Lagrangian multiforms, the Darboux-KP system and Chern-Simons theory in infinite-dimensional space

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The talk is outlined as follows:

- A brief introduction to *Lagrangian multiform theory* - the basic idea, and an example (of a Lagrangian 2-form structure) in 1+1 dimension;
- Present the Darboux- Kadomtsev-Petviashvili (KP) system as a generating system for the entire KP hierarchy, and its Lagrangian 3-form structure;
- Interpretation as a Chern-Simons theory in infinite-dimensional space;
- Comparison with 1+1-dimensional field theories (e.g. 4D Chern-Simons theory).

I will not go into the discrete theory, even though it has been a motivator, and the discrete systems are present in the background\(^1\)

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Why Lagrangian multiform theory?

**Key question:** How to capture the property of multidimensional consistency (MDC) within a Lagrange formalism?

**Multidimensional consistency:** We know that many "integrable" equations, discrete and continuous possess the property of multidimensional consistency.

- **continuous:** commuting flows, higher symmetries & master symmetries, hierarchies;
- **discrete:** consistency-around-the-cube, Bäcklund transforms, higher continuous symmetries, commuting discrete flows

In all these cases we can think of the *dependent* variable $a$ (possibly vector-valued) function of many (discrete and continuous) variables

$$u = u(n_1, n_2, \ldots; x, t_1, t_2, \ldots)$$

on which we can impose many equations simultaneously, and it is the *compatibility* of those equations that makes the integrability manifest.

**Key problem:** In a variational approach, the Euler-Lagrange (EL) equations, only produce one equation per component of the dependent variables; not an entire system of compatible equations on one and the same dependent variable!

**Answer:** Lagrangians of an MDC integrable theory must be differential- or difference forms in space of multi-variables!

Thus, a new variational approach to integrability was initiated by the paper:

An example of a Lagrangian 2-form

Let us denote $u = u(x_1, x_2, x_3)$, $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, etc.

Consider the Lagrangians (Suris, 2012):

$$
\mathcal{L}_{12} = \frac{1}{2} u_1 u_2 - \cos u ,
$$
$$
\mathcal{L}_{13} = \frac{1}{2} u_1 u_3 + \gamma \left( \frac{1}{2} u_{11}^2 - \frac{1}{8} u_1^4 \right) ,
$$
$$
\mathcal{L}_{23} = -\frac{1}{2} u_2 u_3 + \gamma \left( \frac{1}{2} u_1^2 \cos u + u_{11} (u_{12} - \sin u) \right) .
$$

Then the usual Euler-Lagrange (EL) equations for these three Lagrangians yield

respectively:

$$
\frac{\delta \mathcal{L}_{12}}{\delta u} = -u_{12} + \sin u \quad \Rightarrow \quad u_{12} = \sin u \quad \text{sine-Gordon eq.}
$$
$$
\frac{\delta \mathcal{L}_{13}}{\delta u} = \partial_1 \left( -u_3 + \frac{1}{2} \gamma u_1^3 + \gamma u_{111} \right) \quad \Rightarrow \quad v_3 = \gamma (v_{111} + \frac{3}{2} v^2 v_1) \quad \text{mKdV eq}
$$
$$
\frac{\delta \mathcal{L}_{23}}{\delta u} = u_{23} - \gamma \left( \frac{1}{2} u_1^2 \sin u + u_{11} \cos u \right) \quad \text{consistency relation} .
$$

Together with variations w.r.t. 'alien derivatives':

$$
\frac{\delta \mathcal{L}_{23}}{\delta u_1} = \gamma (u_1 \cos u - u_{112}) = 0 , \quad \frac{\delta \mathcal{L}_{23}}{\delta u_{11}} = \gamma (u_{12} - \sin u) = 0 .
$$

In fact,

$$
\partial_2 \text{ (mKdV eq) } \Leftrightarrow \partial_1 \text{ (consist rel) } \Leftrightarrow \partial_3 \text{ (sG eq) } .
$$

This suggest that the Lagrangians are components of a Lagrangian 2-form

$$
\mathcal{L} = \mathcal{L}_{12} \, dx_1 \wedge dx_2 + \mathcal{L}_{23} \, dx_2 \wedge dx_3 + \mathcal{L}_{31} \, dx_3 \wedge dx_1 ,
$$

where we set $\mathcal{L}_{ji} = -\mathcal{L}_{ij}$.
Closure property and generalised EL eqs

The Lagrangian 2-form $L$ has the following remarkable property:

$$dL = (\partial_1 L_{23} + \partial_2 L_{31} + \partial_3 L_{12}) \, dx_1 \wedge dx_2 \wedge dx_3$$

$$= (\sin u - u_{12}) \left( u_3 - \gamma u_{111} - \frac{1}{2} \gamma u_1^3 \right) \, dx_1 \wedge dx_2 \wedge dx_3$$

which has a 'double zero' when $u$ satisfies the EL equations! Thus we have the closure property:

$$dL|_{EL} = 0$$

Action functional:

$$S[u(x); \sigma] = \int_\sigma L = \int_\sigma L_{12} \, dx_1 \wedge dx_2 + L_{23} \, dx_2 \wedge dx_3 + L_{31} \, dx_3 \wedge dx_1$$

is a functional of both the field variables $u(x)$ as well as of the surface in the space of independent variables $\sigma$ over which to integrate.

