3D integrability in gauge theories

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Introduction

Motivation

Understand connections between

$$\text{gauge theories} \quad \longleftrightarrow \quad \text{integrable systems}\,.$$

Integrability appears in many gauge theory contexts:

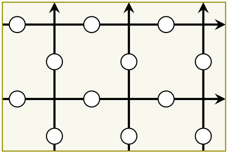
- Gauge/Bethe correspondence [Nekrasov–Shatashvili,. . .]
- Gauge/YBE correspondence [Spiridonov, Yamazaki,. . .]
- 4D Chern–Simons theory [Costello–Yamazaki–Witten,. . .]
- AdS/CFT integrability [Many people]

All of these are related to 2D integrability.

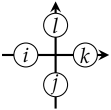
What about 3D integrability?

2D Lattice Models

Consider a 2D classical spin model on a square lattice:



4 spins surrounding a vertex interact, with energy

$E_{ij}^{kl} =$ 

$, \quad i, j, k, l \in \{1, \ldots, N\}\,.$

Assign an N-dim local Hilbert space V to each line.

We can think of i, j, k, l as state vectors in V.

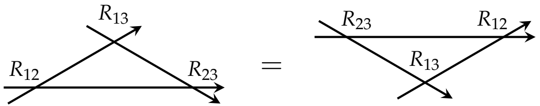
Yang–Baxter Equation

The local Boltzmann weights define the R-matrix

$$R \in \mathrm{End}(V \otimes V)\,, \quad R_{ij}^{kl} := e^{-E_{ij}^{kl}/k_B T}\,.$$

If R satisfies the Yang–Baxter equation

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23} \in \mathrm{End}(V \otimes V \otimes V)$$



then the model is integrable and can be solved (perhaps).

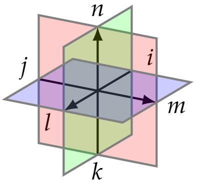
YBE is hard to solve: R has N^4 components, YBE has N^6.

3D Lattice Models & Tetrahedron Equation

A 3D cubic lattice is made of intersecting planes.

Assign V to the intersections of planes and introduce

$$R \in \mathrm{End}(V \otimes V \otimes V)\,, \quad R_{ijk}^{lmn} =$$



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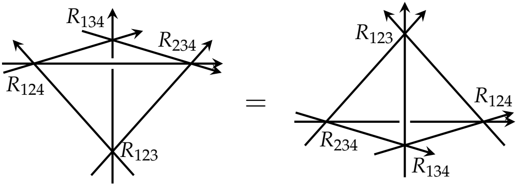
Zamolodchikov tetrahedron equation implies integrability:

$$R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234}\,.$$



State of the Art

TE has a relatively long history [Zamolodchikov '80], but far less developed than YBE. (Only one book [Kuniba '22] on TE!)

Very complicated: R has N^6 components, TE has N^{12}.
(Zamolodchikov wrote down first nontrivial solutions "by what appears to be an extraordinary feat of intuition" (Baxter).)

But important: we live in 3D space!

Recent progress:

- Solutions from cluster algebras, related to 3D gauge theories [Sun–Y '22, Inoue–Kuniba–Terashima '23, IKSTY '24; also Gavrylenko–Semenyakin–Zenkevich '20].
- Solutions from equivalence of 2D gauge theories [Y '24].
- Construction using branes in M-theory [Y '23].

Tetrahedron Equation from Cluster Algebras

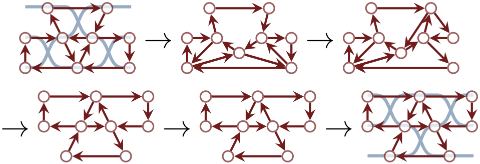
[SY '22, IKT '23, IKSTY '24]

Wires & Quivers

To a wiring diagram we assign a quiver:

A braid move induces a transformation of quivers:

The two quivers are related by a sequence of mutations:



