Noncommutative $G_{2,4}(\mathbb{C})$ as Deformation Quantization with Separation of Variables

Taika Okuda[1]

(Joint work with Akifumi Sako[1,2], based on arXiv:2401.00500 [math-ph])

[1]Graduate School of Science, Tokyo University of Science
[2]Faculty of Science Division II, Tokyo University of Science

International Seminar-Type Online Workshop on Noncommutative Integrable Systems
(March 4-15, 2024, Online)

Outline

1. Introduction (Deformation Quantization/Complex Grassmannian)

2. Main Result: Star Product with Separation of Variables on $G_{2,4}(\mathbb{C})$

3. Summary and Outlook

Definition of Deformation Quantization

Definition 1 (Deformation Quantization)

Let $(M, \{\ ,\ \})$ be a Poisson manifold and $C^\infty(M)[\![\hbar]\!]$ be the ring of formal parameter series over $C^\infty(M)$. Let $*$ be the star product denoted by $f * g = \sum_k C_k(f,g)\hbar^k$ satisfying the following conditions :

1. For any $f, g, h \in C^\infty(M)[\![\hbar]\!]$, $f * (g * h) = (f * g) * h$.
2. For any $f \in C^\infty(M)[\![\hbar]\!]$, $f * 1 = 1 * f = f$.
3. For any $f, g \in C^\infty(M)$, $C_k(f,g) = \sum_{I,J} a_{I,J} \partial^I f \partial^J g$, where I, J are multi-indices.
4. $C_0(f,g) = fg, \quad C_1(f,g) - C_1(g,f) = \{f, g\}$.

$(C^\infty(M)[\![\hbar]\!], *)$ called a deformation quantization for Poisson manifold M.

Deformation Quantization with Separation of Variables

For Kähler manifolds, Karabegov proposed one of the deformation quantizations.

Definition 2 (D. Q. with separation of variables(Karabegov(1996)))

Let M be an N-dimensional Kähler manifold. The star product $*$ on M is separation of variables if $*$ satisfies the two conditions for any open set U and $f \in C^{\infty}(U)$:

1. For a holomorphic function a on U, $a * f = af$.
2. For an anti-holomorphic function b on U, $f * b = fb$.

The deformation quantization by the star product $*$ such that separation of variables $(C^{\infty}(M)[\![\hbar]\!], *)$ is called a deformation quantization with separation of variables for Kähler manifold M.

Historical Background of Deformation Quantization(1)

Deformation quantization for symplectic manifolds

- de Wilde-Lecomte (1983)
- Omori-Maeda-Yoshioka (1991)
- Fedosov (1994)

Deformation quantization for Poisson manifolds

- Kontsevich (2003)

Deformation quantization for contact manifolds

- Elfimov-Sharapov (2022)

Historical Background of Deformation Quantization(2)

Deformation quantization for Kähler manifolds

- Moreno (1986)
- Omori-Maeda-Miyazaki-Yoshioka (1998)
- Reshetikhin-Takhtajan (2000)

Deformation quantization with separation of variables for Kähler manifolds

- Karabegov (1996), Gammelgaard (2014)
  $\longrightarrow$ For the case of Kähler manifolds.
- Sako-Suzuki-Umetsu (2012), Hara-Sako (2017)
  $\longrightarrow$ For the case of locally symmetric Kähler manifolds

Deformation Quantization and Modern Physics

Deformation quantization (or star product) has applications in modern physics, for example, quantum field theory, string theory and quantum gravity.

Applications of deformation quantization for physics

1. Kontsevich's star product interpretation using a path integral on the Poisson-sigma model. (Cattaneo-Felder (2000))
2. Noncommutative solitons on Kähler manifolds via a star product. (Spradlin-Volovich (2002))
3. Extension of soliton theories and integrable systems using a star product and quasideterminant. (Hamanaka (2010,2014))
4. Deformed (or Noncommutative) gauge theories on homogeneous Kähler manifolds. (Maeda-Sako-Suzuki-Umetsu (2014))

Construction Method proposed by Hara-Sako

Hara and Sako proposed the method for locally symmetric Kähler manifolds.

