[Nagoya Math-Phys Seminar 2024 Feb.8]

1. BV formalism and homotopy algebras in field theories (review)

2. QFT calculations via BV Lagrangian's Homotopy algebra

Osaka Metropolitan Univ. College of Tech.

Hiroaki Matsunaga

[Plan]

0. Motivation

“BV” & “Homotopy alg”

Why do we need them in QFT?

1. BV formalism and homotopy algebras
in field theories (review)

2. QFT calculations via
BV Lagrangian’s Homotopy algebra

[Plan]

0. *Motivation*

1. *BV formalism and homotopy algebras in field theories (review)*

A crash review course of
* BV formalism
* Homotopy/Homology alg.
is delivered.

2. *QFT calculations via BV Lagrangian's Homotopy algebra*

[Plan]

0. Motivation

1. BV formalism and homotopy algebras in field theories (review)

2. QFT calculations via BV Lagrangian's Homotopy algebra

My recent & on-going works

→ Today's goal

Why homotopy alg. ?

Roughly speaking..

- Each quantum field theory that has the path-integral description has own "homotopy algebraic structure μ" .

  → Your can quickly find a cyclic A_∞ (and L_∞) algebra in your e.o.m. and Lagrangian and

  quantum A_∞/L_∞ algebra in correlation fnc . $\langle \dots \rangle = \int \nu_\phi (\dots)$ where $\nu_\phi = \mathscr{D}\phi \, e^{S[\phi]} / Z$.

- Path-integral *P* gives a morphism of this algebra.

$$P \mu = \mu' P$$

(*P* : homotopy alg. of the original QFT→ homotopy alg. of its effective QFT)

Why homotopy alg. ?

A useful tool for QFT computations.

- All quantities written by the path-integral, as well as Lagrangians, may be written in terms of this algebra.

- We can apply this technique to obtain

  effective theories, S-matrix, current recursion relations, …

  or to study

  realization of symmetry, anomalies, flow of ERG… .

- Some instanton effects can be derived by using this package.

Why BV ?

A useful tool in field theories.

- The BV formalism is equivalent to this homotopy algebra.

BV master eq. $\overset{dual}{\longleftrightarrow}$ quantum homotopy alg.

like . . . Component : $\partial_\mu F^{\mu\nu} = \mu_0 \, j^\nu$ $\overset{dual}{\longleftrightarrow}$ Form : $d\star F = \star J$.

- BV tells us how to find homotopy algebras.

- **We can always apply** the BV formalism to given Lagrangians :

BV formalism is a generating fnc. of Noether id./Schwinger-Dyson eq.

Today, I will explain

See also Supplement 1 & 2

1. Why every path-integrable QFT have a homotopy algebra

2. Why the path-integral preserves such a homotopy algebra

   (In particular, one can obtain Wick's theorem in QFT exactly by using this method.)

with the help of the BV formalism.

[Nagoya Math-Phys Seminar 2024 Feb.8]

1. BV formalism and homotopy algebras in field theories (review)

Osaka Metropolitan Univ. College of Tech.

Hiroaki Matsunaga

Outline

0. What is homotopy algebra ?

1. BV formalism

2. BV Lagrangian's homotopy algebra

0. What is homotopy algebra ?

Definition :

- We consider a vector space H and multilinear maps $\mu_n : H^{\otimes n} \longrightarrow H$.
- Homotopy algebra (H, μ) is a set of multilinear maps $\mu = \{\mu_n\}_{n=1}^{\infty}$ with

$$\mu_1 \cdot \mu_1(\phi) = 0$$

$$\mu_1 \cdot \mu_2(\phi_1, \phi_2) + \mu_2(\mu_1(\phi_1), \phi_2) + \mu_2(\phi_1, \mu_1(\phi_2)) = 0$$

$$\vdots$$

where $\phi \in H$.

$$\sum_{k+l=n} \sum_{m=0}^{k} \mu_{k+1}(\underbrace{\ldots, \phi_m}_{m}, \mu_l(\phi_{m+1}, \ldots, \phi_{m+l}) \underbrace{, \ldots, \phi_n}_{k-m}) = 0$$

Put it more simply . . .