Multiform principle: The action $S$ is critical w.r.t. variations $u \rightarrow u + \delta u$ of the fields, as well as w.r.t. variations $\sigma \rightarrow \sigma + \delta \sigma$ of the surfaces of integration.

Generalised EL equations: Considering a closed $d$-dim surface $\sigma = \partial B$ for some $d + 1$-dimensional volume $B$, in the language of the differential bi-complex, using Stokes’ theorem:

$$\int_B dL = \int_{\sigma=\partial B} L = S \quad \Rightarrow \quad \delta S = \int_B \delta dL = 0$$

for all volumes $B$. Hence, we have

generalised EL eqs: $\delta S = 0 \iff \delta dL = 0$
Lagrangian multi-form theory (LMFT), provides a variational approach to integrability in the sense of multidimensional consistency (MDC).

LMFT differs from the conventional variational approach in a number of respects:

▶ Lagrangians are differential- (or difference) forms (with co-dimension nonzero) in the space of independent variables;

▶ the action is a functional of the dependent variables (the "fields") as well as of the surfaces in the space of independent variables;

▶ the EL equations form a MDC (i.e. integrable) system of equations;

▶ the critical point of the action, i.e. solutions of a system of generalized EL equations, the action is independent on local variations of the surface in the space of multi-variables;

▶ the Lagrangians are no longer input (from tertiary considerations) but can be viewed as solutions of the system of generalized EL equations.

This new approach was initiated by the paper:


Seminal work at TU Berlin (A. Bobenko & Yu. Suris and collabs.) have contributed to the development of the theory, which was there also called theory of pluri-Lagrangian systems.

First step to a quantum theory of LMFT, in terms of Feynman propagators, was undertaken in:

Lagrangian 3-form and KP type systems

In the case of three-dimensional equations the relevant variational structure is that of Lagrangian 3-forms. This includes the Kadomtsev-Petviashvili (KP) type systems and its generalisations. The generalised Darboux system is in this class as well.

**Generalised Darboux system**

The original Darboux system\(^2\) describes conjugate nets in the theory of orthogonal curvilinear coordinates. The *generalised Darboux system* reads

\[
\begin{align*}
\frac{\partial B_{qr}}{\partial \xi_p} &= B_{qp} B_{pr}, & \frac{\partial B_{rq}}{\partial \xi_p} &= B_{rp} B_{pq}, \\
\frac{\partial B_{pr}}{\partial \xi_q} &= B_{pq} B_{qr}, & \frac{\partial B_{rp}}{\partial \xi_q} &= B_{rq} B_{qp}, \\
\frac{\partial B_{pq}}{\partial \xi_r} &= B_{pr} B_{rq}, & \frac{\partial B_{qp}}{\partial \xi_r} &= B_{qr} B_{rp}.
\end{align*}
\]

where the \(B_{pq}\), etc., are scalar functions (but can be readily generalised to matrices) of the independent variables \(\xi_p, \xi_q\) and \(\xi_r\), which are continuous variables labelled by parameters \(p, q\) and \(r\) respectively (*where* \(p \neq q \neq r \neq p\)).

**Remark:** The integrability aspects of the Darboux system has been investigated by many authors mostly in the late 1980s and 1990s (Zakharov, Manakov, Doliwa, Santini, Konopelchenko and Bogdanov, Martinez-Alonso, etc.)

Multidimensional consistency of the Darboux system

A main feature of the Darboux system is the following.

**Proposition:** The PDE system (7) for the quantities $B_{..}$ is multidimensionally consistent.

The proof is by direct computation, introducing a fourth variable $\xi_s$ and associated lattice direction with parameter $s$, such that the system of independent variables is extended to include $B_{ps}, B_{qs}, B_{rs}$ and $B_{sp}, B_{sq}, B_{sr}$ obeying relations of the form

$$\frac{\partial B_{ps}}{\partial \xi_q} = B_{pq}B_{qs}, \quad \frac{\partial B_{pq}}{\partial \xi_s} = B_{ps}B_{sq},$$

etc. We then establish by direct computation from the extended system of equations comprising (7) and the PDEs w.r.t. $\xi_s$, the relation

$$\frac{\partial}{\partial \xi_s} \left( \frac{\partial}{\partial \xi_p} B_{qr} \right) = \frac{\partial}{\partial \xi_p} \left( \frac{\partial}{\partial \xi_s} B_{qr} \right),$$

by direct computation. Similarly all relations obtained from cross-differentiation hold by the same token.

**Remark:** The MDC property suggests that here is a Lagrangian multiform structure behind the Darboux system.

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Lagrangian 3-form structure for the Darboux system

We now introduce the Lagrangian structure. Let us consider the following Lagrangian components

\[
L_{pqr} = \frac{1}{2} \left( B_{rq} \partial_{ξ_p} B_{qr} - B_{qr} \partial_{ξ_p} B_{rq} \right) + \frac{1}{2} \left( B_{qp} \partial_{ξ_r} B_{pq} - B_{pq} \partial_{ξ_r} B_{qp} \right) \\
+ \frac{1}{2} \left( B_{pr} \partial_{ξ_q} B_{rp} - B_{rp} \partial_{ξ_q} B_{pr} \right) + B_{rp} B_{pq} B_{qr} - B_{rq} B_{qp} B_{pr}.
\] (1.1)

