E. Noether's variational symmetries and pluri-Lagrangian systems

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Part 1. 1D pluri-Lagrangian systems, and variational symmetries in classical mechanics

Based on:

Yu. S. *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*. J. Geometric Mechanics, 2013, **5**, 365–379.

M. Petrera, Yu. S. *Variational symmetries and pluri-Lagrangian systems in classical mechanics*. J. Nonlin. Math. Phys., 2017, **24**, Sup. 1, 12–145.

1-dimensional pluri-Lagrangian problem

- Multi-time $T = \mathbb{R}^m$, configuration space $X = \mathbb{R}^N$.
- A multi-time 1-form: a 1-form on T with coefficients depending on $(x, u_1, \ldots, u_m) \in X \times X^m$:

$$\mathcal{L}(x, u_1, \ldots, u_m) = \sum_{k=1}^{m} L_k(x, u_1, \ldots, u_m) dt_k.$$

- *1D pluri-Lagrangian problem:* find functions $x : T \to X$ delivering critical points to the functionals

$$S_\Gamma = \int_\Gamma \mathcal{L}(x, x_{t_1}, \ldots, x_{t_m})$$

along *any* smooth curve Γ in T (connecting two fixed endpoints).

Multi-time Euler-Lagrange equations

Theorem. *$x : T \to X$ delivers a critical point for S_Γ for any smooth curve Γ, iff:*

- $\dfrac{\partial L_k}{\partial u_\ell}(x, x_{t_1}, \ldots, x_{t_m}) = 0, \quad 1 \le k \ne \ell \le m;$
- $\dfrac{\partial L_1}{\partial u_1}(x, x_{t_1}, \ldots, x_{t_m}) = \ldots = \dfrac{\partial L_m}{\partial u_m}(x, x_{t_1}, \ldots, x_{t_m}) =: p;$
- $\dfrac{\partial p}{\partial t_k} = \dfrac{\partial L_k}{\partial x}(x, x_{t_1}, \ldots, x_{t_m}), \quad k = 1, \ldots, m.$

Remark. These conditions have to be fulfilled *on-shell* (not identically). They constitute *multi-time EL equations*.

Remark. Multi-time EL equations highly overdetermined, they are compatible (admit general solutions) only for very special $\mathcal{L}$.

Almost closedness of the pluri-Lagrangian 1-form

Theorem. On solutions of multi-time EL equations, we have:

$$\mathsf{D}_{t_k} L_\ell - \mathsf{D}_{t_\ell} L_k = c_{k\ell} = \mathrm{const}.$$

In particular, if all $c_{k\ell} = 0$, then the pluri-Lagrangian 1-form $\mathcal{L}$ is closed on solutions of multi-time EL equations, so that the action functional S_Γ does not depend on the choice of the curve Γ connecting two given points in T.

Example: Toda chain, $m = 2$

$$L_1 = \frac{1}{2}\sum_{i=1}^{N}((x_i)_{t_1})^2 - \sum_{i=1}^{N} e^{x_{i+1}-x_i},$$

$$L_2 = \sum_{i=1}^{N}(x_i)_{t_1}(x_i)_{t_2} - \sum_{i=1}^{N}\Big(\frac{1}{3}((x_i)_{t_1})^3 + \big(e^{x_{i+1}-x_i} + e^{x_i-x_{i-1}}\big)(x_i)_{t_1}\Big).$$

Then $\partial L_1/\partial(x_i)_{t_2} = 0$ is trivially satisfied, while $\partial L_2/\partial(x_i)_{t_1} = 0$ reads:

$$(x_i)_{t_2} = ((x_i)_{t_1})^2 + e^{x_{i+1}-x_i} + e^{x_i-x_{i-1}}.$$

Further $\partial L_1/\partial(x_i)_{t_1} = \partial L_2/\partial(x_i)_{t_2} = (x_i)_{t_1} =: p_i$. Finally:

$$\begin{aligned}(x_i)_{t_1 t_1} &= e^{x_{i+1}-x_i} - e^{x_i-x_{i-1}},\\ (x_i)_{t_1 t_2} &= e^{x_{i+1}-x_i}\big((x_i)_{t_1} + (x_{i+1})_{t_1}\big) - e^{x_i-x_{i-1}}\big((x_{i-1})_{t_1} + (x_i)_{t_1}\big).\end{aligned}$$

Remarks. The first EL equation evolutionary! The last equation a corollary of previous two (compatibility).

