New Soliton Solutions of Anti-Self-Dual Yang-Mills (ASDYM) Equations

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参考文献： [HH] arXiv:2004.09248
[GHHN] arXiv:2004.01718

Introduction

Brief Content of this Talk

- Darboux transformation solution of (Anti-self-dual Yang-Mils) ASDYM on Complex Space ($G = GL(2)$).

  $\Longrightarrow$ Under the Reduction conditions

- Soliton Solutions on 4D Real Spaces.
  (Euclidean(+,+,+,+), Minkowski(+,-,-,-), Ultrahyperbolic (+,+,-,-)).
  $\Longrightarrow$ **1-Soliton Solutions** (Not Instantion)

  (take a reduced form to remove the singularities)

  $\Longrightarrow$ **Domain Wall type Soliton** (Interpretation)

Motivation, Difficulty, and Objective

- 'tHooft Ansatz solution ($G = SU(2)$) $\Longrightarrow$ Tr$F_{\mu\nu}F^{\mu\nu}$=0
  Atiyah-Ward Ansatz solution ($G = GL(2)$) $\Longrightarrow$ Tr$F_{\mu\nu}F^{\mu\nu}$=0
- For $G = GL(2)$, Action Density is Complex-Valued in general.
- $G = U(2)$ (For physical purpose)

Achievement

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \propto \left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right), \text{ and } \int \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} d^4 x = 0.$$

- Nontrivial Action Denstiy:
  $\Longrightarrow$ New formulation of ASDYM for Darboux Transformation.
- Real-Valued Action Density for $G = GL(2)$:
  $\Longrightarrow$ Suitable construction of Yang's J-matrix.
- Pure 1-Soliton:
  $\Longrightarrow$ No Singularites, No Periodic Fluctuation.
- Domain Wall type Soliton
- $G = U(2) \Longrightarrow$ Ultrahyperbolic Space.

	E	M	U
Real-valued $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$	○	○	○
$J \in SU(2)$	○	○	○
$G = SU(2)$	×	×	◎

Outline

- Introduction
- Darboux Transformation Solution of ASDYM
- Soliton Solutions of ASDYM on Complex Space
- Soliton Solutions of ASDYM on 4D Real Spaces (Euclidean, Minkowski, and Ultrahyperbolic Signature)
- Conditions of Unitary J-matrix on each Space
- Discussion of Unitary Gauge Group
- Conclusion and Future Work

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ASDYM on 4D Complex Space with Coordinates $(z, \tilde{z}, w, \tilde{w})$

- Metric :

$$ds^2 = g_{mn} dz^m dz^n = 2(dzd\tilde{z} - dwd\tilde{w}), \quad g_{mn} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- Field Strength: $F_{mn} := \partial_m A_n - \partial_n A_m + [A_m, A_n]$
- ASDYM

$$F_{zw} = [D_z, D_w] = 0, \;\; F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}] = 0,$$
$$F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}] - [D_w, D_{\tilde{w}}] = 0.$$

- Complex Action Density

$$\text{Tr}F^2 := \text{Tr}F_{mn}F^{mn} = -2\text{Tr}(F_{w\tilde{w}}^2 + F_{z\tilde{z}}^2 + 2F_{\tilde{z}w}F_{z\tilde{w}} + 2F_{zw}F_{\tilde{z}\tilde{w}}),$$

where $F^{mn} := g^{mk}g^{nl}F_{kl}$.

Usual Formulation of ASDYM

ASDYM equation holds

$\iff$ The Lax pair $L := D_w - \zeta D_{\tilde{z}}, \quad M := D_z - \zeta D_{\tilde{w}} \quad s.t \ \ [L, M] = 0.$

$\iff$ Yang equation holds (J: Yang's J-matrix)

$$\partial_{\tilde{z}}(J^{-1}\partial_z J) - \partial_{\tilde{w}}(J^{-1}\partial_w J) = 0.$$

(That is, J-matrix of Yang equation $\iff$ solution of ASDYM.)