Then we have the following main statement

**Theorem**

*The differential of the Lagrangian 3-form*

\[
L := L_{pqr} \, dξ_p \wedge dξ_q \wedge dξ_r + L_{qrs} \, dξ_q \wedge dξ_r \wedge dξ_s + \\
+ L_{rsp} \, dξ_r \wedge dξ_s \wedge dξ_p + L_{spq} \, dξ_s \wedge dξ_p \wedge dξ_q,
\] (1.2)

has a “double zero” on the solutions of the set of generalised Darboux equations (7), i.e. \(dL\) can be written as

\[
dL = A_{pqr} \, dξ_p \wedge dξ_q \wedge dξ_r \wedge dξ_s
\] (1.3)

with the coefficient \(A_{pqr}\) being a sum of products of factors which vanish on solutions of the EL equations.
Proof.
Computing the components of the differential $dL$ we obtain

$$\partial_{s}L_{pqr} - \partial_{p}L_{sqr} + \partial_{q}L_{rsp} - \partial_{r}L_{spq} =$$

$$\Gamma_{s;r q p;q r} - \Gamma_{p;r q s;q r} + \Gamma_{s;q p r;q s} - \Gamma_{r;q p s;q r}$$

$$+ \Gamma_{s;p r q;r p} - \Gamma_{q;p r s;r p} + \Gamma_{q;s r p;s r} - \Gamma_{r;s q p;s q}$$

$$+ \Gamma_{p;q s r;r s q} - \Gamma_{r;q s p;r s p} + \Gamma_{q;p s r;r s p} - \Gamma_{r;p s q;r s q} ,$$

where

$$\Gamma_{p;q s} = \partial_{p}B_{q s} - B_{q p}B_{p s} ,$$

and similarly for the other indices. The set of generalised EL equations in this case are obtained from $\delta \mathcal{A}_{p q r s} = 0$, repeating the general arguments\(^4\), for deriving the EL equations from the differential of the Lagrangian multiform. Thus, since all the variations $\delta B_{p q}$ etc. and their first derivatives, are independent, the coefficients are precisely all the combinations $\Gamma_{r;p q}$, etc. which will have to vanish at the critical point for the action

$$S[B(\xi);\mathcal{V}] = \int_{\mathcal{V}} L = \int_{\mathcal{V}} dL ,$$

integrated over any arbitrary 3-dimensional closed hypersurfaces $\mathcal{V}$ in the multivariable space of all the $\xi_{p}$’s, such that $\mathcal{V} = \partial \mathcal{V}$. 

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Lax multiplet

The generalised Darboux system of B-equations possesses a Lax multiplet\(^5\).

**Proposition:** The system of B-equations arises as the compatibility conditions for the linear overdetermined system of the form

\[
\frac{\partial \Phi_q}{\partial \xi_p} = B_{qp} \Phi_p, \quad p \neq q, \quad \text{or} \quad \frac{\partial \Psi_r}{\partial \xi_p} = \Psi_p B_{pr} \quad \forall p \neq r,
\]

**Remark:** The Lax multiplets can be obtained from the Darboux system itself, relying on MDC, by identifying the Lax wave functions \(\Phi = B_{pk}\) and \(\Psi = B_{lp}\) fixing two directions in the space of independent variables, \(\xi_k\) and \(\xi_l\), say (where \(k\) and \(l\) play the role of spectral parameters).

Furthermore, \(\Phi\) and \(\Psi\) obey a linear homogeneous set of equations of the form

\[
\partial_p \partial_q \Phi_r = (\partial_p \ln \Phi_q) \partial_q \Phi_r + (\partial_q \ln \Phi_p) \partial_p \Phi_r,
\]

\[
\partial_p \partial_q \Psi_r = (\partial_p \ln \Psi_q) \partial_q \Psi_r + (\partial_q \ln \Psi_p) \partial_p \Psi_r.
\]

A corollary to the multiform structure is a Lagrangian description of the Lax pair:

**Corollary:** The Lagrangian components

\[
\mathcal{L}_{pq(k)} = \frac{1}{2} \left( \Psi_q \partial_{\xi_p} \Phi_q - (\partial_{\xi_p} \Psi_q) \Phi_q \right) - \frac{1}{2} \left( \Psi_p \partial_{\xi_q} \Phi_p - (\partial_{\xi_q} \Psi_p) \Phi_p \right)
\]

\[
+ \frac{1}{2} \left( B_{qp} \partial_{\xi_k} B_{pq} - B_{pq} \partial_{\xi_k} B_{qp} \right) + \Psi_p B_{pq} \Phi_q - \Psi_q B_{qp} \Phi_p,
\]

and the corresponding Lagrangian 3-form, fixing the direction given by \(x_k\), reduces to a 2-form:

\[
\mathcal{L}_{(k)} := \mathcal{L}_{pq(k)} \, d\xi_p \wedge d\xi_q + \mathcal{L}_{qr(k)} \, d\xi_q \wedge d\xi_r + \mathcal{L}_{rp(k)} \, d\xi_r \wedge d\xi_p.
\]

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Discrete Darboux system

A discrete analogue of the Darboux system of orthogonal coordinate systems, foes back to Bogdanov & Konopelchenko, and Doliwa & Santini. The corresponding discrete analogue of the generalised Darboux system (7) reads

\[ \Delta_p B_{qr} = B_{qp} T_p B_{pr} , \quad \Delta_p B_{rq} = B_{rp} T_p B_{pq} , \]
\[ \Delta_q B_{rp} = B_{rq} T_q B_{qp} , \quad \Delta_q B_{pr} = B_{pq} T_q B_{qr} , \]
\[ \Delta_r B_{pq} = B_{pr} T_r B_{rq} , \quad \Delta_r B_{qp} = B_{qr} T_r B_{rp} , \]

where the difference operator \( \Delta_p = T_p - \text{id} \). This system is related to other multidimensional lattice systems of matrix KP type (FWN, 1985). Now:

- The above system of difference equations is multidimensionally consistent, and furthermore, it is consistent with the differential Darboux-KP system.

