fukushima-Sakamoto-KY, 2112.11276]

We have derived the Faddeev-Reshetikhin model and non-abelian Toda field theories including (complex) sine-Gordon model and Liouville theory.

The derivation of sine-Gordon model was a long-standing problem from the original CY paper (2019).

- Derivation of Integrable $T^{1,1}$ sigma model from 4D CS

[Fukushima-Sakamoto-KY, 2105.14920]

It is well known that the usual $T^{1,1}$ sigma model is non-integrable.

[Basu-Pando Zayas, 1103.4107]

But recently, a modified $T^{1,1}$ has been shown to be integrable.

[Arutyunov-Bassi-Lacroix, 2010.05573]

We have studied classical chaos apart from integrable points.

[Ishii-Kushiro-KY, 2103.12416]

3. Summary and Discussion

3) Summary and Discussion

We have discussed how to derive 2D ISMs from 4D CS.

In particular, superstring on $AdS_5 x S^5$ is also included.

→ The origin of kappa-symmetry?

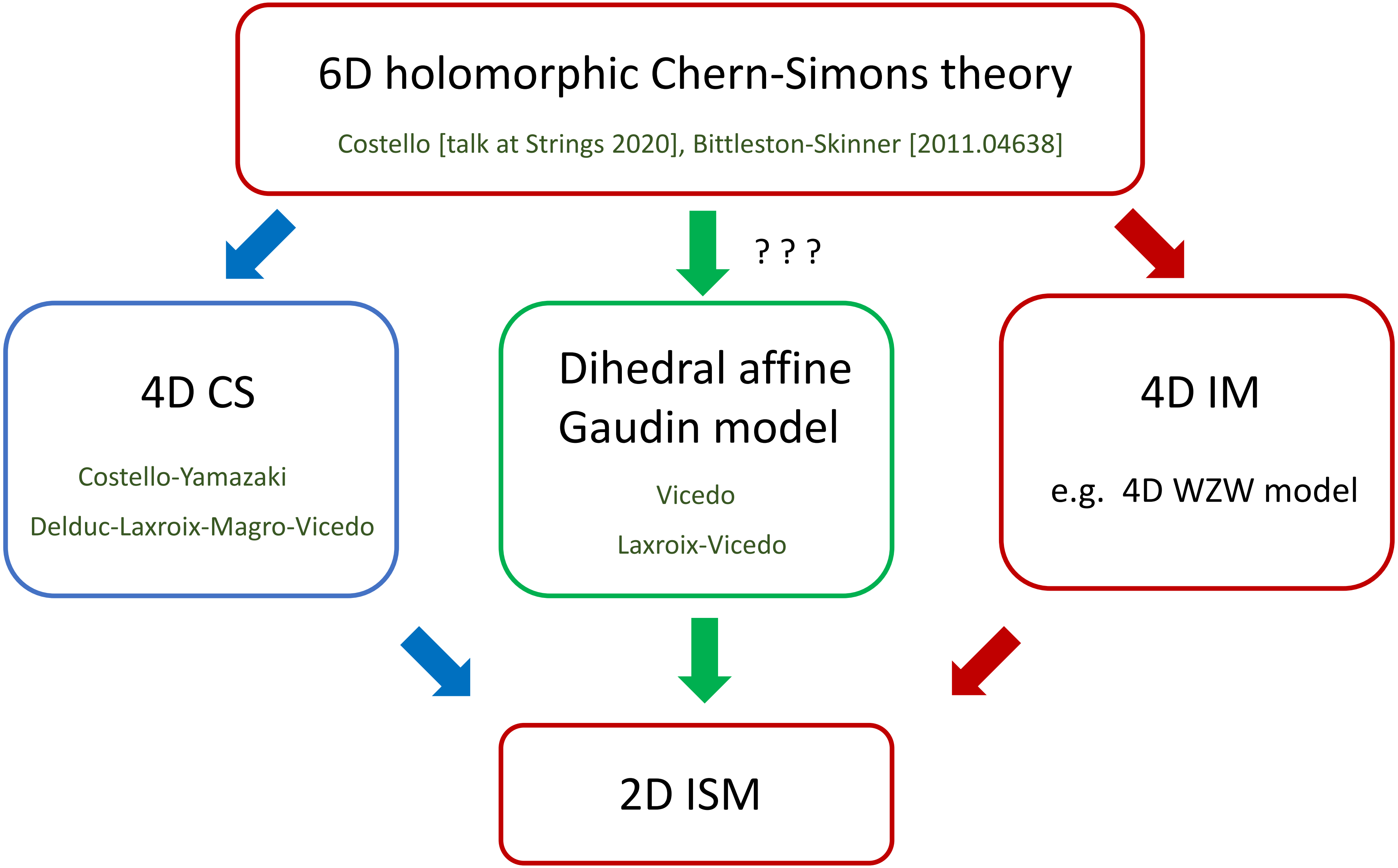
Kappa symmetry: A fermionic gauge symmetry in the Green-Schwarz formulation of superstring theory which is based on space-time fermions. It is necessary to remove the redundant space-time fermions. But it was introduced in a heuristic way and its origin is unclear.