Quivers to Quantum Mechanics

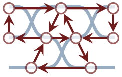
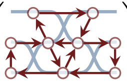
The theory of quantum cluster algebras [Fock–Goncharov] gives

$$\begin{aligned} \text{quivers} &\to \text{quantum mechanical systems} \\ Q &\mapsto \mathrm{FG}(Q) \end{aligned}$$

such that

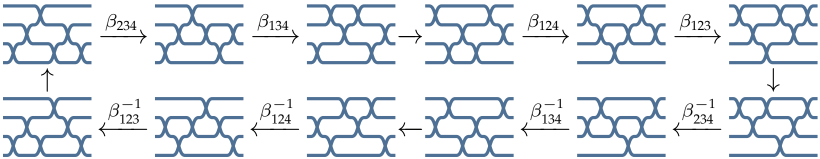
$$Q \xleftrightarrow{\text{mutation}} Q' \iff \mathrm{FG}(Q) \cong \mathrm{FG}(Q')\,.$$

A braid move induces an isomorphism of two Hilbert spaces:

$R\colon \mathcal{H}\Big($  $\Big) \xrightarrow{\sim} \mathcal{H}\Big($  $\Big)$

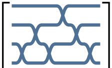
Quantum Mechanics to TE

The loop of braid moves



shows

$$R_{234}R_{134}R_{124}R_{123}R_{234}^{-1}R_{134}^{-1}R_{124}^{-1}R_{123}^{-1} \in \mathrm{End}\left(\mathcal{H}\left[\;\cdot\;\right]\right).$$



It turns out this operator is 1. Hence, R solves TE.

R coincides with the partition function of a 3D $\mathcal{N} = 2$ SUSY gauge theory on S^3.

The two sides of TE are related by a duality of 3D theories.

Tetrahedron Equation from 2D Duality [Y '24]

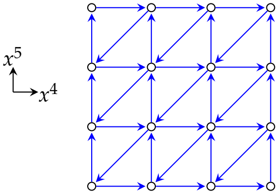
Gauge/YBE correspondence [Yamazaki '12, Y '15, ...]

Brane box model [Hanany–Zaffaroni '98]

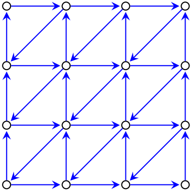
	$\mathbb{R}^4_{0123}$	$\times$	$\mathbb{R}^2_{45}$	$\times$	$\mathbb{R}^2_{67}$	$\times$	$\mathbb{R}^2_{89}$
N D5	$\mathbb{R}^4_{0123}$	$\times$	$\mathbb{R}^2_{45}$	$\times$	$\{0\}$	$\times$	$\{0\}$
$\mathrm{NS5}_{X_l}, l \in \mathbb{Z}$	$\mathbb{R}^4_{0123}$	$\times$	$\{x^4 = l + \frac{1}{2}\}$	$\times$	$\{x^6 = 0\}$	$\times$	$\{0\}$
$\mathrm{NS5}_{Y_m}, m \in \mathbb{Z}$	$\mathbb{R}^4_{0123}$	$\times$	$\{x^5 = m + \frac{1}{2}\}$	$\times$	$\{x^7 = 0\}$	$\times$	$\{0\}$

NS5s make a square lattice in $\mathbb{R}^2_{45}$.

BBM produces a 4D $\mathcal{N} = 1$ quiver gauge theory at low energies.

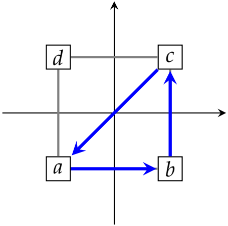


Gauge/YBE correspondence



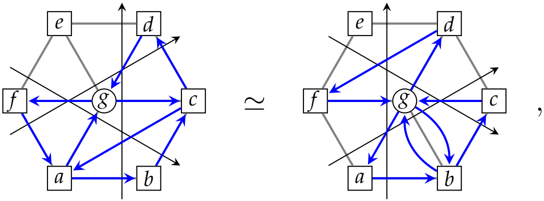
- $\circ$: $\mathrm{SU}(N)$ gauge group
- $a \to b$: bifundamental chiral
- $\mathrm{SU}(N)_{(l,m)}$ arises from the part of D5s containing $(l, m) \in \mathbb{R}^2_{45}$.
- $\mathrm{U}(1)_R$ from rotation symmetry of $\mathbb{R}^2_{89}$, can mix with $\mathrm{U}(1)$ flavor symmetries.