Theorem 3 (Hara-Sako(2017))

Let M be an N-dimensional locally symetric Kähler manifold, i.e. a Kähler manifold such that $\nabla R^{\nabla} = 0$, and U be an open set of M. Then, for any $f, g \in C^{\infty}(U)$, there exists a star product with separation of variables $*$ such that

$$f * g = \sum_{n=0}^{\infty} \sum_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} \left(D^{\overrightarrow{\alpha_n}} f \right) \left(D^{\overrightarrow{\beta_n^*}} g \right).$$

Here $D^i := g^{i\bar{j}} \frac{\partial}{\partial \bar{z}^j}$, $D^{\bar{i}} = \overline{D^i}$, $\overrightarrow{\alpha_n} = (\alpha_1^n, \cdots, \alpha_N^n)$, $\overrightarrow{\beta_n} = (\beta_1^n, \cdots, \beta_N^n)$ are the multi-indices such that the sum of each component is n, and the coefficient $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$ is a formal power series satisfying some recurrence relations.

Obtained star product: $\mathbb{C}^N$, $\mathbb{C}P^N$, $\mathbb{C}H^N$, arbitrary 1- and 2-dimensional ones.

Complex Grassmannian

Definition 4 (Complex Grassmannian)

The complex Grassmannian $G_{p,p+q}(\mathbb{C})$ is defined by

$$G_{p,p+q}(\mathbb{C}) := \left\{ V \subset \mathbb{C}^{p+q} \mid V : \text{complex vector subspace } s.t. \ \mathrm{dim} V = p \right\}.$$

We take the local coordinates of $G_{p,p+q}(\mathbb{C})$. Let

$$U := \left\{ Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \in M(p+q, p; \mathbb{C}) \mid Y_0 \in GL_p(\mathbb{C}), \ Y_1 \in M(q, p; \mathbb{C}) \right\}$$

be an open set of $G_{p,p+q}(\mathbb{C})$, and $\phi : U \to M(q, p; \mathbb{C})$ be a holomorphic map such that $Y \mapsto \phi(Y) := Y_1 Y_0^{-1}$. By using ϕ, we can choose

$$Z := \left(z^I \right) = \left(z^{ii'} \right) = Y_1 Y_0^{-1}$$

as the local coordinates, where $I := ii'$ $(i = 1, \cdots, q, \ i' = 1', \cdots, p')$.

Recurrence Relations for $G_{2,4}(\mathbb{C})$

We focus on $G_{2,4}(\mathbb{C})$. The recurrence relations for $G_{2,4}(\mathbb{C})$ are given by

$$
\begin{aligned}
&\hbar \sum_{D \in \mathcal{I}} g_{\overline{I} D} T^{n-1}_{\overrightarrow{\alpha_n} - \overrightarrow{e_D}, \overrightarrow{\beta^*_n} - \overrightarrow{e^*_I}} \\
&= \hbar \beta^n_I \left(\tau_n + \beta^n_{\not I} \right) T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta^*_n}} - \hbar \left(\beta^n_{i \not i'} + 1 \right) \left(\beta^n_{\not i i'} + 1 \right) T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta^*_n} - \overrightarrow{e^*_I} + \overrightarrow{e^*_{i \not i'}} + \overrightarrow{e^*_{\not i i'}} - \overrightarrow{e^*_{\not I}}}, \qquad (1)
\end{aligned}
$$

where $\tau_n := 1 - n + \frac{1}{\hbar}$, $\overrightarrow{\beta^*_n} = (\beta^n_I, \beta^n_{\not I}, \beta^n_{i \not i'}, \beta^n_{\not i i'})$, and $\mathcal{I} := \left\{ I, \not I, i \not i', \not i i' \right\}$.

Remark

$\not i$ (or $\not i'$) is the other index which is not i (or not i'). $\not i$ (or $\not i'$) is uniquely determined when i (or i') is fixed. For example, if $I = 11'$, then $i \not i' = 12'$, $\not i i' = 21'$, and $\not I = 22'$. $I = ii'$ may take $12'$, $21'$ and $22'$ as well as $11'$.

Solving the Recurrence Relations for $G_{2,4}(\mathbb{C})$ (1)

First problem

The number of variables in (1) increases combinatorially **with increasing** n. For this reason, it is difficult to obtain the general term $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}}$ from (1) in a straightforward way.

$$\Downarrow$$

Solutions to first problem

1. We transform (1) into **the (equivalent) ones** such that the only term of order n appearing in the expression is $\beta^n_I T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}}$.
2. We derive **new recurrence relations** satisfied by $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}}$ from equivalent recurrence relations. By using the obtained ones, we can explicitly determine $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}}$.