Lagrangian mechanics in 1D

- Time $\mathbb{R}$, configuration space $X = \mathbb{R}^N$, Lagrange function $L : TX = X \times X \to \mathbb{R}$;
- Hamilton's principle: motions $x : [t_0, t_1] \to X$ are critical points of the action functional

$$S[x] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t))dt$$

  (with fixed endpoints $x(t_0)$, $x(t_1)$).
- Euler-Lagrange equations: $\mathscr{E}_i(x, \dot{x}, \ddot{x}) = 0,\ i = 1, \ldots, N$, where

$$\mathscr{E}_i = \frac{\delta L}{\delta x_i} := \frac{\partial L}{\partial x_i} - \mathrm{D}_t\left(\frac{\partial L}{\partial \dot{x}_i}\right)$$

  are the *variational derivatives*.

Generalized vector fields and their prolongations

- A *generalized vector field* v on X:

$$v = \sum_{i=1}^{N} V_i(x, \dot{x}) \frac{\partial}{\partial x_i} \ ,$$

$(V_1, \dots, V_N)$ its *characteristic*.

- *Prolongation* of v (acts on differential polynomials of x):

$$\mathrm{D}_v = \sum_{i=1}^{N} V_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{N} (\mathrm{D}_t V_i) \frac{\partial}{\partial \dot{x}_i} + \sum_{i=1}^{N} (\mathrm{D}_t^2 V_i) \frac{\partial}{\partial \ddot{x}_i} + \dots$$

Variational symmetries and integrals of EL equations

Definition. A generalized vector field v is a *variational symmetry* of the Lagrange function $L(x,\dot{x})$ if

$$\mathrm{D}_v L(x,\dot{x}) = \mathrm{D}_t F(x,\dot{x})$$

for some $F = F(x,\dot{x})$, called the *flux* of the variational symmetry v.

Definition. A function $J(x,\dot{x})$ is an *integral* of the EL equations $\mathscr{E}_i = 0$ with the *characteristic* $\big(V_1(x,\dot{x}),\ldots,V_N(x,\dot{x})\big)$, if

$$\mathrm{D}_t J(x,\dot{x}) = -\sum_{i=1}^{N} V_i(x,\dot{x})\mathscr{E}_i.$$

As a consequence, $\mathrm{D}_t J = 0$ on solutions of $\mathscr{E}_i = 0$ (on shell).

E. Noether's theorem

Theorem.

• Let v be a variational symmetry of the Lagrange function $L(x,\dot{x})$, with the flux $F(x,\dot{x})$. Then the function

$$J(x,\dot{x})=\sum_{i=1}^{N}\frac{\partial L(x,\dot{x})}{\partial \dot{x}_i}V_i(x,\dot{x})-F(x,\dot{x})$$

is an integral of EL equations $\mathscr{E}_i=0$ with the characteristic $(V_1,\ldots,V_N)$.

• Conversely, let $J(x,\dot{x})$ be an integral of EL equations $\mathscr{E}_i=0$, with the characteristic $(V_1,\ldots,V_N)$. Then the generalized vector field v with this characteristic is a variational symmetry of the Lagrange function $L(x,\dot{x})$, with the flux

$$F(x,\dot{x})=\sum_{i=1}^{N}\frac{\partial L(x,\dot{x})}{\partial \dot{x}_i}V_i(x,\dot{x})-J(x,\dot{x}).$$

Proof of E. Noether's theorem

Both parts are consequences of the following statement:

$$\mathrm{D}_v L - \sum_{i=1}^{N} V_i \mathscr{E}_i = \mathrm{D}_t(A),$$

where

$$A = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i} V_i .$$

Indeed, then

$$\mathrm{D}_v L = \mathrm{D}_t F \quad \text{and} \quad \mathrm{D}_t J = -\sum_{i=1}^{N} V_i \mathscr{E}_i$$

are equivalent, provided $J + F = A$. ■

Proof of the key identity – integration by parts

$$
\begin{aligned}
& \mathrm{D}_v L - \sum_{i=1}^{N} V_i \frac{\delta L}{\delta x_i} \\
= & \left(\sum_{i=1}^{N} V_i \frac{\partial L}{\partial x_i} + \sum_{i=1}^{N} (\mathrm{D}_t V_i) \frac{\partial L}{\partial \dot{x}_i} \right) - \sum_{i=1}^{N} V_i \left(\frac{\partial L}{\partial x_i} - \mathrm{D}_t \Big(\frac{\partial L}{\partial \dot{x}_i} \Big) \right) \\
= & \sum_{i=1}^{N} (\mathrm{D}_t V_i) \frac{\partial L}{\partial \dot{x}_i} + \sum_{i=1}^{N} V_i \, \mathrm{D}_t \Big(\frac{\partial L}{\partial \dot{x}_i} \Big) \\
= & \mathrm{D}_t \left(\sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i} V_i \right). \blacksquare
\end{aligned}
$$