A "unit cell" of BBM is a theory with 3 bifund chirals:



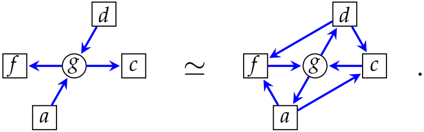
Gauge/YBE correspondence

The unit cell solves YBE for interaction-round-a-face models:



where $\simeq$ means IR equivalence. Indeed, using

we can simplify YBE to Seiberg duality:



SUSY index on $S^1 \times S^3$ gives an ordinary solution of YBE.

Gauge/YBE correspondence

	$S^1 \times S^3 \times$	$\mathbb{R}^2_{45}$	$\times$ $\mathbb{R}^2_{67}$	$\times$ $\mathbb{R}^2_{89}$
N D5	$S^1 \times S^3 \times$	$\mathbb{R}^2_{45}$	$\times$ $\{0\}$	$\times$ $\{0\}$
$\mathrm{NS5}_{X_l}, l \in \mathbb{Z}$	$S^1 \times S^3 \times$	$\{x^4 = l + \frac{1}{2}\}$	$\times$ $\{x^6 = 0\}$	$\times$ $\{0\}$
$\mathrm{NS5}_{Y_m}, m \in \mathbb{Z}$	$S^1 \times S^3 \times$	$\{x^5 = m + \frac{1}{2}\}$	$\times$ $\{x^7 = 0\}$	$\times$ $\{0\}$

D5s on $\mathcal{M} \times \Sigma \subset (K_{\mathcal{M}} \oplus K_{\mathcal{M}}^{-1}) \times T^*\Sigma$ produces a theory that is holomorphic on $\mathcal{M} = S^1 \times S^3$ and topological on $\Sigma = \mathbb{R}^2_{45}$.

T-dualize on S^1 and lift to M-theory:

	S^1	$\times S^3 \times$	$\mathbb{R}^2_{45}$	$\times$ $\mathbb{R}^2_{67}$	$\times$ $\mathbb{R}^2_{89}$	$\times$ S^1_{10}
N M5	$\{\mathrm{pt}\}$	$\times S^3 \times$	$\mathbb{R}^2_{45}$	$\times$ $\{0\}$	$\times$ $\{0\}$	$\times$ S^1_{10}
$\mathrm{M5}_{X_l}, l \in \mathbb{Z}$	S^1	$\times S^3 \times$	$\{x^4 = l + \frac{1}{2}\}$	$\times$ $\{x^6 = 0\}$	$\times$ $\{0\}$	$\times$ $\{\mathrm{pt}\}$
$\mathrm{M5}_{Y_m}, m \in \mathbb{Z}$	S^1	$\times S^3 \times$	$\{x^5 = m + \frac{1}{2}\}$	$\times$ $\{x^7 = 0\}$	$\times$ $\{0\}$	$\times$ $\{\mathrm{pt}\}$

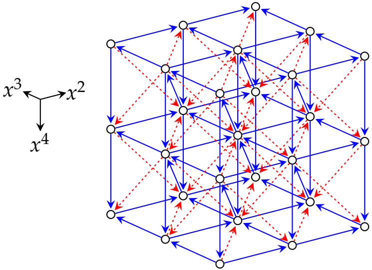
TQFT + line defects + extra dim leads to YBE [Costello '13, Y '15].