Solving the Recurrence Relations for $G_{2,4}(\mathbb{C})$ (2)

$\Downarrow$ Some technical calculations

Proposition 2.1 (O.-Sako(2024))

$T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$ for $G_{2,4}(\mathbb{C})$ is expressed using the solution of order $(n-1)$ as follows :

$$T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}} = \frac{\sum_{J,D\in\mathcal{I}} \sum_{k=1}^{2} \left\{ \left(\tau_n \delta_{jk} + \beta^n_{j\not{j}'} + 1 \right) g_{\overline{kj'},D} T^{n-1}_{\overrightarrow{\alpha_n}-\overrightarrow{e_D},\overrightarrow{\beta_n^*}-\overrightarrow{e_J^*}-\delta_{\not{j}k}\left(\overrightarrow{e^*_{\not{J}}}-\overrightarrow{e^*_{j\not{j}'}}\right)} \right\}}{\tau_n \left\{ n(\tau_n+1) + 2\left(\beta^n_I + \beta^n_{\not{i}i'}\right)\left(\beta^n_{\not{I}} + \beta^n_{i\not{i}'}\right) \right\}}. \tag{2}$$

That is, $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta_n^*}}$ given by (2) gives the star product with separation of variables.

Solving the Recurrence Relations for $G_{2,4}(\mathbb{C})$ (3)

Second problem

1. If $\overrightarrow{\alpha_n}$ or $\overrightarrow{\beta_n^*}$ has at least one negative component, we define $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} := 0$.
2. The multi-index $\overrightarrow{\beta_n^*} - \overrightarrow{e_J^*} - \delta_{\not{j}k}\left(\overrightarrow{e_{\not{j}}^*} - \overrightarrow{e_{j\not{j}'}^*}\right)$ appearing on the right-hand side of (2) in Proposition 2.1 includes not only subtraction but also **addition**.
3. We should determine $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$ from (2) taking into account the above problems. However, it is **difficult to obtain** $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$ straightforwardly from (2) due to the above problems.

Solutions to second problem

- The property " $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}} = 0$ when $\overrightarrow{\alpha_n}$ or $\overrightarrow{\beta_n^*}$ contain negative components" corresponds well to **the Fock representation**.
  $\longrightarrow$ By using the Fock representation, the above problems are eliminated.
  This make it **possible to determine** $T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$ straightforwardly from (2).

Solving the Recurrence Relations for $G_{2,4}(\mathbb{C})$ (4)

We introduce the following operator:

$$T_n : \text{a linear operator on a Fock space } s.t. \ \langle \overrightarrow{\alpha_n} | T_n | \overrightarrow{\beta_n^*} \rangle = T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}.$$

Corresponding table:

Notations appearing in rec. rel. (1)	$\longleftrightarrow$	Fock representations
$T^n_{\overrightarrow{\alpha_n}, \overrightarrow{\beta_n^*}}$	$\longleftrightarrow$	T_n
β^n_I	$\longleftrightarrow$	$N_I (:= a_I^\dagger a_I)$
$+\overrightarrow{e_I^*}$	$\longleftrightarrow$	$a_I^\dagger \frac{1}{\sqrt{N_I+1}}$
$-\overrightarrow{e_I^*}$	$\longleftrightarrow$	$a_I \frac{1}{\sqrt{N_I}}$
Scalar (not β^n_I)	$\longleftrightarrow$	Scalar multiplication

Here $a_I^\dagger$, a_I are creation and annihilation operators defined as

$$a_I^\dagger | \overrightarrow{\beta_n} \rangle := \sqrt{\beta^n_I + 1} | \overrightarrow{\beta_n} + \overrightarrow{e_I^*} \rangle, \qquad a_I | \overrightarrow{0} \rangle = 0, \qquad a_I | \overrightarrow{\beta_n} \rangle := \sqrt{\beta^n_I} | \overrightarrow{\beta_n} - \overrightarrow{e_I^*} \rangle.$$