Hamiltonian side of the picture

Introduce the conjugate momenta $p=(p_1,\ldots,p_N)$ by the Legendre transformation $TX \ni (x,\dot{x}) \mapsto (x,p) \in T^*X$,

$$p_i = p_i(x,\dot{x}) = \frac{\partial L(x,\dot{x})}{\partial \dot{x}_i}, \quad i=1,\ldots,N.$$

Under the non-degeneracy condition $\det\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\right) \neq 0$, this defines a local diffeomorphism $TX \to T^*X$. The dynamics on T^*X, equivalent to EL equations on TX, is given by a Hamiltonian system with the Hamilton function

$$H(x,p) = \sum_{i=1}^{N} p_i \dot{x}_i - L(x,\dot{x})\Big|_{\dot{x}=\dot{x}(x,p)}.$$

Hamiltonian side of E. Noether's theorem

Theorem

• Let v be a variational symmetry of the Lagrange function $L(x,\dot{x})$, with the flux $F(x,\dot{x})$ and with the Noether integral $J(x,\dot{x})$. Set $H_1(x,p)=J(x,\dot{x})|_{\dot{x}=\dot{x}(x,p)}$. Then H_1 Poisson commutes with H, $\{H,H_1\}=0$.

• Let H_1 Poisson commute with H. Then

$$V_i(x,\dot{x})=\frac{\partial H_1(x,p)}{\partial p_i}\bigg|_{p=p(x,\dot{x})},\quad J(x,\dot{x})=H_1(x,p)\Big|_{p=p(x,\dot{x})}$$

defines the characteristic of a generalized vector field which is a variational symmetry of the Lagrange function $L(x,\dot{x})$, with the Noether integral $J(x,\dot{x})$.

Commuting variational symmetries

A possible framework for integrability in the Lagrangian context.

Assume that $L(x,\dot x)$ admits two variational symmetries $v_k=\sum_{i=1}^N V_i^{(k)}(x,\dot x)\frac{\partial}{\partial x_i}$, with the corresponding fluxes $F_k(x,\dot x)$, $k=1,2$. Then $[\mathrm{D}_{v_1},\mathrm{D}_{v_2}]=\mathrm{D}_{[v_1,v_2]}$, the prolongation of the generalized vector field

$$[v_1,v_2]=\sum_{i=1}^N\Big(\mathrm{D}_{v_1}V_i^{(2)}-\mathrm{D}_{v_2}V_i^{(1)}\Big)\frac{\partial}{\partial x_i}.$$

Definition. Variational symmetries v_1, v_2 *commute* if

$$\mathrm{D}_{v_1}V_i^{(2)}-\mathrm{D}_{v_2}V_i^{(1)}=\sum_{j=1}^N r_{ij}(x,\dot x)\mathscr{E}_j\,,\quad i=1,\ldots,N$$

with some functions $r_{ij}(x,\dot x)$, so that $[v_1,v_2]=0$ on shell (on solutions of EL equations $\mathscr{E}_j=0$).

Commutativity: Lagrangian vs. Hamiltonian

Theorem. Let v_k, $k = 1, 2$, be two variational symmetries for $L(x, \dot{x})$, with the fluxes F_k and Noether integrals J_k. If v_1, v_2 commute, then their fluxes satisfy:

$$\mathrm{D}_{v_1} F_2 - \mathrm{D}_{v_2} F_1 = c_{12} + \sum_{i=1}^{N} p_i \left(\mathrm{D}_{v_1} V_i^{(2)} - \mathrm{D}_{v_2} V_i^{(1)} \right),$$

where $c_{12} = \mathrm{const}$. In particular, on solutions of EL equations $\mathscr{E}_j = 0$ we have:

$$\mathrm{D}_{v_1} F_2 - \mathrm{D}_{v_2} F_1 = c_{12}.$$

Set $H_k(x, p) = J_k(x, \dot{x})|_{\dot{x} = \dot{x}(x,p)}$. Then

$$\{H_1, H_2\} = c_{12}.$$

Thus, Hamiltonian flows with the Hamilton functions H_k commute.