Take-home message

The unified theory of 2D ISMs may reveal the fundamental symmetry of String Theory.

But 4D CS scenario might be a tip of the iceberg!

Current understanding:



Question: Are these three ways equivalent?

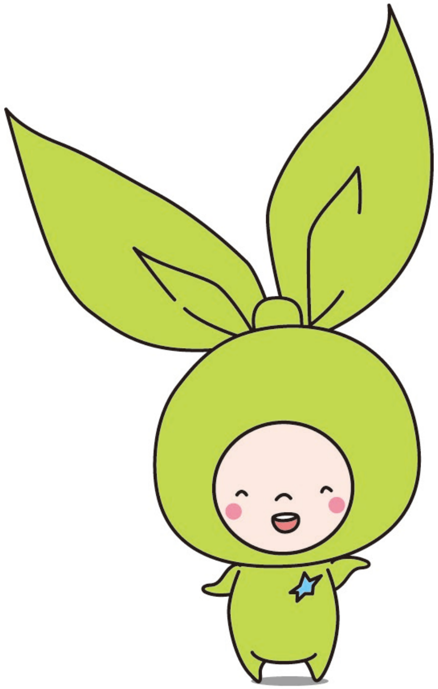
No?

According to a paper [2105.06826] by Bin Chen, Yi-jun He and Jia Tian,

Homogeneous YB sigma models, λ-models, generalized λ-models
CANNOT be obtained from 4D IM, though these are derived from 4D CS.

These models may be counter-examples for the equivalence.

Thank you for your attention!



延長戦

Non-Abelian Toda Field Theory (NATFT)

$$S_{\rm NATFT}[h] = \frac{1}{2\pi\beta^2}\left(S_{\rm WZW}[h] - \int_{\mathcal{M}} d^2\sigma\, V(h)\right)\,, \qquad V(h) := -m^2\langle \Lambda_+\,,\, h^{-1}\Lambda_- h\rangle\,,$$

$$S_{\rm WZW}[h] := -\frac{1}{2}\int_{\mathcal{M}} d^2\sigma\,\langle h^{-1}\partial_+ h\,, h^{-1}\partial_- h\rangle - \int_{\mathcal{M}\times[0,R_x]} I_{\rm WZ}[h]\,,$$

$$I_{\rm WZ}[h] := \frac{1}{3}\langle h^{-1}dh\,, h^{-1}dh\wedge h^{-1}dh\rangle\,, \qquad \partial_\mp\Lambda_\pm = 0\,.$$

h is a smooth function $\mathcal{M}\to G$, and $\Lambda_\pm$ are generators of $\mathfrak{g}$.