0. What is homotopy algebra ?

Equivalent definition :

. We can consider its tensor algebra $T(H) = \mathbb{C} \oplus H \oplus H^{\otimes 2} \oplus \cdots \oplus H^{\otimes n} \oplus \cdots$ and linear maps $\boldsymbol{\mu}_n : T(H) \longrightarrow T(H)$.

. Homotopy algebra (H, μ) is a set of **linear maps**

$$\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_1 = 0$$

$$\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 + \boldsymbol{\mu}_2 \cdot \boldsymbol{\mu}_1 = 0$$

$$\vdots$$

$$\sum_{k+l=n} \boldsymbol{\mu}_{k+1} \cdot \boldsymbol{\mu}_l = 0$$

⟶ **Blackboard**

0. What is homotopy algebra ?

Simplest definition :

. We can consider its tensor algebra $T(H) = \mathbb{C} \oplus H \oplus H^{\otimes 2} \oplus \cdots \oplus H^{\otimes n} \oplus \cdots$

and linear maps $\boldsymbol{\mu_n} : T(H) \longrightarrow T(H)$.

. Homotopy algebra (H, μ) is a nilpotent linear map on $T(H)$

$$\boldsymbol{\mu_1} \cdot \boldsymbol{\mu_1} = 0$$

$$\boldsymbol{\mu_1} \cdot \boldsymbol{\mu_2} + \boldsymbol{\mu_2} \cdot \boldsymbol{\mu_1} = 0$$

$$\vdots$$

$$\sum_{k+l=n} \boldsymbol{\mu_{k+1}} \cdot \boldsymbol{\mu_l} = 0$$

Homotopy algebraic relation

$$\boldsymbol{\mu} \cdot \boldsymbol{\mu} = 0$$

where $\boldsymbol{\mu} = \boldsymbol{\mu_1} + \boldsymbol{\mu_2} + \cdots$.

0. What is homotopy algebra ?

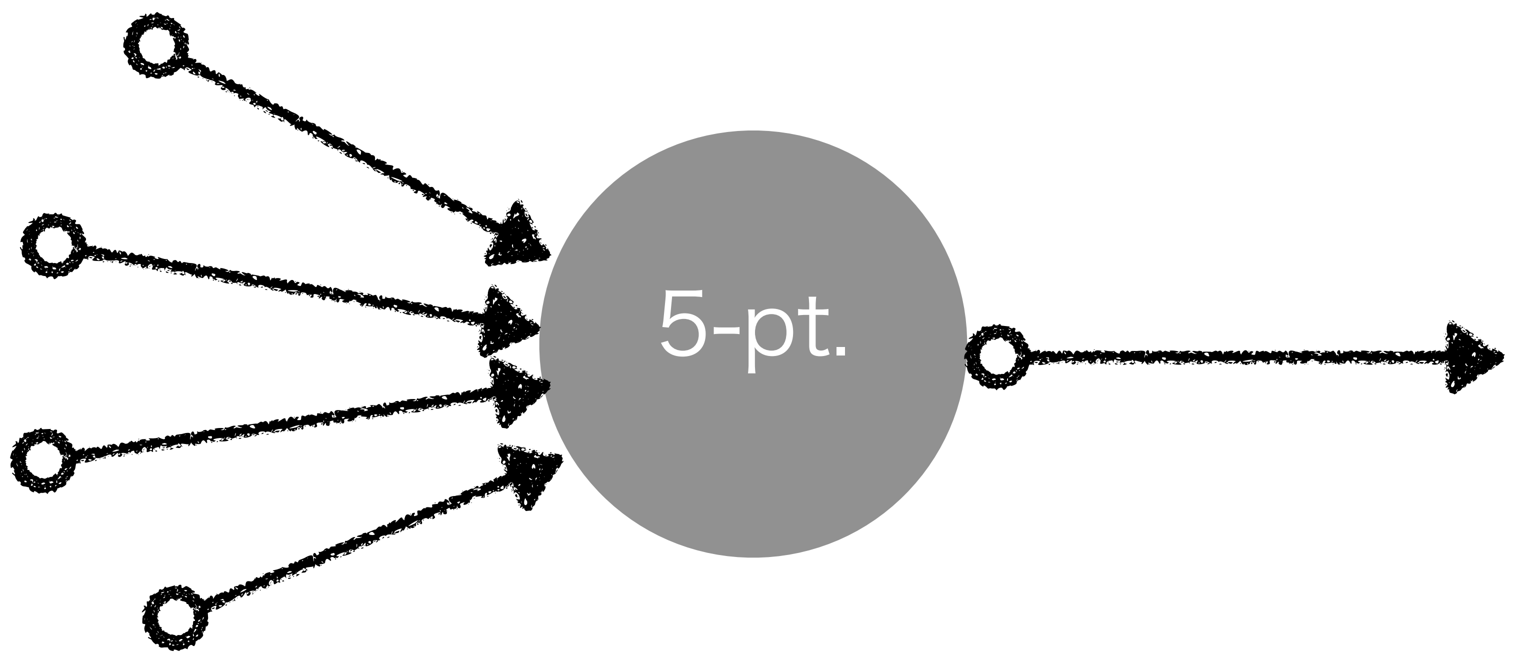
Homotopy Algebras in QFT :