Commutativity: Lagrangian vs. Hamiltonian

Conversely, let $H_1, H_2 : T^*X \to \mathbb{R}$ be integrals of motion of the Hamiltonian flow with the Hamilton function H, such that $\{H_1, H_2\} = c_{12}$. Define variational symmetries v_1, v_2 of $L(x, \dot{x})$ by their characteristics

$$V_i^{(k)}(x, \dot{x}) = \frac{\partial H_k(x, p)}{\partial p_i}\bigg|_{p=p(x,\dot{x})}, \quad k = 1, 2.$$

Then v_1, v_2 commute on solutions of EL equations $\mathscr{E}_j = 0$.

Proof is based on the formula

$$\mathrm{D}_{v_1} V_i^{(2)} - \mathrm{D}_{v_2} V_i^{(1)} = \frac{\partial}{\partial p_i}\{H_1, H_2\} + \sum_{j=1}^{N} r_{ij}\mathscr{E}_j.$$

From variational symmetries to a multi-time 1-form

Change point of view. Till now considered D_{v_k} as acting on differential polynomials of a function x of a single time t. Suppose that $L(x, \dot{x})$ admits m pairwise commuting variational symmetries $v_1, \ldots, v_m$. We interpret them as m flows

$$(x_i)_{t_k} = V_i^{(k)}(x, \dot{x}), \quad i = 1, \ldots, N, \quad k = 1, \ldots, m,$$

commuting when restricted to solutions of EL equations. Interpret x as function of $m+1$ independent variables, $x = x(t, t_1, \ldots, t_m)$. Operators D_t, D_{v_k} are replaced by

$$\mathrm{D}_{t_k} = \sum_{i=1}^{N} \sum_{K} (x_i)_{K+e_k} \frac{\partial}{\partial (x_i)_K},$$

using multi-indices $K = (i_0, i_1, \ldots, i_m) \in (\mathbb{Z}_{\geq 0})^{m+1}$.

From variational symmetries to a multi-time 1-form

The defining formula of variational symmetries becomes

$$\mathrm{D}_{t_k} L(x,\dot{x}) - \mathrm{D}_t F_k(x,\dot{x}) = 0,$$

while the commutativity of symmetries (on shell) would read

$$\mathrm{D}_{t_k} F_\ell(x,\dot{x}) - \mathrm{D}_{t_\ell} F_k(x,\dot{x}) = c_{k\ell}.$$

Observation: these relations are nothing but the coefficients of $\mathrm{d}\mathcal{L}$ for the 1-form

$$\mathcal{L} = L(x,\dot{x})\mathrm{d}t + F_1(x,\dot{x})\mathrm{d}t_1 + \ldots + F_m(x,\dot{x})\mathrm{d}t_m.$$

Problem: for this 1-form the multi-time EL equations are inconsistent (for instance, $\partial F_k/\partial(x_i)_{t_k} = 0 \neq \partial L/\partial \dot{x}_i$).

Solution: adjustments in coefficients, vanishing on shell.

Repaired pluri-Lagrangian 1-form

Theorem. Let v_k, $k = 1, \dots, m$, be commuting variational symmetries for the Lagrange function $L(x, \dot{x})$, with the fluxes $F_k(x, \dot{x})$, and with the Noether integrals $J_k(x, \dot{x})$. Define

$$
\begin{aligned}
L_k(x, \dot{x}, x_{t_k}) &= \sum_{i=1}^{N} \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} (x_i)_{t_k} - J_k(x, \dot{x}) \\
&= \sum_{i=1}^{N} \frac{\partial L(x, \dot{x})}{\partial \dot{x}_i} \big((x_i)_{t_k} - V_i^{(k)}(x, \dot{x})\big) + F_k(x, \dot{x}).
\end{aligned}
$$

Then for the 1-form

$$
\mathcal{L} = L(x, \dot{x})\mathrm{d}t + L_1(x, \dot{x}, x_{t_1})\mathrm{d}t_1 + \ldots + L_m(x, \dot{x}, x_{t_m})\mathrm{d}t_m,
$$

the multi-time EL equations are equivalent to $\mathscr{E}_i = 0$ coupled with $(x_i)_{t_k} = V_i^{(k)}$. Observe that $L_k(x, \dot{x}, x_{t_k}) = F_k(x, \dot{x})$ on shell.