Decomposition of Lie algebra

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\,, \qquad \sigma(\mathsf{x}) = (-1)^p\mathsf{x}\,, \qquad {}^\forall\mathsf{x}\in\mathfrak{g}_p\,, \quad p = 0,1\,.$$

The Lie group G_0 is a subgroup of G associated with $\mathfrak{g}_0$.

$$\begin{cases} h\in G_0\,, & \Lambda_\pm\in\mathfrak{g}_0 \quad : \text{ a homogeneous sine-Gordon model} \\ h\in G_0\,, & \Lambda_\pm\in\mathfrak{g}_1 \quad : \text{ a symmetric space sine-Gordon model} \end{cases}\,.$$

EOM

$$0 = \partial_+(h^{-1}\partial_- h) + m^2[h\Lambda_+ h^{-1}\,,\Lambda_-]\,.$$

Zero curvature condition and Lax pair

$$0 = [\partial_+ + \mathcal{L}_+\,, \partial_- + \mathcal{L}_-]\,,$$

$$\mathcal{L}_+ = h^{-1}\partial_+ h + imw\Lambda_+\,, \qquad \mathcal{L}_- = \frac{im}{w} h^{-1}\Lambda_- h\,,$$

where *w* is a spectral parameter

Example 1: sine-Gordon model (a symmetric space case)

Let us take the generators of $\mathfrak{su}(2)$ T^a $(a = 1, 2, 3)$ as

$$[T^a, T^b] = \epsilon^{abc} T^c \,, \qquad \langle T^a \,, T^b \rangle = -\frac{1}{2}\delta^{ab}$$

$$\mathfrak{g} = \mathfrak{su}(2) \text{ and } \mathfrak{g}_0 = \mathfrak{u}(1) \,.$$

$$h = \exp\big(\beta\phi T^3\big) \,, \qquad \Lambda_+ = \Lambda_- = T^1 \,, \qquad \phi \in \mathbb{R}$$

Classical action and Lax pair

$$S_{\text{SG}}[\phi] = \frac{1}{4\pi} \int_{\mathcal{M}} d^2\sigma \left(\frac{1}{2}\partial_+\phi\partial_-\phi - \frac{m^2}{\beta^2}\cos(\beta\phi) \right) \,,$$

$$\mathcal{L}_+ = imwT^1 + \beta\partial_+\phi T^3 \,,$$

$$\mathcal{L}_- = \frac{im}{w}\Big(\cos(\beta\phi)T^1 - \sin(\beta\phi)T^2 \Big) \,.$$

Example 2: Liouville theory (a symmetric space case)

$$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \text{ and } \mathfrak{g}_0 = \mathbb{R}$$

$$[T^1, T^2] = T^3\,, \quad [T^0, T^1] = -T^2\,, \quad [T^0, T^2] = T^1\,,$$
$$\langle T^0, T^0\rangle = -\frac{1}{2}\,, \quad \langle T^1, T^1\rangle = \frac{1}{2}\,, \quad \langle T^2, T^2\rangle = \frac{1}{2}\,, \quad \text{otherwise} = 0\,.$$

$$h = \exp\left(2\beta\phi T^1\right)\,, \quad \Lambda_+ = T^2 + T^0\,, \quad \Lambda_- = T^2 - T^0\,.$$

Classical action and Lax pair

$$S_{\text{Liouville}}[\phi] = \frac{1}{\pi}\int_{\mathcal{M}} d^2\sigma\left(-\frac{1}{2}\partial_+\phi\partial_-\phi + \frac{m^2}{2\beta^2}e^{2\beta\phi}\right)\,,$$

$$\mathcal{L}_+ = imwT^0 + 2\beta\partial_+\phi T^1 + imwT^2\,,$$
$$\mathcal{L}_- = -\frac{im}{w}e^{2\beta\phi}T^0 + \frac{im}{w}e^{2\beta\phi}T^2\,.$$

Example 3: complex sine-Gordon model (a homogeneous case)

$$\mathfrak{g}_0 = \mathfrak{su}(2) \qquad \Lambda_+ = T^3\,, \quad \Lambda_- = -T^3\,.$$

$$h = \exp\left(\frac{\chi+\theta}{2}T^3\right)\exp\left((\phi-\pi)T^1\right)\exp\left(\frac{\chi-\theta}{2}T^3\right) \qquad (\phi\,,\chi\,,\theta\in\mathbb{R})\,.$$

