Qusideterminants. Lecture 2

Noncommutative Plücker and Flag Coordinates

Commutative Plücker coordinates. Let A be a $k\times n$-matrix, $k\leq n$ over a commutative ring R. Denote by $A(i_1,\dots,i_k)$ the $k\times k$-submatrix of A consisting of columns with indices $i_1,\dots,i_k$. The elements

$$p_{i_1\dots i_k}(A)=\det A(i_1,\dots,i_k)$$

are called Plücker coordinates of A.

Properties Plücker coordinates:

(i) (invariance) $p_{i_1\dots i_k}(XA)=\det X\cdot p_{i_1\dots i_k}(A)$ for any $k\times k$-matrix X over R;

(ii) (skew-symmetry) $p_{i_1\dots i_k}(A)$ are skew-symmetric in indices;

(iii) (Plücker relations) Let $i_1,\dots,i_{k-1}$ and $j_1,\dots,j_{k+1}$ be sets of distinct numbers. Then

$$\sum_{s=1}^{k}(-1)^s p_{i_1\dots i_k j_s}(A)p_{j_1\dots\widehat{j_s}\dots j_{k+1}}(A)=0$$

Set $p_{ij} := p_{ij}(A)$. For $k = 2$, $n = 4$ we have

$$p_{12}p_{34} + p_{14}p_{23} = p_{13}p_{24}$$

Left quasi-Plčker coordinates

Assume now that $A = (a_{ij})$ be a $k \times n$-matrix over noncommutative ring R. For any set of distinct elements $I = \{i_1, \dots, i_{k-1}\}$ and $i \notin I$ **set** $q^I_{ij}(A)$ **to be equal to**

$$\begin{vmatrix} a_{1i} & a_{1i_1} & \dots & a_{1,i_{k-1}} \\ & \dots & \dots & \\ \boxed{a_{si}} & a_{si_1} & \dots & a_{s,i_{k-1}} \\ & \dots & \dots & \\ a_{ki} & a_{ki_1} & \dots & a_{k,i_{k-1}} \end{vmatrix}^{-1} \begin{vmatrix} a_{1j} & a_{1i_1} & \dots & a_{1,i_{k-1}} \\ & \dots & \dots & \\ \boxed{a_{sj}} & a_{si_1} & \dots & a_{s,i_{k-1}} \\ & \dots & \dots & \\ a_{kj} & a_{ki_1} & \dots & a_{k,i_{k-1}} \end{vmatrix}$$

This ratio does not depend on s and on the ordering of I. In the commutative case

$$q^I_{ij} = \frac{\det A(j, i_1, \dots, i_k)}{\det A(i, i_1, \dots, i_{k-1})}$$

Properties of quasi-Plc̆ker coordinates

Note: $q^I_{ij} = 0$ if $j \in I$; $q^I_{ij} \cdot q^I_{ji} = 1$ if $i, j \notin I$.

(i) (invariance) For any invertible $k \times k$-matrix X

$$q^I_{ij}(XA) = q^I_{ij}(A)$$

(ii) (skew-symmetry) Let $|N| = k + 1$ and $i, j, \ell \in N$.

$$q^{N\setminus\{i,j\}}_{ij} \cdot q^{N\setminus\{j,\ell\}}_{j\ell} \cdot q^{N\setminus\{\ell,i\}}_{\ell i} = -1$$

(iii) (Plücker relations) $M = \{m_1, \ldots, m_{k-1}\}$, $L = \{\ell_1, \ldots, , \ell_k\}$, $i \notin M$. Then

$$\sum_{j \in L} q^M_{ij} \cdot q^{L\setminus\{j\}}_{ji} = 1$$

Basic examples: $N = \{1, 2, 3\}$. Then

$$q^3_{12} \cdot q^1_{23} \cdot q^2_{31} = -1$$

Commutative case:

$$\frac{p_{23}}{p_{13}} \cdot \frac{p_{31}}{p_{21}} \cdot \frac{p_{12}}{p_{32}} = -1$$

Plücker: $i = 1, M = \{3\}, L = \{2, 4\}$.

$$q^3_{12} \cdot q^4_{21} + q^3_{14} \cdot q^2_{41} = 1$$

Interpretation: Sum of two noncommutative cross-ratios equals 1. Ptolemy identity.

Commutative case:

$$\frac{p_{23}}{p_{13}} \cdot \frac{p_{14}}{p_{24}} + \frac{p_{43}}{p_{13}} \cdot \frac{p_{12}}{p_{42}} = 1$$

Right quasi-Plücker coordinates. Let $B = (b_{pq})$, $1 \le p \le n$, $1 \le q \le k$ over a division ring. Let $I = \{i_1, \dots, i_{k-1}\}$, $j \notin I$. Set

$$r^I_{ij}(B) = \begin{vmatrix} b_{i1} & \dots & \boxed{b_{it}} & \dots & b_{ik} \\ b_{i_1 1} & \dots & b_{i_1 t} & \dots & b_{i_1 k} \\ & \dots & \dots & \dots & \\ b_{i_{k-1} 1} & \dots & b_{i_{k-1} t} & \dots & b_{i_{k-1} k} \end{vmatrix} \times$$

$$\times \begin{vmatrix} b_{j1} & \dots & \boxed{b_{jt}} & \dots & b_{jk} \\ b_{i_1 1} & \dots & b_{i_1 t} & \dots & b_{i_1 k} \\ & \dots & \dots & \dots & \\ b_{i_{k-1} 1} & \dots & b_{i_{k-1} t} & \dots & b_{i_{k-1} k} \end{vmatrix}^{-1}$$

Right quasi-Plücker coordinates are dual to left quasi-Plücker coordinates.