Solution of the Recurrence Relations (1)

$$T_n = \sum_{J,D\in\mathcal{I}} a^\dagger_D \frac{1}{\sqrt{N_D+1}} T_{n-1} \left\{ a_J \frac{1}{\sqrt{N_J}} \left(\tau_n + N_{j\not{j}'} + 1\right) g_{\overline{J}D} + a_J a_{\not{J}} a^\dagger_{j\not{j}'} \frac{N_{j\not{j}'}+1}{\sqrt{N_J N_{\not{J}} \left(N_{j\not{j}'}+1\right)}} g_{\overline{\not{j}j'},D} \right\}$$
$$\cdot\, \tau_n^{-1} \left\{ n\left(\tau_n+1\right) + 2\left(N_I + N_{\not{i}i'}\right)\left(N_{\not{I}} + N_{i\not{i}'}\right)\right\}^{-1}. \tag{3}$$

$\Downarrow$ By sequentially substituting lower-order ones...

Theorem 5 (O.Sako(2024))

A linear operator T_n is explicitly given by

$$T_n = \sum_{\substack{J_i\in\{J_i\}_n \\ D_i\in\{D_i\}_n}} \sum_{\substack{k_i=1 \\ k_i\in\{k_i\}_n}}^{2} \frac{g_{\overline{k_1 j_1{}'},D_1} \cdots g_{\overline{k_n j_n{}'},D_n}}{\tau_1\cdots\tau_n} a^\dagger_{D_n} \frac{1}{\sqrt{N_{D_n}+1}} \cdots a^\dagger_{D_1} \frac{1}{\sqrt{N_{D_1}+1}}$$
$$\cdot\, T_0 \mathcal{A}_{J_1,k_1} \cdots \mathcal{A}_{J_n,k_n} \mathcal{C}_{1,\{J_i\}_n,\{k_i\}_n} \cdots \mathcal{C}_{n,\{J_i\}_n,\{k_i\}_n}$$
$$\cdot\, \mathcal{F}_{1,\{J_i\}_n,\{k_i\}_n} \cdots \mathcal{F}_{n,\{J_i\}_n,\{k_i\}_n}. \tag{4}$$

Here, for $l = 1, \cdots, n$, $\{J_i\}_n := \{J_1, \cdots J_n\}$ and $\{k_i\}_n := \{k_1, \cdots k_n\}$,

$$\sum_{J_i \in \{J_i\}_n} := \sum_{J_1 \in \mathcal{I}} \cdots \sum_{J_n \in \mathcal{I}}$$

$$\sum_{D_i \in \{D_i\}_n} := \sum_{D_1 \in \mathcal{I}} \cdots \sum_{D_n \in \mathcal{I}}$$

$$\sum_{\substack{k_i = 1 \\ k_i \in \{k_i\}_n}}^{2} := \sum_{k_1 = 1}^{2} \cdots \sum_{k_n = 1}^{2}$$

$$\mathcal{A}_{J_l, k_l} := a_{J_l} \frac{1}{\sqrt{N_{J_l}}} \left(a_{\cancel{J_l}} \frac{1}{\sqrt{N_{\cancel{J_l}}}} a^{\dagger}_{j_l \cancel{j_l'}} \frac{1}{\sqrt{N_{j_l \cancel{j_l'}} + 1}} \right)^{\delta_{\cancel{j_l} k_l}},$$

$$\mathcal{C}_{l, \{J_i\}_n, \{k_i\}_n} := \tau_l \delta_{j_l k_l} + N_{j_l \cancel{j_l'}} + 1 - \cdots,$$

$$\mathcal{F}_{l, \{J_i\}_n, \{k_i\}_n} = \left\{ l (\tau_l + 1) + 2 \left(N_I + N_{\cancel{i} i'} - \cdots \right) \left(N_{\cancel{I}} + N_{i \cancel{i'}} - \cdots \right) \right\}^{-1}.$$

Solution of the Recurrence Relations (2)

Theorem 6 (O.-Sako(2024))