Take the gauge $\theta = 0\,.$

Classical action and Lax pair

$$\begin{aligned} S_{\rm CSG}[\psi] &= \frac{1}{2\pi\beta^2}\int d^2\sigma\left(\frac{\partial_+\bar\psi\partial_-\psi+\partial_-\bar\psi\partial_+\psi}{2(1-|\psi|^2)} - m^2\left(\frac{1}{2}-|\psi|^2\right)\right) \\ &= \frac{1}{4\pi\beta^2}\int d^2\sigma\left(\frac{1}{2}\partial_+\phi\partial_-\phi - m^2\cos\phi + \frac{\tan^2(\phi/2)}{2}\partial_+\chi\partial_-\chi\right)\,. \end{aligned}$$

$$0 = \left[\partial_+ + h^{-1}\partial_+ h + imw\Lambda_+ + h^{-1}\mathcal{A}_+ h\;,\;\partial_- + \frac{im}{w}h^{-1}\Lambda_- h + \mathcal{A}_-\right]$$

$$\mathcal{A}_+ = \frac{\partial_+\chi}{2}\tan^2\left(\frac{\phi}{2}\right)T^3\,, \quad \mathcal{A}_- = -\frac{\partial_-\chi}{2}\tan^2\left(\frac{\phi}{2}\right)T^3$$

4D Chern-Simons theory with order defects

(In the previous talk, we have considered the 4D CS theory with disorder defect.)

$$S[\mathcal{G}_{(\pm)}, A] = S_{4\mathrm{dCS}}[A] + S_{\mathrm{defect}}[\mathcal{G}_{(\pm)}, A] \,,$$

$$S_{4\mathrm{dCS}}[A] := \frac{i}{4\pi} \int_{\mathcal{M}\times C} \omega \wedge \left\langle A\,,\, dA + \frac{2}{3} A \wedge A \right\rangle ,$$

$$S_{\mathrm{defect}}[\mathcal{G}_{(\pm)}, A] := C_{(+)} \int_{\mathcal{M}\times\{+z_1\}} d\sigma^+ \wedge d\sigma^- \left\langle \mathcal{G}_{(+)}^{-1} D_+ \mathcal{G}_{(+)} \,,\, \mathcal{G}_{(+)}^{-1} D_- \mathcal{G}_{(+)} \right\rangle$$

$$+ C_{(-)} \int_{\mathcal{M}\times\{-z_1\}} d\sigma^+ \wedge d\sigma^- \left\langle \mathcal{G}_{(-)}^{-1} D_+ \mathcal{G}_{(-)} \,,\, \mathcal{G}_{(-)}^{-1} D_- \mathcal{G}_{(-)} \right\rangle ,$$

4d manifold $\mathcal{M} \times C$ $\quad$ $\mathcal{M}$ is 2d Minkowski spacetime and $C := \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$

The two defects are located at $z = \pm z_1$ on C. $\quad$ $z_1 \in \mathbb{R}$ and $C_{(\pm)} \in \mathbb{R}$.

$\mathcal{G}_{(\pm)} : \mathcal{M} \times C \to G^{\mathbb{C}}$. $\quad$: group-valued smooth functions

Covariant derivative:

$$D_\pm := \partial_\pm + A_\pm \,.$$

A meromorphic 1-form:

$$\omega = \varphi(z) dz := \frac{1}{(z - z_1)(z + z_1)} dz$$

The bulk eom

$$F_{+-} = 0\,,$$

$$\delta A_- \cdot \varphi(z) F_{\bar{z}+} = +\,2\pi i\, \delta A_- \cdot \sum_{x=\pm z_1} C_{(x)} \delta(z-x) D_+ \mathcal{G}_{(x)} \cdot \mathcal{G}_{(x)}^{-1}\,,$$

$$\delta A_+ \cdot \varphi(z) F_{\bar{z}-} = -\,2\pi i\, \delta A_+ \cdot \sum_{x=\pm z_1} C_{(x)} \delta(z-x) D_- \mathcal{G}_{(x)} \cdot \mathcal{G}_{(x)}^{-1}\,,$$