This can be checked by direct computation.

Similarly to the continuous case we have a Lax system, and its adjoint, given by

\[ \Delta_p \Phi_q = B_{qp} T_p \Phi_p , \quad \Delta_p \Psi_q = \Psi_p T_p B_{pq} , \]

and the homogeneous linear difference system for an eigenfunctions \( \Phi_r, \Psi_r \) respectively,

\[ \Delta_p \Delta_q \Phi_r = \frac{\Delta_p (T_q \Phi_q)}{T_q \Phi_q} \Delta_q \Phi_r + \frac{\Delta_q (T_p \Phi_p)}{T_p \Phi_p} \Delta_p \Phi_r , \]
\[ \Delta_p \Delta_q \Psi_r = \frac{\Delta_p \Psi_q}{T_p \Psi_q} \Delta_q (T_p \Psi_r) + \frac{\Delta_q \Psi_p}{T_q \Psi_p} \Delta_p (T_q \Psi_r) . \]

This is the discrete analogue to the Lamé system of equations arising in the theory of conjugate nets of curvilinear coordinates.\(^6\)

Connection with KP system

To understand the structure of the KP system it is necessary to consider the lattice KP system\(^7\) and the KP hierarchy\(^8\) as part of one and the same system.

In the direct linearisation (DL) approach to KP, the dynamics is governed by plane-wave factors which take the form

\[
\rho_k = \left[ \prod_{\nu} (p_{\nu} - k)^{n_{\nu}} \right] \exp \left\{ k\xi - \sum_{\nu} \frac{\xi_{p_{\nu}}}{p_{\nu} - k} \right\},
\]

\[
\sigma_{k'} = \left[ \prod_{\nu} (p_{\nu} - k')^{-n_{\nu}} \right] \exp \left\{ -k'\xi + \sum_{\nu} \frac{\xi_{p_{\nu}}}{p_{\nu} - k'} \right\}.
\]

in the construction of the \(\tau\)-function which obeys the Hirota equation:

\[
(p - q)(T_p T_q \tau) T_r \tau + (q - r)(T_q T_r \tau) T_p \tau + (r - p)(T_r T_p \tau) T_q \tau = 0,
\]

Here \(T_{p_{\nu}}\) \((p, q, r\) being any three of the \(p_{\nu}\)) denotes the elementary shift in the variable \(n_{\nu}\) associated with \(p_{\nu}\) (which in this context) has the interpretation of a lattice parameter measuring the grid width in the discrete direction labelled by \(n_{\nu}\).

The interplay between discrete and continuous variables turns out to be an essential feature of the structure:

\[
\frac{\partial \tau}{\partial \xi_p} = - \left( T_{p_{\nu}}^{-1} \frac{d}{dp} T_p \right) \tau := \lim_{\varepsilon \to 0} \frac{T_{p_{\nu}}^{-1} T_{p_{\nu} - \varepsilon} \tau - \tau}{\varepsilon \tau},
\]

for any of the parameters \(p_{\nu} = p\).


\(^\text{M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, RIMS Kōkyūroku **439** (1981) 30–46.\)
τ-function relations

Using the identification between lattice shifts and derivatives, we can perform a limit $r \to p$ on Hirota’s equation and thus obtain the following differential-difference equation for $\tau$

$$(p - q) \left(\tau T_q \frac{\partial \tau}{\partial \xi_p} - (T_q \tau) \frac{\partial \tau}{\partial \xi_p}\right) = \tau T_q \tau - (T_p \tau) T_q T_p^{-1} \tau.$$

Furthermore, the $\tau$-function also obeys the differential-difference equation

$$1 + (p - q)^2 \frac{\partial^2 \ln \tau}{\partial \xi_p \partial \xi_q} = \frac{(T_p T_q^{-1} \tau) T_q T_p^{-1} \tau}{\tau^2},$$

which is the bilinear form of the 2D Toda equation (with the discrete variable along the skew-diagonal lattice direction in the lattice generated by the $T_p$ and $T_q$ shifts).

Miwa variables: The KP hierarchy can be obtained by the expansions$^9$

$$t_j = \delta_{j,1} \xi + \sum_{\nu} \left(\frac{\xi_{p_{\nu}}}{p_{\nu}^{j+1}} + \frac{1}{j} \frac{n_{\nu}}{p_{\nu}^j}\right)$$

$$\Rightarrow T_{p_{\nu}} \tau = \tau \left\{t_j + \frac{1}{j p_{\nu}^j}\right\} \quad \text{and} \quad \frac{\partial \tau}{\partial \xi_{p_{\nu}}} = \sum_{j=1}^{\infty} \frac{1}{p_{\nu}^{j+1}} \frac{\partial \tau}{\partial t_j},$$

where the $t_j$ are the usual independent time-variables in the hierarchy.