- We consider **the state space** H, so $T(H)$ is the Fock space.

- In most cases, linear maps $\mu_n : H^{\otimes n} \to H$ are proportional to **vertices**,

  so that $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ can give, for example, *the BRST operator, symmetry generators, S-matrix, effective Lagrangians, etc.*

$$\mu_4 : H^{\otimes 4} \longrightarrow H \qquad \sim$$



0. What is homotopy algebra ?

Homotopy Algebras in QFT :

- We consider **the state space** H, so $T(H)$ is the Fock space.

- In most cases, linear maps $\mu_n : H^{\otimes n} \to H$ are proportional to **vertices**,

  so that $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ can give, for example, *the BRST operator, symmetry generators, S-matrix, effective Lagrangians, etc.*

- Then, the relation $\boldsymbol{\mu} \cdot \boldsymbol{\mu} = 0$ tells us physics described by this $\boldsymbol{\mu}$:

  E.g. *Decoupling of gauge & unphysical degrees, realization of nonlinear symmetry, unitarity, relations between currents, flow of ERG, etc.*

0. What is homotopy algebra ?

Why is the homotopy algebra usable in QFT ?

. We can always extract such $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ from a given QFT.

— *Your Lagrangian has its* $\boldsymbol{\mu}_{S[\phi]}$.

— *Symmetry of your QFT has* $\boldsymbol{\mu}_{sym}$.

⟶ ***Blackboard***

0. What is homotopy algebra ?

Why is the homotopy algebra usable in QFT ?

. We can always extract such $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ from a given QFT.

— *Your Lagrangian has its* $\boldsymbol{\mu}_{S[\phi]}$; *Its "effective" theory also has .*

— *Symmetry of your QFT has* $\boldsymbol{\mu}_{sym}$; *"effective" one exists .*

For ***general splitting*** $\phi = \phi' + \phi''$, we call $A[\phi'] \equiv \ln \int \mathscr{D}\phi'' \, e^{S[\phi'+\phi'']}$ ***effective theory***.

This is a slight abuse of terminology :
We are not integrating out "high energy modes" in a conventional sense.

0. What is homotopy algebra ?

This $\mu = \mu_1 + \mu_2 + \cdots$ is useful because the path-integral preserves it.

- *Once you find* $S_{original}[\varphi] = \sum_n \frac{1}{n+1} \langle \varphi, \mu_n(\varphi, \ldots, \varphi) \rangle$ *with* $(\mu)^2 = 0$, then its "effective" theories take $S_{eff}[\varphi'] = \sum_n \frac{1}{n+1} \langle \varphi', \mu'_n(\varphi', \ldots, \varphi') \rangle'$ with $(\mu')^2 = 0$.

- ***The Feynman graph expansion*** preserves $(\mu)^2 = 0$.

- So, we can apply the same method to the S-matrix, Wilsonian, gauge-fixed QFT, etc.