Then ASD gauge fields can be given in terms of the decompositions of J-matrix, $J = \tilde{h}^{-1}h$:

$$A_z = -(\partial_z h)h^{-1}, \ A_w = -(\partial_w h)h^{-1}, \ A_{\tilde{z}} = -(\partial_{\tilde{z}}\tilde{h})\tilde{h}^{-1}, \ A_{\tilde{w}} = -(\partial_{\tilde{w}}\tilde{h})\tilde{h}^{-1}.$$

New Formulation of ASDYM (Nimmo-Gilson-Ohta 2000)

Let Lax pair $L\phi := D_w\phi - (D_{\widetilde{z}}\phi)\zeta$, $M\phi := D_z\phi - (D_{\widetilde{w}}\phi)\zeta$,

then $LM\phi - ML\phi = 0 \iff$ ASDYM equations holds.

- ζ : Right-multiplicative Constant Matrix

A special gauge $\widetilde{h} = 1$

Gauge fields become a simpler form in terms of J:

$$A_z = J^{-1}\partial_z J, \quad A_w = J^{-1}\partial_w J, \quad A_{\widetilde{z}} = A_{\widetilde{w}} = 0,$$

Darboux Transformation for $n = 1$ (Nimmo-Gilson-Ohta 2000)

Let $(\phi, \zeta) = (Q, \Lambda)$ be an eigenfunction-eigenvalue pair of linear system

$$L\phi = D_w\phi - (\partial_{\tilde{z}}\phi)\zeta = (\partial_w + J^{-1}\partial_w J)\phi - (\partial_{\tilde{z}}\phi)\zeta = 0,$$
$$M\phi = D_z\phi - (\partial_{\tilde{w}}\phi)\zeta = (\partial_z + J^{-1}\partial_z J)\phi - (\partial_{\tilde{w}}\phi)\zeta = 0.$$

Then this linear system is "form-invariant" under the transformation:

$$\widetilde{J} = -Q\Lambda^{-1}Q^{-1}J, \quad \widetilde{\phi} = \phi\zeta - Q\Lambda Q^{-1}\phi \; .$$

That is, $\widetilde{J}$, $\widetilde{\phi}$ satisfy the linear system

$$\widetilde{L}\widetilde{\phi} = (\partial_w + \widetilde{J}^{-1}\partial_w \widetilde{J})\widetilde{\phi} - (\partial_{\tilde{z}}\widetilde{\phi})\zeta = 0,$$
$$\widetilde{M}\widetilde{\phi} = (\partial_z + \widetilde{J}^{-1}\partial_z \widetilde{J})\widetilde{\phi} - (\partial_{\tilde{w}}\widetilde{\phi})\zeta = 0.$$

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Soliton solution for $n = 1$ Darboux transf and matrix size N=2

Take seed solution $J_0 = I_{2\times 2}$ for simplicity, Then the $(\phi, \zeta) = (Q, \Lambda)$ must satisfy the linear system

$$\partial_w \phi = (\partial_{\tilde{z}} \phi)\zeta, \ \partial_z \phi = (\partial_{\tilde{w}} \phi)\zeta.$$

Soliton Solution

$$Q = \begin{pmatrix} a_1 e^L + a_2 e^{-L} & b_1 e^M + b_2 e^{-M} \\ c_1 e^L + c_2 e^{-L} & d_1 e^M + d_2 e^{-M} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

$$L := \lambda\beta z + \alpha\tilde{z} + \lambda\alpha w + \beta\tilde{w}, \quad M := \mu\delta z + \gamma\tilde{z} + \mu\gamma w + \delta\tilde{w},$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, \alpha, \beta, \gamma, \delta, \lambda, \mu$ are complex constants.