The solution $T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}}$ *for* $G_{2,4}(\mathbb{C})$ *is given by*

$$
\begin{aligned}
&T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}} \\
&= \sum_{\substack{J_i \in \{J_i\}_n \\ D_i \in \{D_i\}_n}} \sum_{\substack{k_i=1 \\ k_i \in \{k_i\}_n}}^{2} \delta_{\overrightarrow{\alpha_n}, \sum_{m=1}^{n} \overrightarrow{e_{D_m}}} \delta_{\overrightarrow{\beta^*_n}, \sum_{P \in \mathcal{I}} \sum_{m=1}^{n} d_{P,J_m,k_m} \overrightarrow{e^*_P}} \\
&\quad \times \left(\prod_{S \in \mathcal{I}} \prod_{r=1}^{n} \theta \left(\beta^n_S - \sum_{m=r}^{n} d_{S,J_m,k_m} \right) \right) \left(\prod_{l=1}^{n} \frac{g_{\overline{k_l j'_l}, D_l}}{\tau_l} \right) \\
&\quad \times \left\{ \prod_{l=1}^{n} \frac{\tau_l \delta_{j_l k_l} + \beta^n_{j_l \cancel{j_l'}} + 1 - \Lambda_{l, j_l \cancel{j_l'}, \{J_i\}_n, \{k_i\}_n}}{l(\tau_l + 1) + 2 \left(\beta^n_I + \beta^n_{\cancel{i} i'} - \Delta_{I, \cancel{i} i', l, \{J_i\}_n} \right) \left(\beta^n_{\cancel{I}} + \beta^n_{i \cancel{i'}} - \Delta_{\cancel{I}, i \cancel{i'}, l, \{J_i\}_n} \right)} \right\}.
\end{aligned}
\tag{5}
$$

Note that for any $\overrightarrow{m} = m_I \overrightarrow{e_I} + m_{\not{I}} \overrightarrow{e_{\not{I}}} + m_{i\not{i}'} \overrightarrow{e_{i\not{i}'}} + m_{\not{i}i'} \overrightarrow{e_{\not{i}i'}}$, $a_J|\overrightarrow{m}\rangle$ can also be expressed as

$$a_J|\overrightarrow{m}\rangle = \left(\prod_{L\in\mathcal{I}} \theta(m_L - \delta_{LJ})\right)\sqrt{m_J}|\overrightarrow{m} - \overrightarrow{e_J}\rangle \ (J \in \mathcal{I}),$$

where $\theta : \mathbb{R} \to \{0, 1\}$ is the step function defined by

$$\theta(x) := \begin{cases} 1 & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

Star product with separation of variables on $G_{2,4}(\mathbb{C})$

Hence, we eventually obtained the explicit star product $f * g$ on $G_{2,4}(\mathbb{C})$.

Theorem 7 (O.-Sako)

For $f, g \in C^{\infty}(G_{2,4}(\mathbb{C}))$, the star product with separation of variables on $G_{2,4}(\mathbb{C})$ is given by

$$
\begin{aligned}
& f * g \\
&= \sum_{n=0}^{\infty} \sum_{\substack{J_i \in \{J_i\}_n \\ D_i \in \{D_i\}_n}} \sum_{\substack{k_i=1 \\ k_i \in \{k_i\}_n}}^{2} \left(\prod_{l=1}^{n} \frac{g_{\overline{k_l j_l'}, D_l} \Upsilon_{l, \{J_i\}_n, \{k_i\}_n}}{\tau_l} \right) \\
& \quad \times \left(\prod_{S \in \mathcal{I}} \prod_{r=1}^{n} \theta \left(\sum_{m=1}^{r-1} d_{S, J_m, k_m} \right) \right) \left(D^{\sum_{m=1}^{n} \overrightarrow{e_{D_m}}} f \right) \left(D^{\sum_{P \in \mathcal{I}} \sum_{m=1}^{n} d_{P, J_m, k_m} \overrightarrow{e_P^*}} g \right),
\end{aligned}
\tag{6}
$$

Here

$$\Upsilon_{l,\{J_i\}_n,\{k_i\}_n} := \frac{\tau_l \delta_{j_l k_l} + 1 + \sum_{m=1}^{l} d_{m, j_l \not{j}_l', J_m, k_m}}{l(\tau_l + 1) + 2\left\{\sum_{m=1}^{l}\left(\delta_{I J_m} + \delta_{\not{i} i', J_m}\right)\right\}\left\{\sum_{m=1}^{l}\left(\delta_{\not{I} J_m} + \delta_{i \not{i}', J_m}\right)\right\}}$$

for $l, r = 1, \cdots, n$, $\{J_i\}_n := \{J_1, \cdots, J_n\}$, $\{D_i\}_n := \{D_1, \cdots, D_n\}$, $\{k_i\}_n := \{k_1, \cdots, k_n\}$ and $\mathcal{I} := \left\{I, \not{I}, i\not{i}', \not{i}i'\right\}$.