Part 2. 2D pluri-Lagrangian systems, and variational symmetries in PDEs

Based on:

Yu. S. *Variational symmetries and pluri-Lagrangian systems*. – In: Dynamical Systems, Number Theory and Applications, World Scientific, 2016, p. 255–266.

Yu. S., M. Vermeeren. *On the Lagrangian structure of integrable hierarchies*. – In: Advances in Discrete Differential Geometry, Springer, 2016, p. 347–378.

2-dimensional pluri-Lagrangian problem

- Multi-space-time $T = \mathbb{R}^m$, configuration space $X = \mathbb{R}$.
- A multi-time 2-form: a 2-form on T with coefficients depending on $(u, u_{t_i}, u_{t_i t_j} \ldots) \in J^\infty X$:

$$\mathcal{L}[u] = \sum_{i<j}^{m} L_{ij}[u] dt_i \wedge dt_j.$$

- *2D pluri-Lagrangian problem:* find functions $u : T \to \mathbb{R}$ delivering critical points to the functionals

$$S_\Gamma = \int_\Gamma \mathcal{L}[u]$$

over *any* smooth 2-dim surface Γ in T (with a fixed boundary).

Multi-time Euler-Lagrange equations

Theorem. *$u : T \to \mathbb{R}$ is a critical point for S_Γ for any smooth surface Γ, iff the following* multi-time EL equations *are fulfilled:*

- $\dfrac{\delta_{ij} L_{ij}}{\delta u_K} = 0 \quad \text{for all} \quad K \not\ni t_i, t_j;$
- $\dfrac{\delta_{ij} L_{ij}}{\delta u_{Kt_j}} = \dfrac{\delta_{ik} L_{ik}}{\delta u_{Kt_k}} \quad \text{for all} \quad K \not\ni t_i;$
- $\dfrac{\delta_{ij} L_{ij}}{\delta u_{Kt_i t_j}} + \dfrac{\delta_{jk} L_{jk}}{\delta u_{Kt_j t_k}} + \dfrac{\delta_{ki} L_{ki}}{\delta u_{Kt_k t_i}} = 0 \quad \text{for all} \quad K.$

Here $K \in (\mathbb{Z}_{\geq 0})^m$ are multi-indices, and

$$\frac{\delta_{ij} L_{ij}}{\delta u_K} = \sum_{\alpha,\beta \geq 0} (-1)^{\alpha+\beta} \mathrm{D}_i^\alpha \mathrm{D}_j^\beta \left(\frac{\partial L_{ij}}{\partial u_{Kt_i^\alpha t_j^\beta}} \right)$$

is the variational derivative in the (t_i, t_j)-plane.

Example

Pluri-Lagrangian structure for the system consisting of Sine-Gordon (SG) and potential modified KdV (pMKdV) equations:

$$u_{xy} = \sin u, \quad u_t = u_{xxx} + \frac{1}{2}u_x^3.$$

Denote $t_1 = x$, $t_2 = y$, $t_3 = t$, and consider

$$\mathcal{L} = L_{12}[u]dx \wedge dy + L_{13}[u]dx \wedge dt + L_{23}[u]dy \wedge dt,$$

where

$$\begin{aligned} L_{12} &= \frac{1}{2}u_x u_y - \cos u, \\ L_{13} &= \frac{1}{2}u_t u_x - \frac{1}{8}u_x^4 + \frac{1}{2}u_{xx}^2, \\ L_{23} &= -\frac{1}{2}u_t u_y + \frac{1}{2}u_x^2 \cos u + u_{xx}(u_{xy} - \sin u). \end{aligned}$$

Multi-time EL equations for Sine-Gordon equation

Then the system of multi-time EL equations is *equivalent* to

$$u_{xy} = \sin u, \quad u_t = u_{xxx} + \frac{1}{2} u_x^3.$$

Indeed:

- the first equation (SG) appears from $\delta_{12} L_{12} / \delta u = 0$, and also from $\delta_{23} L_{23} / \delta u_{xx} = 0$;
- the second equation (potential MKdV) appears from $\delta_{13} L_{13} / \delta u_x = \delta_{23} L_{23} / \delta u_y$.