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Identification between KP and Darboux

Consider the quantities

\[ S_{a,b} = \frac{T_a^{-1} T_b}{\tau} \quad \Rightarrow \quad B_{pq} = \frac{\sigma_p \rho_q S_{p,q}}{q-p} = \frac{\sigma_p \rho_q T_p^{-1} T_q}{(q-p)\tau}, \quad (q \neq p). \]

The quantities \( S \), as a consequence of the Hirota and the differential-difference equation, obey the relations

\[
(p - b) T_p S_{a,b} - (p - a) S_{a,b} = (a - b) S_{a,p} T_p S_{p,b},

(p - a)(p - b) \frac{\partial S_{a,b}}{\partial \xi_p} = (a - b) \left( S_{a,p} S_{p,b} - S_{a,b} \right).
\]

These relations are compatible for all parameters \( p \) and corresponding shifts and derivatives w.r.t. the corresponding Miwa variables \( \xi_p \). Furthermore, the quantity \( S = S_{a,b} \) obeys the following 3-dimensional partial difference equation

\[
\frac{[(p - b) T_p T_q S - (p - a) T_q S]}{[(p - b) T_p T_r S - (p - a) T_r S]} \frac{[(q - b) T_q T_r S - (q - a) T_r S]}{[(q - b) T_p T_q S - (q - a) T_p S]}
\times \frac{[(r - b) T_p T_r S - (r - a) T_p S]}{[(r - b) T_q T_r S - (r - a) T_q S]} = 1
\]

which is essentially the lattice Schwarzian KP equation.

\[ ^{10} \text{We can also identify } B_{pp} = \mathcal{C} \partial_{\xi_p} (\ln \tau), \text{ where } \mathcal{C} \text{ is some constant normalisation factor.} \]

\[ ^{11} \text{Similar relations also appeared in:} \]


A matrix generalisation of the Darboux system is given by
\[ \partial_i G_{jk} = G_{ik} J_i G_{ji}, \quad i \neq j \neq k \neq i. \]

Here the \( G_{ij} \) are \( N \times N \) matrix functions of dynamical variables
\[ x_i = \xi_{i_l}^j, \quad x_j = \xi_{j_l}^j, \quad x_k = \xi_{l_k}^j, \ldots, \] which are labelled by a continuous parameter \( l \) and also by constant matrices \( J_i, J_j, J_k \) which commute among themselves. \(^{13}\)

In fact, i.e., \([J_i, J_j] = [J_j, J_k] = [J_k, J_i] = 0\), and we have denoted \( \partial / \partial \xi_{lj} =: \partial_j \), etc. for the sake of brevity.

The matrices \( J \) 'tune' a hierarchy of associated PDEs.

A Lagrangian for the matrix Darboux system reads \(^{14}\)
\[ \mathcal{L}_{ijk} = \frac{1}{2} \text{tr} \left\{ G_{ij} J_i (\partial_k G_{ji}) J_j - (\partial_k G_{ij}) J_i G_{ji} J_j + \text{cycl.} \ (ijk) \right\} - \text{tr} \left\{ G_{ij} J_i G_{ki} J_k G_{jk} J_j - G_{ji} J_j G_{kj} J_k G_{ik} J_i \right\}, \]

which is a matrix generalisation of the Darboux Lagrangian.

\(^{13}\) In fact, one can also consider the non-commutative case \([J_i, J_j] = \Gamma_{ij}^{k} J_k\), in which case we get non-commuting flows on a loop group, for which a Lagrangian description was proposed, for \((1+1)\)-dimensional hierarchies, recently in: V. Caudrelier, F.W. Nijhoff, D. Sleigh and M. Vermeeren, Lagrangian multiforms on Lie groups and noncommuting flows, ArXiv:2204.09663.

Matrix 3-form structure

The Lagrangian $L_{ijk}$ can be viewed as a components of a Lagrangian 3-form:

$$L = \sum_{i < j < k} L_{ijk} \, dx_i \wedge dx_j \wedge dx_k ,$$

which obeys the following property: • The Lagrangian 3-form $L$ has a double zero on solutions of the set of matrix Darboux equations.

The proof is computational, and in essence similar to the scalar case, (differing only in the matrix ordering within the trace). Computing the differential of $L$ we get:

$$dL = \sum_{i,j,k,l} A_{ijkl} \, dx_i \wedge dx_j \wedge dx_k \wedge dx_l ,$$

with

$$A_{ijkl} = \frac{1}{2} \text{tr} \{ \Gamma_{l;i,j} \Gamma_{k;i,j} - \Gamma_{k;i,j} \Gamma_{l;i,j} + \Gamma_{l;k,i} \Gamma_{j;i,k} - \Gamma_{j;i,k} \Gamma_{l;k,i} - \Gamma_{j;i,k} \Gamma_{l;k,i} \pm \text{cycl} \, (ijkl) \} ,$$

where the quantities $\Gamma$ are given by

$$\Gamma_{i;j,k} = \partial_i G_{jk} - G_{ik} J_i G_{ji} .$$

The double zero expansion implies that the generalised EL equations arising from $\delta dL = 0$(for all $G_{ij}$ varied independently, for different indices) gives rise to the entire system of matrix Darboux equations. They yield the critical point of the action

$$S[G_{.,.}(x); \mathcal{V}] = \int_{\mathcal{V}} L ,$$

as a functional of all the matrix fields $G_{.,.}$ for all hypersurfaces $\mathcal{V}$ in the space of independent variables.
Higher-dimensional Chern-Simons actions

The conventional Chern-Simons theory over a Lie algebra $\mathfrak{g}$, with associated gauge group $G$, involves a $\mathfrak{g}$-valued gauge connection 1-form $A$, and the associated curvature 2-form,

$$F = dA + A \wedge A .$$

Here we consider matrix-valued gauge fields only, where the gauge groups of interest are the general linear groups, $GL(n, \mathbb{R})$, endowed with the matrix trace $\text{Tr}$, and where the wedge product $A \wedge A$ is evaluated via the matrix product, and not via the Lie bracket.