Let $A = (a_{ij})$ be $k \times n$-matrix and $B = (b_{rs})$ be $n \times (n-k)$-matrix such that $AB = 0$.

Let $i, j \in [1, n]$, $I \subset [1, n]$, $|I| = k - 1$, and $i \notin I$. Set $J = ([1, n] \setminus I) \setminus \{i, j\}$. Then

$$q_{ij}^{I}(A) + r_{ij}^{J}(B) = 0$$

Applications of quasi-Plücker coordinates

Decompositions along rows and columns

Let $A = (a_{ij})$, $1 \leq i, j \leq n$. By removing the k-th row (resp. ℓ-th column) from A we get $A^{r,\emptyset}$ (resp. $A^{\emptyset,\ell}$). Then

$$|A|_{k\ell} = a_{k\ell} - \sum_{j \neq \ell} a_{kj} \cdot q_{j\ell}^{[1,n]\setminus\{j,\ell\}}(A^{k,\emptyset})$$

$$|A|_{k\ell} = a_{k\ell} - \sum_{i \neq k} r_{ki}^{[1,n]\setminus\{i,k\}}(A^{\emptyset,\ell}) \cdot a_{i\ell}$$

Relations for quasideterminants of a matrix

In the previous notations

$$|A|_{ik} = -|A|_{ij} \cdot q_{jk}^{[1,n]\setminus\{j,k\}}(A^{i,\emptyset})$$

$$|A|_{\ell j} = -r_{\ell i}^{[1,n]\setminus\{i,\ell\}}(A^{\emptyset,j}) \cdot |A|_{ij}$$

LDU-factorization

Let $A = (a_{ij})$, $1 \leq i, j \leq n$. Set

$A(k) = (a_{ij})$, $1 \leq i, j \leq k$,
$B(k) = (a_{ij})$, $1 \leq i \leq n,\ 1 \leq j \leq k$,
$C(k) = (a_{ij})$, $1 \leq i \leq k,\ 1 \leq j \leq n$.

In a generic case

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x_{\beta\alpha} & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & z_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Here $y_k = |A(k)|_{kk}$, $1 \leq \alpha < \beta \leq n$,

$$x_{\beta\alpha} = r_{\beta\alpha}^{[1,\alpha-1]}(B(\alpha)),\ \ z_{\alpha\beta} = q_{\alpha\beta}^{[1,\alpha-1]}(C(\alpha))$$

Example: If a_{11} is invertible then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{21}a_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & |A|_{22} \end{pmatrix} \begin{pmatrix} 1 & a_{11}^{-1}a_{12} \\ 0 & 1 \end{pmatrix}$$

Generalization: Berenstein and R.,

"Noncommutative double Bruhat cells and their factorizations" (IMRN, 2005)

Noncommutative flag coordinates

Let $A = (a_{ij})$, $1 \le i \le k$, $1 \le j \le n$, $k \le n$ be a matrix over a division ring R. Let F_p be the subspace of the left vector space R^n generated by the first p rows of A.

Then $\mathcal{F} = (F_1 \subset F_2 \subset \dots F_k)$ is a flag in R^n. We define *flag coordinates* of $\mathcal{F}$ as

$$f_{j_1 \dots j_k}(\mathcal{F}) = \begin{vmatrix} a_{1j_1} & \dots & a_{1j_k} \\ & \dots & \\ \boxed{a_{kj_1}} & \dots & a_{kj_k} \end{vmatrix}$$

Transformations properties of quasideterminants imply that $f_{j_1 \dots j_k}(\mathcal{F})$ does not depend on the order of indices $j_2, \dots, j_k$.

Flag coordinates are invariant under left multiplication of A by lower-triangular unipotent matrices.