Gauge Fields:

$$A_m = \frac{2(\mu-\lambda)}{\Delta^2}\begin{pmatrix} pBD - qAC & -pB^2 + qA^2 \\ pD^2 - qC^2 & -pBD + qAC \end{pmatrix}$$

$$\begin{cases} (p,\ q) := (\alpha\varepsilon_0,\ \gamma\widetilde{\varepsilon}_0) \ \text{ if } \ m = w, \\ (p,\ q) := (\beta\varepsilon_0,\ \delta\widetilde{\varepsilon}_0) \ \text{ if } \ m = z \\ \varepsilon_0 := a_2c_1 - a_1c_2, \quad \widetilde{\varepsilon}_0 := b_2d_1 - b_1d_2 \end{cases}$$

Complex-Valued Action Density

$$\mathrm{Tr}F^2 = 8(\lambda-\mu)^2(\alpha\delta-\beta\gamma)^2\varepsilon_0\widetilde{\varepsilon}_0\left[\frac{2\varepsilon_1\widetilde{\varepsilon}_1\sinh^2 X_1 - 2\varepsilon_2\widetilde{\varepsilon}_2\sinh^2 X_2 - \varepsilon_0\widetilde{\varepsilon}_0}{\left((\varepsilon_1\widetilde{\varepsilon}_1)^{\frac{1}{2}}\cosh X_1 + (\varepsilon_2\widetilde{\varepsilon}_2)^{\frac{1}{2}}\cosh X_2\right)^4}\right]$$

where

$$X_1 := M + L + \frac{1}{2}\log(\varepsilon_1/\widetilde{\varepsilon}_1), \quad X_2 := M - L + \frac{1}{2}\log(\varepsilon_2/\widetilde{\varepsilon}_2)$$

$$\varepsilon_0 := a_2c_1 - a_1c_2, \quad \widetilde{\varepsilon}_0 := b_2d_1 - b_1d_2,$$

$$\varepsilon_1 := a_1d_1 - b_1c_1, \quad \widetilde{\varepsilon}_1 := a_2d_2 - b_2c_2,$$

$$\varepsilon_2 := a_2d_1 - b_1c_2, \quad \widetilde{\varepsilon}_2 := a_1d_2 - b_2c_1.$$

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Euclidean real slice condition: $\tilde{z} = \bar{z},\ \tilde{w} = -\bar{w}$.

$$z = \frac{1}{\sqrt{2}}(x^0 - ix^1),\ \tilde{z} = \frac{1}{\sqrt{2}}(x^0 + ix^1),\ w = \frac{-1}{\sqrt{2}}(x^2 - ix^3),\ \tilde{w} = \frac{1}{\sqrt{2}}(x^2 + ix^3)$$

satisfying Euclidean metric : $ds^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$.

ASDYM on $\mathbb{E}$: $\quad F_{01} + F_{23} = 0,\quad F_{02} - F_{13} = 0,\quad F_{03} + F_{12} = 0.$

Soliton Solution:

$$Q = \begin{pmatrix} a_1 e^L + a_2 e^{-L} & b_1 e^{\bar{L}} + b_2 e^{-\bar{L}} \\ -\bar{b}_1 e^L - \bar{b}_2 e^{-L} & \bar{a}_1 e^{\bar{L}} + \bar{a}_2 e^{-\bar{L}} \end{pmatrix},\quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -1/\bar{\lambda} \end{pmatrix},$$

where

$$L = (\lambda\beta)z + \alpha\bar{z} + (\lambda\alpha)w - \beta\bar{w}$$

$$= \frac{1}{\sqrt{2}}(\alpha + \lambda\beta,\ i(\alpha - \lambda\beta),\ \beta - \lambda\alpha,\ i(\beta + \lambda\alpha)) \cdot (x^0,\ x^1,\ x^2,\ x^3) := l_\mu x^\mu$$

$$\overline{M} = L \Longrightarrow \gamma = \overline{\lambda\beta},\ \delta = -\bar{\lambda}\bar{\alpha},\ \mu = -1/\bar{\lambda}$$

Action Density of this Soliton Solution is Real-Valued:

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} =$$

$$8\left[(|\alpha|^2+|\beta|^2)(|\lambda|^2+1)\,|\varepsilon_0|\right]^2 \left[\frac{2\varepsilon_1\widetilde{\varepsilon}_1 \sinh^2 X_1 - 2\,|\varepsilon_2|^2 \sinh^2 X_2 - |\varepsilon_0|^2}{\left((\varepsilon_1\widetilde{\varepsilon}_1)^{\frac{1}{2}} \cosh X_1 + |\varepsilon_2| \cosh X_2\right)^4}\right]$$

where

$$\text{Coefficients}\begin{cases} \varepsilon_0 = a_1\overline{b}_2 - a_2\overline{b}_1 \\ \varepsilon_1 = |a_1|^2 + |b_1|^2,\ \widetilde{\varepsilon}_1 = |a_2|^2 + |b_2|^2 \in \mathbb{R} \\ \varepsilon_2 = \overline{a}_1 a_2 + b_1\overline{b}_2 \end{cases}$$

$$\text{Variables}\begin{cases} X_1 = \overline{L} + L + \frac{1}{2}\log(\varepsilon_1/\widetilde{\varepsilon}_1) : \text{Real-valued function} \\ \quad \Longrightarrow \text{Solitonic wave} \sim \text{sech}^2 X_1 \\ X_2 = \overline{L} - L + \frac{1}{2}\log(\varepsilon_2/\overline{\varepsilon}_2) : \text{Pure imaginary function} \\ \quad \Longrightarrow \cosh X_2 = \cos(\mathrm{Im}X_2),\ \ \sinh X_2 = i\sin(\mathrm{Im}X_2) \\ \quad \Longrightarrow \text{Periodic fluctuation} \end{cases}$$

Setting $a_2 = b_1 = 0,\ a_1 = a,\ b_2 = b$, or equivalently,

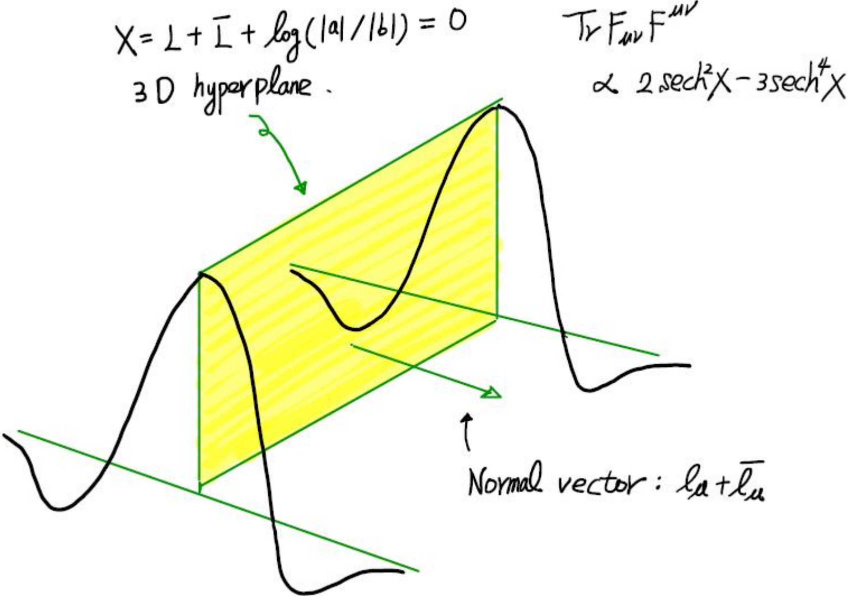
$$Q = \begin{pmatrix} ae^{L} & be^{-\overline{L}} \\ -\overline{b}e^{-L} & \overline{a}e^{-\overline{L}} \end{pmatrix}.$$

Then we can obtain a reduced form of action density without the periodic fluctuation part : (Note that $|\varepsilon_0|^2 = \varepsilon_1\widetilde{\varepsilon}_1 = |ab|^2,\ \ |\varepsilon_2|^2 = 0$.)

$$\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = 8\left[(|\alpha|^2 + |\beta|^2)(|\lambda|^2 + 1)\right]^2 \left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right),$$

where $X = L + \overline{L} + \log(|a|\,/\,|b|)$.