All other EL equations are either trivially satisfied or are differential consequences of these two. For instance:

- eq. $\delta_{13} L_{13} / \delta u = 0$ results in $u_{xt} = u_{xxxx} + \frac{3}{2} u_x^2 u_{xx}$,
- eq. $\delta_{23} L_{23} / \delta u = 0$ results in $u_{yt} = u_{xx} \cos u + \frac{1}{2} u_x^2 \sin u$,
- eq. $\delta_{23} L_{23} / \delta u_x = 0$ results in $u_{xxy} = u_x \cos u$.

Lagrangian 2D (scalar) PDEs

- Space-time $\mathbb{R}^2$, fields with values in $X = \mathbb{R}$, Lagrange function $L : J^\infty X \to \mathbb{R}$;
- Variational principle: fields $u : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$ are critical points of the action functional

$$S[u] = \int_\Omega L[u] dt_1 dt_2$$

(with fixed boundary values $u|_{\partial\Omega}$).
- Euler-Lagrange equation

$$\mathscr{E} = \frac{\delta L}{\delta u} := \sum_{\alpha,\beta \geq 0} (-1)^{\alpha+\beta} \mathrm{D}_1^\alpha \mathrm{D}_2^\beta \left(\frac{\partial L}{\partial u_{t_1^\alpha t_2^\beta}} \right) = 0.$$

Generalized vector fields and their prolongations

- A *generalized vector field* v on $X = \mathbb{R}$:

$$v = V[u]\frac{\partial}{\partial u}\ ,$$

V its *characteristic*.

- *Prolongation* of v (acts on differential polynomials of u):

$$\mathrm{D}_v = \sum_{K\in\mathbb{Z}^2_{\geq 0}} (\mathrm{D}_K V)\frac{\partial}{\partial u_K}.$$

Variational symmetries and integrals of EL equations

Definition. A generalized vector field v is a *variational symmetry* of the Lagrangian $L[u]$ if

$$\mathrm{D}_v L = \mathrm{D}_1 F_1 + \mathrm{D}_2 F_2,$$

for some $F_1[u]$, $F_2[u]$. The pair (F_1, F_2) is called the *flux* of the variational symmetry v.

Definition. A pair of functions $(J_1[u], J_2[u])$ is the *conserved current* of a *conservation law* of the EL equation $\mathscr{E} = \delta L/\delta u = 0$ with the *characteristic* $V[u]$, if

$$\mathrm{D}_1 J_1 + \mathrm{D}_2 J_2 = -V\mathscr{E} = -V\frac{\delta L}{\delta u}.$$

As a consequence, $\mathrm{D}_1 J_1 + \mathrm{D}_2 J_2 = 0$ on solutions of $\mathscr{E} = 0$ (on shell).

E. Noether's theorem

Theorem.

• Let $v = V[u]\,\partial/\partial u$ be a variational symmetry of the Lagrange function $L[u]$, with the flux (F_1, F_2). Then there exists a pair of functions $(A_1[u], A_2[u])$ such that

$$J_1 = A_1 - F_1, \quad J_2 = A_2 - F_2$$

constitute a conserved current of EL equation $\mathscr{E} = \delta L/\delta u = 0$ with the characteristic V.

• Conversely, let (J_1, J_2) be a conserved current of EL equations $\mathscr{E} = \delta L/\delta u = 0$, with the characteristic V. Then the generalized vector field $v = V[u]\,\partial/\partial u$ is a variational symmetry of the Lagrange function $L[u]$, with the flux

$$F_1 = A_1 - J_1, \quad F_2 = A_2 - J_2.$$

Proof of E. Noether's theorem

Both parts are consequences of the following statement:

$$\mathsf{D}_v L - V\mathscr{E} = \mathsf{D}_1 A_1 + \mathsf{D}_2 A_2,$$

with suitable $A_1[u]$, $A_2[u]$. Indeed, then

$$\mathsf{D}_v L = \mathsf{D}_1 F_1 + \mathsf{D}_2 F_2 \quad \text{and} \quad \mathsf{D}_1 J_1 + \mathsf{D}_2 J_2 = -V\mathscr{E}$$

are equivalent, provided $J_1 + F_1 = A_1$ and $J_2 + F_2 = A_2$. ■

Proof of the key identity – integration by parts

From the definition of the variational derivative we have:

$$\frac{\partial L}{\partial u} = \frac{\delta L}{\delta u} + \mathrm{D}_1 \frac{\delta L}{\delta u_{t_1}} + \mathrm{D}_2 \frac{\delta L}{\delta u_{t_2}} + \mathrm{D}_1 \mathrm{D}_2 \frac{\delta L}{\delta u_{t_1 t_2}}.$$