3D brane box model [Garcia-Compean–Uranga '98]

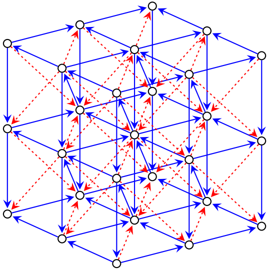
	$\mathbb{R}^2_{01}$ ×	$\mathbb{R}^3_{234}$	× $\mathbb{R}^3_{567}$	× $\mathbb{R}^2_{89}$
N D4	$\mathbb{R}^2_{01}$ ×	$\mathbb{R}^3_{234}$	× $\{0\}$	× $\{0\}$
$\mathrm{NS5}_{X_l}, l \in \mathbb{Z}$	$\mathbb{R}^2_{01}$ ×	$\{x^2 = l + \frac{1}{2}\}$	× $\{x^5 = 0\}$	× $\{0\}$
$\mathrm{NS5}_{Y_m}, m \in \mathbb{Z}$	$\mathbb{R}^2_{01}$ ×	$\{x^3 = m + \frac{1}{2}\}$	× $\{x^6 = 0\}$	× $\{0\}$
$\mathrm{NS5}_{Z_n}, n \in \mathbb{Z}$	$\mathbb{R}^2_{01}$ ×	$\{x^4 = n + \frac{1}{2}\}$	× $\{x^7 = 0\}$	× $\{0\}$

NS5s making a cubic lattice in $\mathbb{R}^3_{234}$.

Produces 2D $\mathcal{N} = (0, 2)$ quiver gauge theory

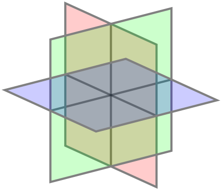


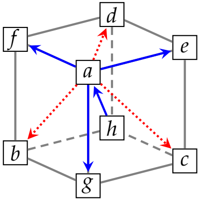
3D brane box model



- $\circ$: $\mathrm{U}(N)$ gauge group
- $a \to b$: bifundamental chiral
- $a \dashrightarrow b$: bifundamental Fermi
- $\mathrm{U}(N)_{(l,m,n)}$ arises from the part of D4s containing $(l, m, n) \in \mathbb{R}^3_{234}$.
- $\mathrm{U}(1)_R$ from rotation symmetry of $\mathbb{R}^2_{89}$, can mix with $\mathrm{U}(1)$ flavor symmetries.

A unit cell is





3D brane box model

	$\mathbb{R}^2_{01} \times$	$\mathbb{R}^3_{234}$	$\times$ $\mathbb{R}^3_{567}$	$\times\ \mathbb{R}^2_{89} \times$	S^1_{10}
N M5	$\mathbb{R}^2_{01} \times$	$\mathbb{R}^3_{234}$	$\times\ \{0\}$	$\times\ \{0\} \times$	S^1_{10}
$\mathrm{M5}_{X_l}, l \in \mathbb{Z}$	$\mathbb{R}^2_{01} \times$	$\{x^2 = l + \frac{1}{2}\}$	$\times\ \{x^5 = 0\}$	$\times\ \{0\} \times$	$\{\mathrm{pt}\}$
$\mathrm{M5}_{Y_m}, m \in \mathbb{Z}$	$\mathbb{R}^2_{01} \times$	$\{x^3 = m + \frac{1}{2}\}$	$\times\ \{x^6 = 0\}$	$\times\ \{0\} \times$	$\{\mathrm{pt}\}$
$\mathrm{M5}_{Z_n}, n \in \mathbb{Z}$	$\mathbb{R}^2_{01} \times$	$\{x^4 = n + \frac{1}{2}\}$	$\times\ \{x^7 = 0\}$	$\times\ \{0\} \times$	$\{\mathrm{pt}\}$

M5s on $\Sigma \times M \times S^1_{10} \subset T^*\Sigma \times T^*M \times S^1_{10}$ produces a theory that is holomorphic on $\Sigma = \mathbb{R}^2_{01}$ and topological on $M = \mathbb{R}^3_{234}$.