Summary

1. We obtained the concrete star product with separation of variables on $G_{2,4}(\mathbb{C})$ by solving the recurrence relations given by Hara-Sako. This means that the noncommutative $G_{2,4}(\mathbb{C})$ as the deformation quantization with separation of variables was constructed.
2. By obtaining the explicit star product with separation of variables on $G_{2,4}(\mathbb{C})$, we can now compute deformations from the commutative product for functions on $G_{2,4}(\mathbb{C})$ with arbitrary precision. For example, when comparing some physical quantity on a commutative $G_{2,4}(\mathbb{C})$ and noncommutative one, it is now possible to compute the difference with arbitrary precision. In this sense, the star product on $G_{2,4}(\mathbb{C})$ is useful.

Outlook of our work

On the other hand, the deformation quantization with separation of variables for $G_{p,p+q}(\mathbb{C})$ in general has not yet been resolved.

Examples of $G_{p,p+q}(\mathbb{C})$ (not yet obtained)

1. $G_{2,2+q}(\mathbb{C}) \quad (p = 2, q > 2)$
   $\longrightarrow$ In this case, we have some prospects. However, several issues remain to be considered.
2. $G_{p,p+q}(\mathbb{C}) \quad (p > 2, q > 2)$
   $\longrightarrow$ In general, the recurrence relations for $G_{p,p+q}(\mathbb{C})$ have been determined. Unfortunately, the concrete star product with separation of variables on $G_{p,p+q}(\mathbb{C})$ has not been determined at the moment.

Thank you for your kind attention!

Appendix (1)

The recurrence relations which give the star product with separation of variables are given as follows:

$$
\begin{aligned}
&\hbar \sum_{d=1}^{N} g_{\bar{i}d} T^{n-1}_{\overrightarrow{\alpha_n}-\overrightarrow{e_d},\,\overrightarrow{\beta^*_n}-\overrightarrow{e^*_i}} \\
&= \beta^n_i T^n_{\overrightarrow{\alpha_n},\,\overrightarrow{\beta^*_n}} \\
&\quad + \hbar \sum_{k=1}^{N} \sum_{\rho=1}^{N} \binom{\beta^n_k - \delta_{k\rho} - \delta_{ik} + 2}{2} R_{\bar{\rho}}{}^{\bar{k}\bar{k}}{}_{\bar{i}} T^n_{\overrightarrow{\alpha_n},\,\overrightarrow{\beta^*_n}-\overrightarrow{e^*_\rho}+2\overrightarrow{e^*_k}-\overrightarrow{e^*_i}} \\
&\quad + \hbar \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \sum_{\rho=1}^{N} \left(\beta^n_k - \delta_{k\rho} - \delta_{ik} + 1\right) \left(\beta^n_{k+l} - \delta_{k+l,\rho} - \delta_{i,k+l} + 1\right) R_{\bar{\rho}}{}^{\overline{k+l}\,\bar{k}}{}_{\bar{i}} \\
&\quad \times T^n_{\overrightarrow{\alpha_n},\,\overrightarrow{\beta^*_n}-\overrightarrow{e^*_\rho}+\overrightarrow{e^*_k}+\overrightarrow{e^*_{k+l}}-\overrightarrow{e^*_i}},
\end{aligned}
$$

where $\overrightarrow{e_k} = (\delta_{1k}, \cdots, \delta_{kk}, \cdots, \delta_{Nk})$, and $R_{\bar{i}}{}^{\bar{j}\,\bar{k}}{}_{\bar{l}} = g^{\bar{j}m} g^{\bar{k}s} R_{\bar{i}ms\bar{l}}$.

Appendix (2)

For $n = 0, 1$, the coefficients are concretely given by

$$T^{0}_{\vec{0}, \vec{0^*}} = 1, \qquad T^{1}_{\vec{e_i}, \vec{e_j^*}} = \hbar g_{i\bar{j}}.$$

In other words, these coefficients are the initial conditions of the recurrence relations.