- Pure 1-Soliton (No sigularities, periodic fluctuation.)
- Integration is possible. In fact, $\int_{\mathbb{E}} \mathrm{Tr}F_{\mu\nu}F^{\mu\nu} d^4x = 0$.
- Domain Wall on $\mathbb{R}^4$:
  The peak of Action Density lies on a 3D hyperplane
  $X = L + \overline{L} + \log|a/b| = 0$ with normal vector $l_\mu + \overline{l}_\mu$.


$X = L + \bar{L} + \log(|a|/|b|) = 0$
3D hyperplane.
$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$
$\propto 2\,\mathrm{sech}^2 X - 3\,\mathrm{sech}^4 X$
Normal vector: $\ell_a + \bar{\ell}_a$

Instantion number = 0

Set $X = X^0$ and introduce 3 independent axes X^1, X^2, X^3 in the directions orthogonal to the X-axis (normal direction of the domain wall (DW)).

$$X^\mu := C_{\mu 0} x^0 + C_{\mu 1} x^1 + C_{\mu 2} x^2 + C_{\mu 3} x^3 + d_\mu .$$

Consider integration on a finite box $-R \leq x^\mu \leq R$. Then under the above linear transformation, the new box is

$$-a_\mu R + d_\mu \leq X^\mu \leq a_\mu R + d_\mu \ , \quad a_\mu := \sum_{\nu=0}^{3} |C_{\mu\nu}|.$$

$$\int_{-R}^{R}\int_{-R}^{R}\int_{-R}^{R}\int_{-R}^{R} (2\text{sech}^2 X - 3\text{sech}^4 X) dx^0 dx^1 dx^2 dx^3$$

$$= |J| \int_{-a_i R + d_i \leq x^i \leq a_i R + d_i} dX^1 dX^2 dX^3 \int_{-a_0 R + d_0}^{a_0 R + d_0} (2\text{sech}^2 X - 3\text{sech}^4 X) dX$$

$$= -|J| \int_{-a_i R + d_i \leq x^i \leq a_i R + d_i} dX^1 dX^2 dX^3 \ (\text{tanh} X \cdot \text{sech}^2 X) \Big|_{-a_0 R + d_0}^{a_0 R + d_0}$$

$$\Longrightarrow \int_{\mathbb{E}} \text{Tr} F_{\mu\nu} F^{\mu\nu} d^4 x \propto \int_{\text{DW}} dX^1 dX^2 dX^3 \ (\text{tanh} X \cdot \text{sech}^2 X) \Big|_{-\infty}^{\infty} = 0.$$

Soliton Solutions of ASDYM on Minkowski Space $\mathbb{M}$

Minkowski real slice condition: $z, \widetilde{z} \in \mathbb{R}, \ \widetilde{w} = \overline{w}$

$$z = \frac{1}{\sqrt{2}}(x^0 - x^1), \ \widetilde{z} = \frac{1}{\sqrt{2}}(x^0 + x^1), \ w = \frac{1}{\sqrt{2}}(x^2 - ix^3), \ \widetilde{w} = \frac{1}{\sqrt{2}}(x^2 + ix^3)$$

Minkowski metric $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.

ASDYM on $\mathbb{M}$ **:** $F_{01} + iF_{23} = 0, \quad F_{02} - iF_{13} = 0, \quad F_{03} + iF_{12} = 0.$

Soliton Solution

$$Q = \begin{pmatrix} a_1 e^{L} + a_2 e^{-L} & b_1 e^{\overline{L}} + b_2 e^{-\overline{L}} \\ -\overline{b}_1 e^{L} - \overline{b}_2^{-L} & \overline{a}_1 e^{\overline{L}} + \overline{a}_2 e^{-\overline{L}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix},$$

where

$$L = (\lambda\overline{\mu}\alpha)z + \alpha\widetilde{z} + (\lambda\alpha)w + (\overline{\mu}\alpha)\overline{w}$$
$$= \frac{1}{\sqrt{2}}\left((1+\lambda\overline{\mu})\alpha, \ (1-\lambda\overline{\mu})\alpha, \ (\overline{\mu}+\lambda)\alpha, \ i(\overline{\mu}-\lambda)\alpha\right) \cdot (x^0, \ x^1, \ x^2, \ x^3)$$

The condition $\overline{M} = L \Longrightarrow$ relations $\beta = \overline{\mu}\alpha,\ \gamma = \overline{\alpha},\ \delta = \overline{\lambda}\overline{\alpha}$
(Relation between λ and μ is not necessary.)