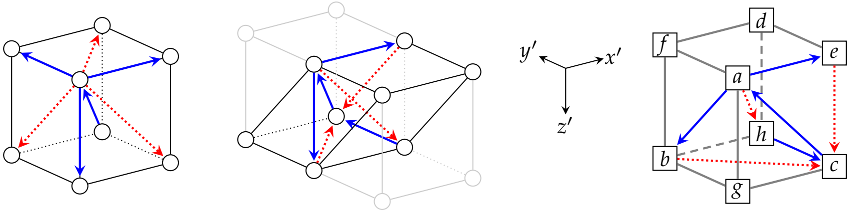
NS5s lift to M5s separated in S^1_{10}, so moving them around in M won't yield anything singular.

The unit cell should satisfy TE as IR equivalence, and elliptic genera should give ordinary solutions of TE

...except they don't (seem to).

Solution of TE by brane box model

It turns out we get a solution for a slanted unit cell:



This is a unit cell in $(x', y', z') = (x^2, -x^3 + x^4, x^4)$.

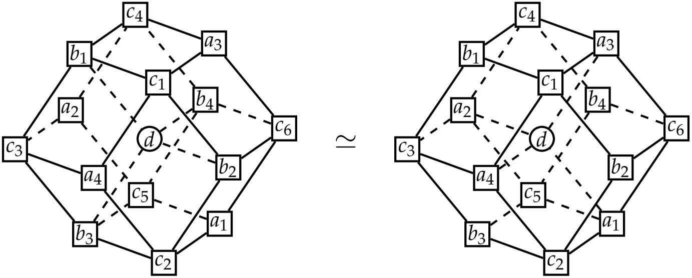
Define the oblique unit cell to be the theory

$$(a|e,f,g|b,c,d|h;s,t,u) = \begin{pmatrix} (a \xrightarrow{t-u} b)(b \overset{1+s-t}{\dashrightarrow} c)(c \xrightarrow{-s+u} a) \\ (a \overset{s+t-u}{\dashrightarrow} h)(h \xrightarrow{1-t} c)(a \xrightarrow{1+s} e)(e \overset{-u}{\dashrightarrow} c) \end{pmatrix}$$

s, t, u are parameters controlling the R-charge assignment.

Solution of TE by brane box model

Oblique unit cell solves TE in interaction-round-a-cube form:



$$\underset{d}{\Gamma}\begin{pmatrix}(a_4|c_2,c_1,c_3|b_1,b_3,b_2|d;s_1,s_3,s_2)(c_1|b_2,a_3,b_1|c_4,d,c_6|b_4;s_1,s_4,s_2)\\(b_1|d,c_4,c_3|a_2,b_3,b_4|c_5;s_1,s_4,s_3)(d|b_2,b_4,b_3|c_5,c_2,c_6|a_1;s_2,s_4,s_3)\end{pmatrix}$$

$$\simeq\underset{d}{\Gamma}\begin{pmatrix}(b_1|c_1,c_4,c_3|a_2,a_4,a_3|d;s_2,s_4,s_3)(c_1|b_2,a_3,a_4|d,c_2,c_6|a_1;s_1,s_4,s_3)\\(a_4|c_2,d,c_3|a_2,b_3,a_1|c_5;s_1,s_4,s_2)(d|a_1,a_3,a_2|c_4,c_5,c_6|b_4;s_1,s_3,s_2)\end{pmatrix}$$

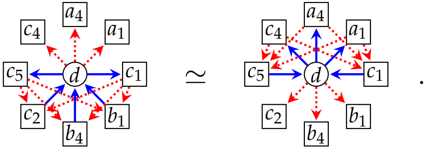
where Γ_d is gauging of $\mathrm{U}(N)_d$.

Solution of TE by brane box model

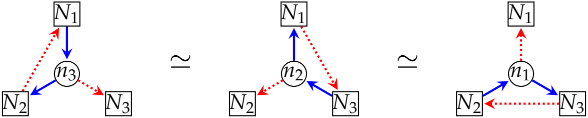
Using relations

$$a \overset{r}{\dashrightarrow} b \simeq b \overset{-r}{\dashrightarrow} a\,, \qquad (a \overset{r}{\rightarrow} b)(b \overset{1-r}{\dashrightarrow} a) \simeq 1\,,$$

we can simplify TE to



This holds by (0, 2) triality [Gadde–Gukov–Putrov '13]:



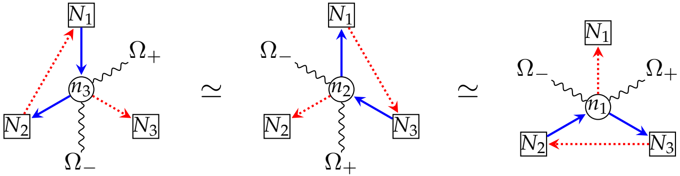
$n_i = \frac{1}{2}(N_1 + N_2 + N_3) - N_i$ for SU gauge anomalies cancellation.

SUSY indices of oblique unit cell solve TE in the ordinary sense.

Solution of TE by brane box model

Caveat: triality requires cancellation of U(1) gauge anomaly.

This is usually done with determinant Fermi multiplets:



In BBM, the anomaly is expected to be cancelled by a Green–Schwarz mechanism.

Formula for SUSY indices are not known in such a case.

We can't write down explicit solutions!

Solution by SUSY QM

We can reduce BBM to 1D – no chiral anomaly then.

Witten index of the resulting SUSY QM solves TE.

To compute Witten index, we assign [Benini–Eager–Hori–Tachikawa '13, Hori–Kim–Yi '14]

$$W(a \xrightarrow{s} b) = \prod_{i,j} \frac{-1}{\phi(e^{4\pi \mathrm{i} s} b_i/a_j)} \,, \quad W(a \overset{s}{\dashrightarrow} b) = \prod_{i,j} \phi(e^{4\pi \mathrm{i} s} b_i/a_j) \,,$$

where

$$\phi(z) = z^{-1/2} - z^{1/2} \,,$$

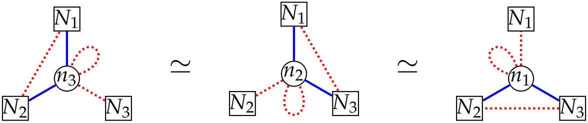
and integrate over the gauge fugacities a with measure

$$\mu(a) = \frac{1}{N!} \prod_i \frac{\mathrm{d}a_i}{2\pi \mathrm{i} a_i} \prod_{i \neq j} \phi(a_i/a_j) \,.$$

Choose to pick poles from $W(b \xrightarrow{s} a)$ but not form $W(a \xrightarrow{s} b)$.

Solution by $(0,1)$ model

Another solution is given by $\mathcal{N}=(0,1)$ theory with $\mathrm{SO}(2N)$ gauge groups, which also enjoy triality [Gukov–Pei–Putrov '19]:



In this case the SUSY index has been computed in the sigma model description. I found it can be written in the same form:

$$W(a\to b)=\prod_{i,j}\frac{-1}{\hat\theta(b_i a_j)\hat\theta(b_i/a_j)}\,,\qquad W(a\dashrightarrow b)=\prod_{i,j}\hat\theta(b_i a_j)\hat\theta(b_i/a_j)\,,$$

with $\hat\theta(z)=\mathrm{i}\theta_1(z;q)/\eta(q)$, and measure

$$\mu(a)=\frac{1}{2^{N-1}N!}\prod_i\frac{\mathrm{d}a_i}{2\pi\mathrm{i}a_i}\eta(q)^2\hat\theta(a_i^2)\prod_{i\neq j}\hat\theta(a_i a_j)\hat\theta(a_i/a_j)\,.$$

Conclusions & Outlook

Conclusions

TE is a 3D analog of YBE and underlies 3D integrability.

It is important but not well-understood.

Solutions of TE arise from

- Cluster algebras and 3D $\mathcal{N} = 2$ SUSY gauge theories
- Duality of 2D $\mathcal{N} = (0, 2)$ SUSY gauge theories
- Brane systems in M-theory

Outlook

How are these constructions related?

Can we construct more solutions along these lines?

Can we get models with discrete spin variables?

Can we say anything about 3D Stat Mech systems?

Relations to 3-manifolds and 3D gravity? [Work in progress]