Appendix (3)

We denote the recurrence relations, equivalent to (1), as follows:

$$\beta^n_I T^n_{\overrightarrow{\alpha_n},\overrightarrow{\beta^*_n}} = \frac{\sum_{D\in\mathcal{I}}\sum_{k=1}^{2}\left(\tau_n\delta_{ik}+\beta^n_{i\not{i}'}+1\right) g_{\overline{ki'},D}\, T^{n-1}_{\overrightarrow{\alpha_n}-\overrightarrow{e_D},\overrightarrow{\beta^*_n}-\overrightarrow{e^*_I}-\delta_{\not{i}k}\left(\overrightarrow{e^*_{\not{I}}}-\overrightarrow{e^*_{i\not{i}'}}\right)}}{\tau_n\left(\tau_n+\beta^n_{\not{I}}+\beta^n_{i\not{i}'}+1\right)}, \quad (7)$$

where $I = ii^{'}$ is fixed. From (7), we can obtain Proposition 2.1.

Appendix (4)

The detailed notations in Theorem 5 are given as follows:

$$\mathcal{A}_{J_l,k_l} := a_{J_l}\frac{1}{\sqrt{N_{J_l}}}\left(a_{\not{J}_l}\frac{1}{\sqrt{N_{\not{J}_l}}}a^{\dagger}_{j_l\not{j}_l}\frac{1}{\sqrt{N_{j_l\not{j}_l}+1}}\right)^{\delta_{\not{j}_l k_l}},$$

$$\mathcal{C}_{l,\{J_i\}_n,\{k_i\}_n} := \left(\tau_l\delta_{j_l k_l} + N_{j_l\not{j}_l} - \sum_{m=1}^{n} d_{j_l\not{j}_l,J_m,k_m} + \sum_{m=1}^{l} d_{j_l\not{j}_l,J_m,k_m} + 1\right),$$

$$\mathcal{F}_{l,\{J_i\}_n,\{k_i\}_n}$$
$$:= \left\{l\left(\tau_l+1\right) + 2\left(N_I + N_{\not{i}i'} - \Delta_{I,\not{i}i',l,\{J_i\}_n}\right)\left(N_{\not{I}} + N_{i\not{i}'} - \Delta_{\not{I},i\not{i}',l,\{J_i\}_n}\right)\right\}^{-1},$$

$$d_{j_l\not{j}_l,J_m,k_m} := \delta_{j_l\not{j}_l,J_m} + \delta_{\not{j}_m k_m}\left(\delta_{j_l\not{j}_l,\not{J}_m} - \delta_{j_l\not{j}_l,j_m\not{j}_m}\right),$$

$$\Delta_{I,\not{i}i',l,\{J_i\}_n} := \sum_{m=1}^{n}\left(\delta_{IJ_m} + \delta_{\not{i}i',J_m}\right) - \sum_{m=1}^{l}\left(\delta_{IJ_m} + \delta_{\not{i}i',J_m}\right),$$

$$\Delta_{\not{I},i\not{i}',l,\{J_i\}_n} := \sum_{m=1}^{n}\left(\delta_{\not{I}J_m} + \delta_{i\not{i}',J_m}\right) - \sum_{m=1}^{l}\left(\delta_{\not{I}J_m} + \delta_{i\not{i}',J_m}\right).$$

Appendix (5)

The detailed notations in Theorem 6 and Theorem 7 are given by

$$\Upsilon_{l,\{J_i\}_n,\{k_i\}_n} := \frac{\tau_l \delta_{j_l k_l} + 1 + \sum_{m=1}^{l} d_{m,j_l \cancel{j_l'},J_m,k_m}}{l\left(\tau_l + 1\right) + 2\left\{\sum_{m=1}^{l}\left(\delta_{IJ_m} + \delta_{\cancel{i}i',J_m}\right)\right\}\left\{\sum_{m=1}^{l}\left(\delta_{\cancel{I}J_m} + \delta_{i\cancel{i'},J_m}\right)\right\}},$$

$$\Lambda_{r,S,\{J_i\}_n,\{k_i\}_n} := \sum_{m=1}^{n} d_{S,J_m,k_m} - \sum_{m=1}^{r} d_{S,J_m,k_m},$$

$$d_{S,J_m,k_m} := \delta_{S,J_m} + \delta_{\cancel{j_m}k_m}\left(\delta_{S,\cancel{J_m}} - \delta_{S,j_m\cancel{j_m'}}\right)$$

for $l, r = 1, \cdots, n$, $\{J_i\}_n := \{J_1, \cdots, J_n\}$, $\{D_i\}_n := \{D_1, \cdots, D_n\}$, $\{k_i\}_n := \{k_1, \cdots, k_n\}$.