Examples of relations for flag coordinates for 2×4-matrices:

Triangular relations

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} \cdot \begin{vmatrix} a_{12} & a_{13} \\ \boxed{a_{22}} & a_{23} \end{vmatrix}^{-1} \cdot \begin{vmatrix} a_{11} & a_{13} \\ \boxed{a_{21}} & a_{23} \end{vmatrix} =$$

$$= - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & \boxed{a_{23}} \end{vmatrix}^{-1} \cdot \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix}$$

Ptolemy relations

$$\begin{vmatrix} a_{12} & a_{13} \\ \boxed{a_{22}} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{12} & a_{14} \\ \boxed{a_{22}} & a_{24} \end{vmatrix}^{-1} \cdot \begin{vmatrix} a_{11} & a_{14} \\ \boxed{a_{21}} & a_{24} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix} \cdot \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & \boxed{a_{24}} \end{vmatrix}^{-1} \cdot \begin{vmatrix} a_{11} & a_{13} \\ \boxed{a_{21}} & a_{23} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ \boxed{a_{21}} & a_{23} \end{vmatrix}$$

Noncommutative cross-ratios

We define cross-ratios over (noncommutative) division ring $\mathcal{R}$ by imitating the definition of calssical cross-ratios in homogeneous coordinates, namely, if the four points are represented in homogeneous coordinates by vectors a, b, c, d such that $c = a + b$ and $d = ka + b$, then their cross-ratio is k.

Let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \ t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

be vectors in $\mathcal{R}^2$. Define **cross-ratio** $\kappa = \kappa(x, y, z, t)$ by equations

$$\begin{cases} t = x\alpha + y\beta \\ z = x\alpha\gamma + y\beta\gamma \cdot \kappa \end{cases}$$

where $\alpha, \beta, \gamma, \kappa \in \mathcal{R}$.

Consider matrix with columns x, y, z, t

$$\begin{pmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \end{pmatrix}$$

Then $\kappa(x, y, z, t) = q^y_{zt} \cdot q^x_{tz}$.

Note that $GL_2(\mathcal{R})$ acts on vectors in $\mathcal{R}^2$ by multiplication from the left: $(g,x) \mapsto gx$,

Group $\mathcal{R}^\times$ of invertible elements in $\mathcal{R}$ acts by multiplication from the right:
$(\lambda, x) \mapsto x\lambda^{-1}$.

It defines the action of $GL_2(\mathcal{R}) \times T_4(\mathcal{R})$ on $P_4 = (\mathcal{R}^2)^4$ where $T_4(\mathcal{R}) = (\mathcal{R}^\times)^4$.

The cross-ratios are *relative invariants* of the action.

The following theorem generalizes the main property of cross-ratios to the noncommutive case.

Theorem. Let $\kappa(x,y,z,t)$ be defined and $\kappa(x,y,z,t) \neq 0,1$. Then 4-tuples (x,y,z,t) and (x',y',z',t') from P_4 belong to the same orbit of $GL_2(\mathcal{R}) \times T_4(\mathcal{R})$ if and only if there exists $\mu \in \mathcal{R}^\times$ such that

$$\kappa(x,y,z,t) = \mu \cdot \kappa(x',y',z',t') \cdot \mu^{-1} \ .$$

Noncommutative cross-ratios satisfy *cocycle conditions* for vectors x, y, z, t, w

$$\kappa(x, y, z, t) = \kappa(w, y, z, t) \cdot \kappa(x, w, z, t)$$
$$\kappa(x, y, z, t) = 1 - \kappa(t, y, z, x)$$

Noncommutative cross-ratios and permutations

There are 24 cross-ratios defined for vectors $x, y, z, t \in \mathcal{R}^2$. They are related by the following formulas:

$$q_{tz}^{x}\kappa(x, y, z, t)q_{zt}^{x} = q_{tz}^{y}\kappa(x, y, z, t)q_{zt}^{y} = \kappa(y, x, t, z);$$

$$q_{xz}^{y}\kappa(x, y, z, t)q_{zx}^{y} = q_{xz}^{t}\kappa(x, y, z, t)q_{zx}^{t} = \kappa(z, t, x, y)$$

$$q_{yz}^{x}\kappa(x, y, z, t)q_{zy}^{x} = q_{yz}^{t}\kappa(x, y, z, t)q_{zy}^{t} = \kappa(t, z, x, y)$$

$$\kappa(x, y, z, t)^{-1} = \kappa(y, x, z, t).$$

Note again the effect of conjugation in the noncommutative case since q_{ij}^{k} and q_{ji}^{k} are inverses to each other.

Relations beetween noncommutative cross-ratio and noncommutative Schwarz derivative were discussed in a joint paper with V. Rubtsov and G. Sharygin (2020).

Other applications to elementary geometry:

Theorem of Menelaus

Let $\mathcal{R}$ be a noncommutative division ring. Consider $\mathcal{R}^2$ as the right vector space over $\mathcal{R}$. For a point $X \in \mathcal{R}^2$ denote by x_i its i-th coordinate, $i = 1, 2$.

Let ABC be a triangle (vertices go anti-clock-wise). Take point R at line AB, point P at line BC, and point Q at line AC. Consider 2×6-matrix with columns
$X = (x_1, x_2)^t$, $X \in \{A, B, C, P, Q, R\}$.

Theorem. Points P, Q, R belong to a straight line if an only if

$$q^Q_{AC} q^P_{CB} q^R_{BA} = 1$$

Applications of elementary geometry to integrable systems were discussed by W. Schief and B. Konopelchenko.