The standard CS Lagrangians in dimensions 3 and 5 read

$$CS_3 = \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) ,$$

$$CS_5 = \text{Tr} \left( A \wedge dA \wedge dA + \frac{3}{2} A \wedge A \wedge A \wedge dA + \frac{3}{5} A \wedge A \wedge A \wedge A \wedge A \right) ,$$

and they are defined through the property that

$$dCS_3 = \text{Tr} (F \wedge F) ,$$

$$dCS_5 = \text{Tr} (F \wedge F \wedge F) .$$

Remark: Coming from the perspective of the multiform variational principle, we recognise in the latter relations a similarity between Chern classes and the double, respectively triple zero conditions for the Euler-Lagrange equations $F = 0$. However, the relation $F = 0$ is too strong for integrability!
General form of (conventional) CS actions

The general higher form of the CS Lagrangians are given by the formula

\[ CS_{2n+1} = (n + 1) \int_0^1 d\lambda \text{Tr} \left( A \wedge F_{\lambda}^{\wedge n} \right) , \]

where \( F_{\lambda} := \lambda dA + \lambda^2 A \wedge A \),

They obey\(^{15}\)

\[ dCS_{2n+1} = \text{Tr} \left( F^{\wedge(n+1)} \right) , \]

where the latter expressions are \(2n + 2\)-forms, whose variational derivative (using the multiform EL equations in the language of the variational bi-complex), are given by

\[ \delta dCS_{2n+1} = (n + 1) \text{Tr} \left( F^{\wedge(n)} \wedge \delta F \right) , \]

which vanishes whenever \( F = 0 \). The latter would correspond to the usual variational equations in conventional CS theory, but in our setting \( F = 0 \) is too stringent a condition. In fact, we will work later with a restricted set of fields, and the corresponding equations of motion are slightly weaker than the standard zero-curvature condition.

In all these conventional CS theories, we fix the dimensionality of the \( (2n+1)\)-dimensional manifold \( \mathcal{M}_{2n+1} \) over which the Lagrangians are integrated through

\[ \mathcal{A}_{2n+1} = \int_{\mathcal{M}_{2n+1}} CS_{2n+1} . \]

\(^{15}\)Note that in dimension 1, we have \( CS_1 = \text{Tr}(A) \), with \( dCS_1 = \text{Tr}(F) \).
Higher Lagrangian Multiforms from CS Lagrangians

In order to make a connection between the CS theory and Lagrangian multiforms we need to specify the gauge field $A$ as:

$$B = \sum_{k,l \in \mathbb{Z}} B_{kl} \, d\xi_k \, E_{k,l} .$$

Here for simplicity $B_{kl} := B_{p_k p_l} \, (\xi_i)_{i \in \mathbb{Z}}$. The $E_{kl}$ are generators of $GL(\infty)$ obeying:\n
$$E_{k,l} \, E_{m,n} = \delta_{l,m} \, E_{k,n} \quad \text{and} \quad \text{Tr}(E_{k,l}) = \delta_{k,l}.$$\n
matrices $(M_{ab})_{a,b \in \mathbb{Z}}$, indexed by integers, with matrices $(E_{kl})_{ab} = \delta_{k,a} \, \delta_{l,b}$. The sum in the definition of $B$ being infinite, can be understood in the 'completed graded sense', for the $d\xi_k \, E_{k,l}$ are linearly independent, and hence we never get infinite sums of real numbers.

$$F_B = \sum_{j,k,l \in \mathbb{Z}} \left( \partial_{\xi_j} B_{kl} - B_{kj} B_{jl} \right) \, d\xi_j \wedge d\xi_k \, E_{k,l} .$$ (1.7)

Note that the coefficients $\partial_{\xi_j} B_{kl} - B_{kj} B_{jl}$ are exactly of the Darboux form.

Main statement: Computing the corresponding CS action, we get exactly the Lagrangian 3-form of the Darboux-KP system:

$$L^{(3)} = CS_3(B) = \sum_{i,j,k \in \mathbb{Z}} L^{(3)}_{ijk} \, d\xi_i \wedge d\xi_j \wedge d\xi_k ,$$

in which the coefficients $L^{(3)}_{ijk} = \frac{2}{3!} L_{p_i p_j p_k}$ of (1.1), including a prefactor for convenience.
Calculating with the special gauge field $B$ the differential of the Lagrangian 3-form, we find

$$dL^{(3)} = \text{Tr}(F_B \wedge F_B) = \sum_{j,k,l,m \in \mathbb{Z}, \text{ all indices different}} \left( \partial_{\xi_j} B_{kl} - B_{kj} B_{jl} \right) \left( \partial_{\xi_m} B_{lk} - B_{lm} B_{mk} \right) d\xi_j \wedge d\xi_k \wedge d\xi_m \wedge d\xi_l.$$ 

In particular, this implies that $\text{Tr}(F_B \wedge F_B)$, has a double zero on the solutions of the generalised Darboux system in (7), which implies that the latter arises as the EL equations of the multiformal action.