Action Density is Real-Valued:

$$\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = 8\left|\alpha(\lambda-\mu)\right|^4\left|\varepsilon_0\right|^2\left[\frac{2\varepsilon_1\widetilde{\varepsilon}_1\sinh^2 X_1 - 2\left|\varepsilon_2\right|^2\sinh^2 X_2 - \left|\varepsilon_0\right|^2}{\left((\varepsilon_1\widetilde{\varepsilon}_1)^{\frac{1}{2}}\cosh X_1 + \left|\varepsilon_2\right|\cosh X_2\right)^4}\right]$$

Reduced Form of Action Density:

$$\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = 8\left|\alpha^2(\lambda-\mu)\right|^2\left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right)$$

Ultrahyperbolic real slice condition for $\mathbb{U}$: $z, \widetilde{z},\ w,\ \widetilde{w} \in \mathbb{R}$

$$z = \frac{1}{\sqrt{2}}(x^0 - x^2),\ \widetilde{z} = \frac{1}{\sqrt{2}}(x^0 + x^2),\ w = -\frac{1}{\sqrt{2}}(x^1 - x^3),\ \widetilde{w} = \frac{1}{\sqrt{2}}(x^1 + x^3).$$

Ultrahyperbolic metric : $ds^2 = (dx^0)^2 + (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$

ASDYM on $\mathbb{U}_2$: $F_{01} + F_{23} = 0, \quad F_{02} + F_{13} = 0, \quad F_{03} - F_{12} = 0.$

Soliton Solution

$$Q = \begin{pmatrix} a_1 e^L + a_2 e^{-L} & b_1 e^{\overline{L}} + b_2 e^{-\overline{L}} \\ -\overline{b}_1 e^L - \overline{b}_2 e^{-L} & \overline{a}_1 e^{\overline{L}} + \overline{a}_2 e^{-\overline{L}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix},$$

where

$$\begin{aligned} L &= (\lambda\beta)z + \alpha\widetilde{z} + (\lambda\alpha)w + \beta\widetilde{w}, \\ &= \frac{1}{\sqrt{2}}(\alpha + \lambda\beta,\ \beta - \lambda\alpha,\ \alpha - \lambda\beta,\ \beta + \lambda\alpha) \cdot (x^0,\ x^1,\ x^2,\ x^3). \end{aligned}$$

The condition $\overline{M} = L \Rightarrow \gamma = \overline{\alpha},\ \delta = \overline{\beta},\ \mu = \overline{\lambda}$.

Action Density is Real-Valued:

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} =$$

$$8\left[(\alpha\overline{\beta} - \overline{\alpha}\beta)(\lambda - \overline{\lambda})\,|\varepsilon_0|\right]^2 \left[\frac{2\varepsilon_1\widetilde{\varepsilon}_1 \sinh^2 X_1 - 2\,|\varepsilon_2|^2 \sinh^2 X_2 - |\varepsilon_0|^2}{\left((\varepsilon_1\widetilde{\varepsilon}_1)^{\frac{1}{2}} \cosh X_1 + |\varepsilon_2| \cosh X_2\right)^4}\right],$$

Reduced form of Action Density :

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} = 8\left[(\alpha\overline{\beta} - \overline{\alpha}\beta)(\lambda - \overline{\lambda})\right]^2 \left(2\mathrm{sech}^2 X - 3\mathrm{sech}^4 X\right),$$

Conditions of Unitary Yang's J-Matrix on each Space

Proposition

Consider

$$Q = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Then Yang's J-matrix is $J = Q\Lambda^{-1}Q^{-1}$

$$= \frac{-1}{|A|^2 + |B|^2} \begin{pmatrix} (1/\lambda)|A|^2 + (1/\mu)|B|^2 & (1/\mu - 1/\lambda)AB \\ (1/\mu - 1/\lambda)\overline{AB} & (1/\mu)|A|^2 + (1/\lambda)|B|^2 \end{pmatrix}.$$

- $J \in U(2) \Leftrightarrow |\lambda| = |\mu| = 1$
  In fact, J is a double-parameter deformation of $U(2)$ group.
- $J \in SU(2) \Leftrightarrow \mu = \overline{\lambda}$ and $|\lambda| = 1$.
  In this case, J is a one parameter deformation of $SU(2)$ group.