0. What is homotopy algebra ?

This $\mu = \mu_1 + \mu_2 + \cdots$ is useful because the path-integral preserves it.

- We presented ***Lagrangian's homotopy algebra*** only.

- For a given homotopy algebraic structure $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ in QFT, there is ***a systematic way*** to give effective one $\boldsymbol{\mu}' = \boldsymbol{\mu}'_1 + \boldsymbol{\mu}'_2 + \cdots$.

  — *Homological perturbation to homotopy algebraic structure*

- *The path-integral also preserves it.*

0. What is homotopy algebra ?

Why is the homotopy algebra usable in QFT ?

. We can always extract such $\mu = \mu_1 + \mu_2 + \cdots$ from a given QFT.

— *We found Lagrangian's* $\boldsymbol{\mu}_{S[\phi]}$ *, which is transferred to "effective" QFT.*

— *Symmetry in QFT has* $\boldsymbol{\mu}_{sym}$ *, which is transferred to "effective" one.*

I might talk about homotopy algebra in symmetry on Feb.19 or March 29-31.

.

. . . *how to extract such* $\mu = \mu_1 + \mu_2 + \cdots$ from a given QFT ?

0. What is homotopy algebra ?

How to extract homotopy algebras from QFT ?

- We can always extract such $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ from a given QFT.

  — *Your Lagrangian has its* $\boldsymbol{\mu}_{S[\phi]}$ *; Its "effective" theory also does.*

  — *Symmetry of your QFT has* $\boldsymbol{\mu}_{sym}$ *; "effective" one also does.*

- Extracting $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \cdots$ from a Lagrangian is equivalent to solving the master equation in **the Batalin-Vilkovisky (BV) formalism**.

  — *Homotopy algebraic technique is widely usable*

  as mach as *the BV formalism !!*

Outline

0. What is homotopy algebra ?

1. BV formalism

2. BV Lagrangian's homotopy algebra

What is BV . . . ?

- BV is one method to get the path-integral quantization.

  Taming gauge redundancy :

  By hand $\subset$ Fadeev-Popov $\subset$ BRST $\subset$ **BV**

- BRST method is a “gauge fixed version” of BV .

  — Very similar to **the standard canonical formalism**.
  — A canonical transformation performs a gauge-fixing.

$$Z = \int \mathscr{D}[\varphi, \varphi^*]\, \delta(\varphi^* - F[\varphi])\, e^{S[\varphi,\varphi^*]}$$

What is BV . . . ?

- BV is one method to get the path-integral quantization.

    Taming gauge redundancy :

    By hand $\subset$ Fadeev-Popov $\subset$ BRST $\subset$ **BV**

- BRST method is a "gauge fixed version" of BV .

    — Very similar to **the standard canonical formalism**.
    — A canonical transformation performs a gauge-fixing.

- BV can be applied to field theories without gauge degree
    and tells us how to extract Lagrangian's homotopy algebra μ_{bv} .

What is BV . . . ?

- A more classical (or modern) view of the BV formalism:

  BV package assigns “a Homology algebra” to “a given space of fields”.

BV formalism : Lagrangian field theory $\longrightarrow$ Derived Geometry
(classical & quantum)

- 以下、日本語のスライド ＋ 板書の補足 で BV 形式を review します。

Batalin-Vilkovisky's antifield formalism

古典力学における観測可能な物理量は、どこに住んでいるか？

- 力学では、作用 $S[\phi]$ の停留点 $\delta S[\phi]=0$ で「実現される運動・場の配位」が決まる。

  Physical states $\Longleftrightarrow$ 運動方程式 $\dfrac{\delta S[\phi]}{\delta\phi}=0$ の解を与える $\phi(x)$