**Remark:** Note that while $\text{Tr}(F_B \wedge F_B)$ indeed has a double zero on the solutions of (7), the form $F_B \wedge F_B$ does not necessarily have such a double zero when (7) holds, as this would require that the Darboux system also extends to the case that all three labelled variables are no longer distinct.
Higher CS multiform actions

Following the connection between the conventional higher CS actions and Chern classes, we can now also postulate higher multiform actions for the Darboux-KP system. Thus, using the same gauge field $B$ in the higher CS action we obtain the Lagrangian 5-form

\[
L^{(5)} = \text{Tr} \left( B \wedge dB \wedge dB + \frac{3}{2} B \wedge B \wedge dB + \frac{3}{5} B \wedge B \wedge B \wedge B \wedge B \right),
\]

\[
= \sum_{j,k,l,m,n \in \mathbb{Z}} \mathcal{L}_{jklmn} d\xi_j \wedge d\xi_k \wedge d\xi_l \wedge d\xi_m \wedge d\xi_n,
\]

with

\[
\mathcal{L}_{jklmn}^{(5)} = \frac{1}{5!} \sum_{j',k',l',m',n' \in \{j,k,l,m,n\}} \varepsilon_{j'k'l'm'n'}
\]

\[
\left[ B_{p_{j'},p_{k'}} (\partial_{\xi_{p_{k'}},p_{n'}} B_{p_{l'},p_{n'}}) (\partial_{\xi_{p_{m'}},p_{n'}} B_{p_{n'},p_{j'}}) \right. \\
\left. + \frac{3}{2} B_{p_{j'},p_{k'}} B_{p_{k'},p_{l'}} B_{p_{l'},p_{n'}} (\partial_{\xi_{p_{m'}},p_{n'}} B_{p_{n'},p_{j'}}) \right. \\
\left. + \frac{3}{5} B_{p_{j'},p_{k'}} B_{p_{k'},p_{l'}} B_{p_{l'},p_{m'}} B_{p_{m'},p_{n'}} B_{p_{n'},p_{j'}} \right],
\]

where $\varepsilon_{jklmn}$ is the 5-dimensional Levi-Civita symbol.

As a consequence of the construction, the Lagrangian 5-form has the property that

\[
dL^{(5)} = \text{Tr} \left( F_B \wedge F_B \wedge F_B \right).
\]

This again leads to the fact that $dL^{(5)}$ has a triple zero on the solutions still of the same Darboux-KP system, as it has the MDC property! (Again $F_B \wedge F_B \wedge F_B$ does not necessarily have a triple zero on the solutions of the Darboux-KP system!)
Generating CS multiform Lagrangian

Similarly, all higher Lagrangian multiforms $L^{(2n+1)}$ of odd degree can be constructed in the same way, leading to the formula:

$$dL^{(2n+1)} = \text{Tr} \left( F_B^n \right) = \sum_{j_1, l_1, j_2, l_2, \ldots, j_n, l_n \in \mathbb{Z}} \left( \partial \xi_{j_1} B_{l_1 l_1} - B_{l_1 j_1} B_{j_1 l_1} \right) \left( \partial \xi_{j_2} B_{l_1 l_2} - B_{l_1 j_2} B_{j_2 l_2} \right) \ldots$$

$$\ldots \left( \partial \xi_{j_n} B_{l_{(n-1)} l_n} - B_{l_{(n-1)} j_n} B_{j_n l_n} \right) d\xi_{j_1} \wedge d\xi_{l_1} \wedge d\xi_{j_2} \wedge d\xi_{l_2} \wedge \ldots \wedge d\xi_{j_n} \wedge d\xi_{l_{(n-1)}},$$

which has a $n$-fold zero on the solutions of the generalised Darboux-KP system. Thus, establishing a hierarchy of Lagrangian multiforms in increasingly higher odd dimensions, but all associated with the same generalised Darboux-KP system. Their action functionals are of the form:

$$\int \mathcal{V}_{2n+1} CS_{2n+1}(B),$$

for each $2n + 1$-dimensional hypersurface $\mathcal{V}_{2n+1}$ embedded in $\mathbb{R}^\mathbb{Z}$. Thus, we can write a generating Lagrangian multiform as the formal sum, in powers of a dummy parameter $\hbar$, 

$$\mathcal{S}_\hbar^{(\infty)}[B; \mathcal{V}_\infty] = \sum_{n=1}^{\infty} \frac{\hbar^n}{n+1} \int_{\mathcal{V}_{2n+1}} L^{(2n+1)}.$$ 

integrated over the disjoint union $\mathcal{V}_\infty = \bigsqcup_{n=1}^{\infty} \mathcal{V}_{2n+1}$ of submanifolds.
Comparison with 4D Chern-Simons theory

In order to depart from the confines of a topological field theory, an action for 1+1-dimensional integrable field theory was proposed by a line of work by Costello et al., going back to earlier ideas by Nekrassov\textsuperscript{17}

The action functional

\[ S[A] = \mathcal{K} \int_{\mathcal{M} = \Sigma \times \mathbb{C}
\mathbb{P}_1} \omega \wedge CS_3(A) \]

extends the usual CS action by integrating over a 4D manifold \( \mathcal{M} \) with coordinates \((\tau, \sigma, z, \bar{z})\), where \((\tau, \sigma)\) are real space-time coordinates and \(z\) is a complex spectral variable. The gauge field is chosen as

\[ A = A_\sigma \, d\sigma + A_\tau \, d\tau + A_\bar{z} \, d\bar{z}. \]

(with component \(A_z = 0\)), and where \(\omega = \varphi(z)dz\) is a meromorphic 1-form.