There follows:

$$\begin{aligned}
\mathrm{D}_v L &= \sum_K (\mathrm{D}_K V)\left(\frac{\delta L}{\delta u_K} + \mathrm{D}_1 \frac{\delta L}{\delta u_{Kt_1}} + \mathrm{D}_2 \frac{\delta L}{\delta u_{Kt_2}} + \mathrm{D}_1 \mathrm{D}_2 \frac{\delta L}{\delta u_{Kt_1 t_2}} \right) \\
&= \sum_K \Big((\mathrm{D}_{Kt_1 t_2} V) + (\mathrm{D}_{Kt_2} V)\mathrm{D}_1 + (\mathrm{D}_{Kt_1} V)\mathrm{D}_2 + (\mathrm{D}_K V)\mathrm{D}_1 \mathrm{D}_2 \Big) \frac{\delta L}{\delta u_{Kt_1 t_2}} \\
&\quad + \sum_{K \not\ni t_2} \Big((\mathrm{D}_{Kt_1} V) + (\mathrm{D}_K V)\mathrm{D}_1 \Big) \frac{\delta L}{\delta u_{Kt_1}} \\
&\quad + \sum_{K \not\ni t_1} \Big((\mathrm{D}_{Kt_2} V) + (\mathrm{D}_K V)\mathrm{D}_2 \Big) \frac{\delta L}{\delta u_{Kt_2}} + \sum_{K \not\ni t_1, t_2} (\mathrm{D}_K V) \frac{\delta L}{\delta u_K}.
\end{aligned}$$

Proof of the key identity – integration by parts

In the last sum, only one multi-index is present, $K=(0,0)$, and everything else simplifies with the help of integration by parts to

$$\begin{aligned} \mathrm{D}_v L - V\frac{\delta L}{\delta u} &= \sum_K \mathrm{D}_1\mathrm{D}_2\Big((\mathrm{D}_K V)\frac{\delta L}{\delta u_{Kt_1t_2}}\Big) \\ &\quad + \sum_{K\not\ni t_2}\mathrm{D}_1\Big((\mathrm{D}_K V)\frac{\delta L}{\delta u_{Kt_1}}\Big) + \sum_{K\not\ni t_1}\mathrm{D}_2\Big((\mathrm{D}_K V)\frac{\delta L}{\delta u_{Kt_2}}\Big). \end{aligned}$$

Obviously, the r.h.s. can be written as $\mathrm{D}_1 A_1 + \mathrm{D}_2 A_2$. ■

Example

For the Lagrangian $L[u] = \frac{1}{2} u_x u_y - \cos u$, we compute EL equation:

$$\mathscr{E} := \frac{\delta L}{\delta u} = \frac{\partial L}{\partial u} - \mathrm{D}_x \frac{\partial L}{\partial u_x} - \mathrm{D}_y \frac{\partial L}{\partial u_y} = \sin u - u_{xy} = 0.$$

Generalized vector field $v = V[u]\, \partial / \partial u$ with $V[u] = u_{xxx} + \frac{1}{2} u_x^3$ is a variational symmetry for the SG equation, with the flux

$$\begin{aligned} F_1[u] &= \frac{1}{2} V u_y - \frac{1}{2} u_x^2 \cos u - u_{xx}(u_{xy} - \sin u), \\ F_2[u] &= \frac{1}{2} V u_x - \frac{1}{8} u_x^4 + \frac{1}{2} u_{xx}^2. \end{aligned}$$