  力学変数 $\phi(x)$ を座標とする "多様体" M を、configuration space と呼んだ。

  作用 $S[\phi]\in C^\infty(M,\mathbb{R})$ をはじめ、物理量は M 上の関数 となる。

- 運動方程式を満たす M の部分空間 (isolated points) を $M|_{\delta S=0}$ と書く。

  物理量の中の "観測可能量 (observables)" は、$M|_{\delta S=0}$ の関数 となる。

  $\mathscr{F}|_{\delta S=0} := \{\, M|_{\delta S=0}$ 上の関数の全体 $\}$ は singular な空間

Batalin-Vilkovisky's antifield formalism

この由来は、自明なゲージ自由度

- 停留作用の式 $\delta S[\phi]=0$ は、ゼロ固有値があると、Noether 恒等式 がみつかる：

$$\text{ゲージ自由度} \quad \Longleftrightarrow \quad \delta S[\phi] = \frac{\delta S[\phi]}{\delta \phi}\cdot \delta\phi = 0 \ \text{ なる 固有ベクトル } \delta\phi \ .$$

- 実はどのような模型にも、on-shell で消える、(自明な) ゲージ自由度 が存在する：

$$\phi \mapsto \phi + \varepsilon\cdot\delta\phi \qquad \text{where} \quad \delta\phi^a = \omega^{ab}\frac{\delta S}{\delta \phi^b} \ , \quad \omega^{ab} = -\,\omega^{ba} \ \ \& \ \ \varepsilon = \varepsilon(x)\ .$$

これは、各 “物理量” を「運動方程式を代入すると消える量」だけずらす操作を生成。

$$\underbrace{\text{観測される物理量の全体}}_{\mathscr{F}|_{\delta S=0}} = \underbrace{\text{物理量の全体}}_{\mathscr{F}_0 = C^\infty(M,\mathbb{R})} \Big/ \underbrace{\text{このイデアル}}_{\mathrm{Sym}\{\,\delta\phi\,\}}$$

Batalin-Vilkovisky's antifield formalism

場の解析力学 → *Derived Geometry*

- このような対象を扱うには、$\mathscr{F}_0$ を含む複体とその **resolution** を考えればよい。

The vector space $\mathscr{F}_0 = \mathrm{Sym}\{\,\phi\,\}$ is embedded into a sequence of vector spaces

$$\cdots \xrightarrow{s|_{-2}} \mathscr{F}_{-1} \xrightarrow{s|_{-1}} \mathscr{F}_0 \xrightarrow{s|_0} 0$$

and then $\mathscr{F}|_{\delta S=0} = \mathscr{F}_0 \big/ \mathrm{Sym}\{\,\delta\phi\,\}$ is realized as a cohomology.

多様体 M の各点 $\phi(x) \in M$ の周りに、座標系に代わり「DGAの列 $\mathscr{F}$」が付与！

- そこで、次数つき線形空間 $\mathscr{F}_{-1}$ と 微分 s をうまく探せばよい。

Batalin-Vilkovisky's antifield formalism

場の解析力学 → *Derived Geometry*

- そのための方法の１つに、**Koszul-Tate resolution** と呼ばれる方法がある。

$\mathscr{F}_{-1} = \mathrm{Sym}\{\,\phi^\star\}$ の作り方 （Antifields の全体）

ϕ^a と同じ数だけ、次数が -1 だけずれた $\phi^\star_a$ を導入する。

微分 s の作り方 （BRST-BV 演算子）

$s\,|_{-1} : \phi^\star_a \mapsto \dfrac{\delta S}{\delta \phi^a}$, $s\,|_0 : \phi^a \mapsto 0$ によって微分 s を定める。

$$\cdots \xrightarrow{s|_{-2}} \mathscr{F}_{-1} \xrightarrow{s|_{-1}} \mathscr{F}_0 \xrightarrow{s|_0} 0$$

観測可能量の全体は、$\mathrm{Im}[\,s\,] = \{\,s\,\phi^\star\} = \{\,\delta\phi\,\}$ および $\mathrm{Ker}[\,s\,] = \{\,\phi\,\}$ より、

$$\mathrm{Cohom}[\,s\,] = \mathrm{Ker}[\,s\,] \big/ \mathrm{Im}[\,s\,] = \mathscr{F}\,|_{\delta S=0}$$

Batalin-Vilkovisky's antifield formalism

BRST-BV 形式： 場の解析力学 → *Derived Geometry*

- BRST-BV 形式 は、このような **resolution** を systematic に与える処方。

各座標関数 $\phi^a : M \to \mathbb{R}$ に対し、antifield $\phi^\star_a \in T^*_\phi[-1]M$ を assign する。