In the classical context one obtains two sets of equations of motion as EL equations:

\begin{itemize}
  \item bulk equations of motion \(\omega \wedge F(A) = 0\),
  \item 'boundary equations' arising from the contours around singularities (defects) of \(\omega\) in \(\mathbb{C}_P^1\): \(d\omega \wedge \text{Tr}(A \wedge \eta)\) (for all variations \(\eta\)).
\end{itemize}

The claim is that the possible 4D CS actions generate integrable 1+1-dimensional field theories, with the gauge field components \(A_\sigma\) and \(A_\tau\) (up to a gauge) acting as Lax connections, and indeed possessing a classical \(r\)-matrix structure for suitable choices of \(\omega\).

\textsuperscript{17}N.A. Nekrassov, \textit{Four Dimensional Holomorphic Theories}, PhD Thesis, Princeton University, 1996.
Connection with $1+1$-dim. integrable field theories

A generalised $N \times N$ matrix Lax system, by ‘compounding’ the usual hierarchy of integrable time-flows, was derived\textsuperscript{18} leading to:

$$\frac{\partial}{\partial \xi_p} \Phi_k = \frac{R_p}{p-k} \Phi_k , \quad \forall \ p,$$

where $k$ is a spectral parameter, $p$ and $\xi_p$ as before, and the matrix coefficients $R_p$ independent of $k$. Imposing, this for all $p$ we get the MDC system

$$\partial_q R_p = \partial_p R_q , \quad p \partial_p R_q - q \partial_q R_p + [R_q , R_p] = 0 , \quad \forall \ p \neq q .$$

(where we abbreviated $\partial / \partial \xi_p = \partial_p$, There are several ways to resolve these relations:

$$R_p = J_p - \partial_p H = p (\partial_p g) g^{-1} ,$$

for some matrices $H$ and $g$ (without label), and where the $J_p$ are commuting constant matrices. This leads to the equations:

$$\partial_p \partial_q H = \frac{[J_p - \partial_p H, J_q - \partial_q H]}{q - p} ,$$

$$p \partial_q \left( (\partial_p g) g^{-1} \right) = q \partial_p \left( (\partial_q g) g^{-1} \right) ,$$

the latter being generalized chiral field equations. Both systems have a Lagrangian structure:

$$\mathcal{L} = -\frac{1}{2} \text{tr} (\partial_p H \cdot \partial_q H) - \frac{1}{2} \text{tr} ([J_p , H] \partial_q H)$$

$$+ \frac{1}{2} \text{tr} ([J_q , H] \partial_p H) - \frac{1}{3} \text{tr} ([\partial_p H , \partial_q H] H) ,$$


and respectively a special Wess-Zumino-Witten-Novikov type action:

\[ \mathcal{L} = \text{tr} \left( \partial_p g \cdot \partial_q g^{-1} \right) + \frac{q + p}{q - p} \int_0^1 dt \text{tr} \left( \left[ \partial_p g \cdot g^{-1}, \partial_q g \cdot g^{-1} \right] \frac{dg}{dt} \cdot g^{-1} \right), \]

(where \( t \) is a dummy variable and \( g = g(t) \) in the second integrand depends on \( t \), s.t. \( g(0) = I \) (unit matrix), \( g(1) = g(\xi_p, \xi_q) \). Curiously, a similar prefactor to the topological term appears in the work on 4D CS theory\(^{19}\).

**Remark:** For neither Lagrangian (for \( H \) and \( g \) fields) a Lagrangian 2-form structure holds, but it was established for the general class of Zakharov-Mikhailov Lagrangians\(^{20}\). A particular case of this structure is given by the Lagrangian components

\[ \mathcal{L}_{pq} = \text{tr} \left( \Phi^{-1} \partial_p \Phi q_j - \Phi^{-1} \partial_q \Phi p_j \right) - \left( \text{tr} \otimes \text{tr} \right) r_{pq} \left( R_p \otimes R_q \right), \]

where the classical \( r \) matrix appears in the ‘potential term’ of the Lagrangian\(^{21}\). Hence the Lagrangian 2-form

\[ L = \sum_{p, q} \mathcal{L}_{pq} d\xi_p \wedge d\xi_q, \]

obeys the closure relation \( dL = 0 \) on solutions of the EL equations

\[ \partial_p R_q = \partial_q R_p = \frac{[R_p, R_q]}{p - q}, \]

as a consequence of the classical Yang-Baxter equation.


Here some points:

- Lagrangian multiform structures seem to form a universal aspect of integrability as it represents the phenomenon of MDC at the variational level;
- Establishing a Lagrangian 3-form structure for the Darboux-KP system seems the most promising route to attain a quantum theory of the KP system;
- The connection with a CS theory in infinite-dimensional space may yield new insights into the connection between topological, conformal and integrable field theories;
- A new departure (within Lagrangian multiform theory) is to develop a variational description of non-commuting flows\(^{22}\), which may yield a variational approach to Lie group actions on manifolds;
- Potentially the quantum version of Lagrangian multiform theory \(^{23}\) may lead to the introduction of a new quantum object, namely the sum over (hyper)surfaces of surface-dependent propagators. embedding space'.

\(^{22}\) V. Caudrelier, FWN, D. Sleigh and M. Vermeeren, Lagrangian multiforms on Lie groups and noncommuting flows, J. Geom. Phys. 187 (2023), ArXiv:2204.09663

THANK YOU FOR YOUR ATTENTION!