- **Euclidean Sapce $\mathbb{E}$:**

  $$J \in U(2) \Leftrightarrow \Lambda = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & -e^{i\theta} \end{pmatrix}, \quad \theta \in \mathbb{R}$$

  $$J \in SU(2) \Leftrightarrow \Lambda = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

- **Minkowski Space $\mathbb{M}$:**

  $$J \in U(2) \Leftrightarrow \Lambda = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}, \quad (\theta_1, \theta_2 \in \mathbb{R})$$

  $$J \in SU(2) \Leftrightarrow \Lambda = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

- **Ultrahyperbolic Space $\mathbb{U}$:**

  $$J \in U(2) \Leftrightarrow J \in SU(2) \Leftrightarrow \Lambda = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

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Discussion of Unitary Gauge Group

Recall that

$$g \in U(2) \iff A_\mu : \text{Anti-Hermitian} \iff F_{\mu\nu} : \text{Anti-Hermitian}.$$

Euclidean Space $\mathbb{E}$

- $F_{\mu\nu}$: Anti-Hermitian $\Longrightarrow$ eignvalues of $F_{\mu\nu}$ are pure imaginary
- $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} = \mathrm{Tr} F_{\mu\nu}^2 < 0$
  (However, our action density is not negative definite.)

Minkowski Space $\mathbb{M}$

- **ASDYM :** $F_{01} + iF_{23} = 0, \quad F_{02} - iF_{13} = 0, \quad F_{03} + iF_{12} = 0.$
  $\Longrightarrow F_{\mu\nu}$ can not be all Anti-Hermitian.

$G=U(2)$ **is realized on Ultrahyperbolic space** $\mathbb{U}$ **successfully.**

- Gauge fields A_z and A_w are anti-hermitian on $\mathbb{U}$ naturally :

$$A_m = \frac{2(\overline{\lambda}-\lambda)}{\Delta^2}\begin{pmatrix} p\overline{A}B+\overline{p}A\overline{B} & -pB^2+\overline{p}A^2 \\ p\overline{A}^2-\overline{p}\overline{B}^2 & -p\overline{A}B-\overline{p}A\overline{B} \end{pmatrix}$$

$$\begin{cases} p:=\alpha\varepsilon_0, \quad \text{if} \quad m=w, \quad p:=\beta\varepsilon_0, \quad \text{if} \quad m=z, \\ \varepsilon_0:=a_1\overline{b}_2-\overline{a}_2 b_1, \end{cases}$$

- From Ultrahyperbolic real slice condition for $\mathbb{U}$,
$\sqrt{2}A_z=A_0+A_2, \sqrt{2}A_{\widetilde{z}}=A_0-A_2, \sqrt{2}A_w=A_1+A_3, \sqrt{2}A_{\widetilde{w}}=A_1-A_3$
- $A_{\widetilde{z}}=A_{\widetilde{w}}=0$

$\Longrightarrow$ All gauge fields A_μ must be anti-hermitian.

Conclusion and Future Work

Conclusion

Doamin Wall type soliton are constructed on each 4D space.

- Action density is Real-Valued.
- Topological charge $= 0$
- $G = U(2)$ is realized in Ultrahyperbolic Space $\mathbb{U}$.

Future Work

- $G = U(2)$ solution on Euclidean and Minkowski Space (under a more general framework)
- $G = GL(3)$ and $G = SU(3)$ solution.
- Soliton type solutions in gravitational field.
  ($n + 2$ dimensional axisymmetric solution of Einstein's vacuum equation $\iff$ J-matrix of $G = GL(n)$ Yang equation)

Thank you for your attendence