One also shows that the conserved charge is given by

$$J_1 = -F_1 + \frac{1}{2} V u_y, \quad J_2 = -F_2 + \frac{1}{2} V u_x.$$

From variational symmetry to pluri-Lagrangan 2-form

Change point of view. Instead of considering D_v as acting on differential polynomials of a function u of two variables x, y, we interpret

$$u_t = V[u] = u_{xxx} + \frac{1}{2}u_x^3$$

as a flow on the space of solutions of $\mathscr{E} = 0$. Thus, we interpret u as function of three independent variables, $t_1 = x$, $t_2 = y$ and $t_3 = t$. We have seen that, if we set

$$L_{12} = L, \quad L_{13} = F_2|_{V \leftarrow u_t}, \quad L_{23} = -F_1|_{V \leftarrow u_t},$$

then the 2-form

$$\mathcal{L} = L_{12}[u]dx \wedge dy + L_{13}[u]dx \wedge dt + L_{23}[u]dy \wedge dt$$

defines a compatible pluri-Lagrangian system, whose multi-time EL equations coincide with

$$u_{xy} = \sin u, \quad u_t = V[u].$$

Variational symmetry vs. closedness of 2-form

In this formulation, operator D_v (acting on $u(x,y)$) is replaced by D_3 (acting on $u(x,y,t)$),

$$\mathrm{D}_3 = \sum_K u_{K+e_3} \frac{\partial}{\partial u_K},$$

with multi-indices $K = (i_1, i_2, i_3) \in (\mathbb{Z}_{\geq 0})^3$. The defining relation of variational symmetry,

$$\mathrm{D}_v L = \mathrm{D}_1 F_1 + \mathrm{D}_2 F_2,$$

becomes replaced by

$$\mathrm{D}_3 L_{12} + \mathrm{D}_1 L_{23} - \mathrm{D}_2 L_{13} = -\Big(u_t - u_{xxx} - \frac{1}{2} u_x^3\Big)(u_{xy} - \sin u).$$

Thus, $\mathcal{L}$ is *closed* on solutions of multi-time EL equations.

KdV hierarchies

- KdV hierarchy: $u_{t_k} = (r_k[u])_x$,

$$r_1 = u, \quad r_2 = u_{xx} + 3u^2, \quad r_3 = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3.$$

- Potential KdV hierarchy: $v_{t_k} = g_k[v]$, where $g_k[v] := r_k[v_x]$,

$$g_1 = v_x, \quad g_2 = v_{xxx} + 3v_x^2, \quad g_3 = v_{xxxxx} + 10v_x v_{xxx} + 5v_{xx}^2 + 10v_x^3.$$

- Differentiated potential KdV hierarchy: $v_{xt_k} = (g_k[v])_x$.

Proposition. *The dpKdV equations are Lagrangian, with the Lagrange functions*

$$L_k[v] = \frac{1}{2} v_x v_{t_k} - h_k[v],$$

where $h_k = \frac{1}{4k+2} g_{k+1}$.

Variational symmetries for KdV

Proposition. For the i-th dpKdV equation (2D Lagrangian system in (x, t_i)-plane), the evolutionary j-th pKdV equation $v_{t_j} = g_j[v]$ is a variational symmetry:

$$\mathrm{D}_{g_j}(L_{1i}) + \mathrm{D}_x(L_{ij}^{(i)}) - \mathrm{D}_i\left(L_{1j}^{(g_j)}\right) = 0.$$

Here, $L_{1j}^{(g_j)}$ is the Lagrangian L_{1j} with v_{t_j} replaced by g_j:

$$L_{1j}^{(g_j)} := \frac{1}{2} v_x g_j - h_j,$$

and $L_{ij}^{(i)}$ depends only on derivatives of v w.r.t. x and t_i.

Proposition. There exists $L_{ij}[v]$ depending on derivatives of v w.r.t. x, t_i and t_j that reduces to $L_{ij}^{(i)}$ and to $L_{ij}^{(j)}$ after the substitutions $v_{t_j} = g_j$ and $v_{t_i} = g_i$, respectively.

Pluri-Lagrangian structure for KdV hierarchy

Theorem. *The multi-time EL equations for*

$$\mathcal{L} = \sum_{i<j}^{N} L_{ij}\, dt_i \wedge dt_j$$

with coefficients given by

$$L_{1i} := L_i = \frac{1}{2} v_x v_{t_i} - h_i$$

and L_{ij} defined above, are

$$v_{t_2} = g_2[v], \quad v_{t_3} = g_3[v], \quad \ldots \quad v_{t_N} = g_N[v]$$

(and their differential consequences). The form $\mathcal{L}$ is closed on the joint solutions of these equations.

Conclusions

Pluri-Lagrangian structure is a recently emerged fundamental feature of integrability. It is intimately related to a much more classical notion of variational symmetries. While the former naturally appear also in the discrete context, the discrete theory of the latter is non-existent. Time for new theories to fertilise the classical ones?