The boundary eom

$$\begin{aligned} 0 &= (\mathrm{Res}_{z=+z_1}\,\omega)\epsilon^{\alpha\beta} \langle A_\alpha|_{+z_1}\,, \delta A_\beta|_{+z_1} \rangle + (\mathrm{Res}_{z=-z_1}\,\omega)\epsilon^{\alpha\beta} \langle A_\alpha|_{-z_1}\,, \delta A_\beta|_{-z_1} \rangle \\ &= \epsilon^{\alpha\beta} \big(\langle A_\alpha|_{+z_1}\,, \delta A_\beta|_{+z_1} \rangle - \langle A_\alpha|_{-z_1}\,, \delta A_\beta|_{-z_1} \rangle \big)\,. \end{aligned}$$

Boundary condition (from the boundary eom)

$$A_+|_{z\to+z_1} = \mathcal{O}(z-z_1)\,, \qquad A_-|_{z\to-z_1} = \mathcal{O}(z+z_1)\,.$$

Furthermore, we suppose that

$$\left(\partial_+ \mathcal{G}_{(+)}\right)\Big|_{z\to+z_1} = \mathcal{O}(z-z_1)\,, \quad \left(\partial_- \mathcal{G}_{(-)}\right)\Big|_{z\to-z_1} = \mathcal{O}(z+z_1)\,.$$

The we can write them as follows:

$$A_+ = (z - z_1)\mathscr{A}_+ \,, \qquad A_- = (z + z_1)\mathscr{A}_- \,,$$
$$\mathcal{G}_{(+)} = (z - z_1)\tilde{\mathcal{g}}_{(+)}(\sigma^+, \sigma^-) + \mathcal{g}_{(+)}(\sigma^-) \,,$$
$$\mathcal{G}_{(-)} = (z + z_1)\tilde{\mathcal{g}}_{(-)}(\sigma^+, \sigma^-) + \mathcal{g}_{(-)}(\sigma^+) \,,$$

where $\mathscr{A}_\pm$, $\tilde{\mathcal{g}}_{(\pm)}$ and $\mathcal{g}_{(\pm)}$ are smooth functions with $\mathscr{A}_\pm : \mathcal{M} \times C \to \mathfrak{g}^{\mathbb{C}}$ and $\tilde{\mathcal{g}}_{(\pm)}, \mathcal{g}_{(\pm)} : \mathcal{M} \to G^{\mathbb{C}}$, respectively.

Regularization:

On the left-hand side of the bulk eom, there are delta functions.

Hence, the gauge field should behave around the location of the defects:

$$A_\pm \sim \frac{1}{z - x} \qquad (x = \pm z_1) \,.$$

For later convenience, we will take the following regularization:

$$A_\pm \sim \frac{1}{z - x}\left(1 - e^{-|z-x|^2/\alpha_x}\right) \,;$$ ⟹ b.c. condition is satisfied.

Lax Form

$$A = -d\hat{g}\hat{g}^{-1} + \hat{g}\mathcal{L}\hat{g}^{-1}\,, \qquad \mathcal{G}_{(\pm)} = \hat{g}g_{(\pm)}\,.$$

Here we take a gauge $\mathcal{L}_{\bar{z}} = 0$ by taking the following gauge field configuration

$$A_{\bar{z}} = -(\partial_{\bar{z}}\hat{g})\,\hat{g}^{-1}\,.$$

Then the Lax form

$$\mathcal{L} = \mathcal{L}_\tau d\tau + \mathcal{L}_\sigma d\sigma = \mathcal{L}_+ d\tau^+ + \mathcal{L}_- d\sigma^-\,.$$

satisfies the bulk eom:

$$0 = \partial_+\mathcal{L}_- - \partial_-\mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-]\,,$$

$$\frac{1}{z^2 - z_1^2}\delta A_-\partial_{\bar{z}}\mathcal{L}_+ = +2\pi i\,\delta A_- \sum_{x=\pm z_1} C_{(x)}\delta(z-x)(\partial_+ + \mathcal{L}_+)g_{(x)}\cdot g_{(x)}^{-1}\,,$$

$$\frac{1}{z^2 - z_1^2}\delta A_+\partial_{\bar{z}}\mathcal{L}_- = -2\pi i\,\delta A_+ \sum_{x=\pm z_1} C_{(x)}\delta(z-x)(\partial_- + \mathcal{L}_-)g_{(x)}\cdot g_{(x)}^{-1}\,.$$

The ansatz for the Lax form:

$$\mathcal{L}_+ = \frac{U_{1,+}z + U_{0,+}z_1}{z + z_1}\,, \qquad \mathcal{L}_- = \frac{U_{1,-}z + U_{0,-}z_1}{z - z_1}\,,$$

Note: the poles of the Lax form are the poles of a meromorphic function.

From the grading constraint, it can be rewritten as

$$\mathcal{L}_+ = \frac{U_{0,+}z + \sigma(U_{0,+})z_1}{z + z_1} = \frac{z - z_1}{z + z_1}\widetilde{V}_+ + V_+\,,$$
$$\mathcal{L}_- = \frac{U_{0,-}z + \sigma(U_{0,-})z_1}{z - z_1} = \frac{z + z_1}{z - z_1}\widetilde{V}_- + V_-\,,$$
$$V_\pm := \frac{U_{0,\pm} + \sigma(U_{0,\pm})}{2}\,, \quad \widetilde{V}_\pm := \frac{U_{0,\pm} - \sigma(U_{0,\pm})}{2}\,.$$

For the Lax pole, we will take the regularization:

$$\frac{1}{z - x} \sim \frac{1}{z - x}\left(1 - \exp\left(-|z - x|^2/\alpha_x\right)\right).$$

Utilizing the gauge transformations, one of natural parametrizations is given by

$$\hat{g}|_{z_1} = h\,, \quad \hat{g}|_{-z_1} = 1\,, \quad g_{(+)}|_{z_1} = h^{-1}\,, \quad g_{(-)}|_{-z_1} = 1\,.$$

Then the boundary eom is rewritten as

$$\begin{aligned}
\epsilon_- \cdot \delta(z+z_1)\widetilde{V}_+ = &- C_{(+)}\epsilon_- \cdot \delta(z-z_1)\left(-(z+z_1)h^{-1}\partial_+ h + (z-z_1)\widetilde{V}_+ + (z+z_1)V_+\right) \\
&- C_{(-)}\epsilon_- \cdot \delta(z+z_1)\left(\frac{1}{2}(z-z_1)\widetilde{V}_+ + (z+z_1)V_+\right)\,, \\
\epsilon_+ \cdot \delta(z-z_1)\widetilde{V}_- = &+ C_{(+)}\epsilon_+ \cdot \delta(z-z_1)\left(-(z-z_1)h^{-1}\partial_- h + \frac{1}{2}(z+z_1)\widetilde{V}_- + (z-z_1)V_-\right) \\
&+ C_{(-)}\epsilon_+ \cdot \delta(z+z_1)\left((z+z_1)\widetilde{V}_- + (z-z_1)V_-\right)\,,
\end{aligned}$$

where the variation of the gauge field is expressed as

$$\delta A_+ = (z-z_1)\epsilon_+\,, \quad \delta A_- = (z+z_1)\epsilon_-\,, \qquad \epsilon_\pm|_{z\to\pm z_1} = \mathcal{O}(1)\,.$$

The following formula is also useful:

$$\frac{ie^{-|z-x|^2/\alpha_x}}{2\pi\alpha_x}\cdot\left(1-e^{-|z-x|^2/\alpha_x}\right) = \frac{ie^{-|z-x|^2/\alpha_x}}{2\pi\alpha_x} - \frac{1}{2}\frac{ie^{-|z-x|^2/(\alpha_x/2)}}{2\pi(\alpha_x/2)} \sim \frac{1}{2}\delta(z-x)\,.$$

Finally, the bulk eom is rewritten as

$$\delta(z+z_1)\widetilde{V}_+ = +\,2z_1 C_{(+)}\delta(z-z_1)\big(-h^{-1}\partial_+ h + V_+\big) - z_1 C_{(-)}\delta(z+z_1)\widetilde{V}_+\,,$$
$$\delta(z-z_1)\widetilde{V}_- = -\,z_1 C_{(+)}\delta(z-z_1)\widetilde{V}_- + 2z_1 C_{(-)}\delta(z+z_1)V_-\,.$$

Solution

$$C_{(+)} = C_{(-)} = \frac{1}{z_1}\,, \qquad V_+ = h^{-1}\partial_+ h\,, \qquad V_- = 0\,.$$

Let us consider a variation $\delta\mathcal{G}_{(\pm)} = \mathcal{G}_{(\pm)}\epsilon_{(\pm)}$, such that $\partial_\pm \epsilon_{(\pm)}|_{z\to\pm z_1} = \mathcal{O}(z\mp z_1)$.

$$\epsilon_{(\pm)} = (z\mp z_1)\tilde{\varepsilon}_{(\pm)}(\sigma^+,\sigma^-) + \varepsilon_{(\pm)}(\sigma^\mp)\,, \qquad \tilde{\varepsilon}|_{z\to\pm z_1} = \mathcal{O}(1)\,.$$

Then the eom with respect the defects is written as

$$0 = (z - z_1)\tilde{\varepsilon}_{(+)}\partial_+ \left(h(\partial_- + \frac{1}{2}\frac{z+z_1}{z-z_1}\widetilde{V}_-)h^{-1} \right)\Big|_{z\to +z_1}$$

$$+ \varepsilon_{(+)}\partial_- \left(h(\partial_+ + h^{-1}\partial_+ h + \frac{z-z_1}{z+z_1}\widetilde{V}_+)h^{-1} \right)\Big|_{z\to +z_1},$$

$$0 = \varepsilon_{(-)}\partial_+ \left(\frac{z+z_1}{z-z_1}\widetilde{V}_- \right)\Big|_{z\to -z_1} + (z+z_1)\tilde{\varepsilon}_{(-)}\partial_- \left(h^{-1}\partial_+ h + \frac{1}{2}\frac{z-z_1}{z+z_1}\widetilde{V}_+ \right)\Big|_{z\to -z_1},$$

$$\partial_+ \left(h\widetilde{V}_- h^{-1} \right) = 0\,, \qquad \partial_- \widetilde{V}_+ = 0 \qquad \Rightarrow \quad \widetilde{V}_- = im\, h^{-1}\Lambda_- h\,, \qquad \widetilde{V}_+ = im\,\Lambda_+\,,$$

The resulting Lax pair is

$$\mathcal{L}_+ = h^{-1}\partial_+ h + im\frac{z-z_1}{z+z_1}\Lambda_+\,, \qquad \mathcal{L}_- = im\frac{z+z_1}{z-z_1}h^{-1}\Lambda_- h\,.$$

By identifying the parameter as $w = (z - z_1)/(z + z_1)$

the Lax pair of the NATFT is reproduced.

The resulting 2D action:

$$S_{2\mathrm{d}}[h;\Lambda_\pm] = \frac{1}{4z_1}\int_{\mathcal{M}} d\sigma^+ \wedge d\sigma^- \,\mathrm{tr}\big(h^{-1}\partial_+ h\, h^{-1}\partial_- h\big) - \frac{1}{4z_1}\int_{\mathcal{M}\times[0,R_x]} I_{\mathrm{WZ}}[\bar{h}]$$
$$- \frac{m^2}{2z_1}\int_{\mathcal{M}} d\sigma^+ \wedge d\sigma^- \,\mathrm{tr}\big(\Lambda_+ h^{-1}\Lambda_- h\big)\,.$$

By identifying the parameter as follows:

$$\frac{1}{2\pi\beta^2} = \frac{1}{4z_1}$$

the NATFT action can be reproduced.

Back Up

Gauge transformation:

$$A \mapsto A^u := uAu^{-1} - duu^{-1}\,, \qquad \mathcal{G}_{(\pm)} \mapsto \mathcal{G}^u_{(\pm)} := u\,\mathcal{G}_{(\pm)}$$

From the b.c.,

$$\partial_+ u|_{+z_1} = 0\,, \qquad \partial_- u|_{-z_1} = 0\,.$$

From the reality condition, u at the defect is parametrized as

$$u|_{\pm z_1} : \mathcal{M} \to G_0\,, \qquad u|_{\pm z_1} = \exp(\alpha_a T^a) \qquad (\alpha : \mathcal{M} \to \mathbb{R})\,.$$