(Antifield $\phi^\star$ は、c-momentum π や source j と同様、最終的な物理には現れない)

Antibracket と呼ばれる、次数 1 の Poisson 括弧 を定める。

$$(A, B) \equiv A \left[\frac{\overleftarrow{\delta}}{\delta \phi^a} \frac{\overrightarrow{\delta}}{\delta \phi^\star_a} - \frac{\overleftarrow{\delta}}{\delta \phi^\star_a} \frac{\overrightarrow{\delta}}{\delta \phi^a} \right] B \quad .$$

微分 $s = (\quad , S)$ が、模型の作用 $S[\phi]$ から定まる。(BRST-BV演算子)

Batalin-Vilkovisky's antifield formalism

BRST-BV 形式： 場の解析力学 → *Derived Geometry*

- 場の古典論を BRST-BV形式 で表すには master eq. $\left(S[\phi], S[\phi]\right) = 0$ を解けばよい

- (非自明な) ゲージ自由度のない模型に対しては、次の classical BV複体 を得る：

A sequence of graded vector spaces $0 \xrightarrow{s|_{-2}} \mathscr{F}_{-1} \xrightarrow{s|_{-1}} \mathscr{F}_0 \xrightarrow{s|_0} 0$,

the BV-BRST differential $s|_{-2} = 0$, $s|_{-1} = -\dfrac{\overleftarrow{\delta}}{\delta\phi_a^\star}\dfrac{\delta S[\phi]}{\delta\phi^a}$, $s|_0 = 0$.

- BRST-BV コホモロジー $\mathscr{F}|_{\delta S=0} = \dfrac{\mathrm{Ker}[s|_0]}{\mathrm{Im}[s|_{-1}]}$ は「 **the fields ϕ^a modulo** $\dfrac{\delta S[\phi]}{\delta\phi^a}$ 」から成る。

$\longrightarrow$ $\phi^a \in \mathscr{F}_0$ の $\mathscr{F}|_{\delta S=0}$ への射影は「 **the fields ϕ^a solving** $\dfrac{\delta S[\phi]}{\delta\phi^a} = 0$ 」となる！

Batalin-Vilkovisky's antifield formalism

まとめ（場の古典論）

- BRST-BV 形式： 場の古典論 → Derived Geometry
- その心は、Physical な量 や 古典解は「**the fields** ϕ^a **modulo** $\frac{\delta S[\phi]}{\delta \phi^a}$」から生成。

⟶ 板書： 簡単な模型で確認してみる

Batalin-Vilkovisky's antifield formalism

まとめ（場の古典論）

- BRST-BV 形式： 場の古典論 → Derived Geometry
- その心は、Physical な量 や 古典解は「 **the fields** ϕ^a **modulo** $\frac{\delta S[\phi]}{\delta \phi^a}$ 」から生成。

この話に「場の量子論」を含めるには？

- 量子論においても、Physical States は「 **fields modulo 何？**」さえ決めれば . . .

(摂動的な) 場の量子論 → Derived Geometry

実は、BRST-BV 形式 はこの問に答え、そのような処方を systematic に与える

Batalin-Vilkovisky's antifield formalism

Quantum BV 形式： 場の量子論 → *Derived Geometry*

- 場の量子論において、観測される物理量 $O_A = O_A[\phi]$ の値 A は、

$$A = \langle O_A \rangle = \int \mathscr{D}[\phi] e^{S[\phi]} O_A \overset{\text{摂動}}{=} \int \mathscr{D}[\phi] e^{S_{\text{free}}[\phi]} \Big[e^{S_{\text{int}}[\phi]} O_A \Big]$$

- このとき与えられた模型が「何を法とするか」を決めるのは、Schwinger-Dyson eq.

$$\int \mathscr{D}[\phi] \frac{\delta}{\delta\phi} (\ldots) = 0 \quad \Longrightarrow \quad \textbf{integrand modulo} \ \frac{\delta}{\delta\phi}(\ldots)$$

- これを**出発点**とするのが、(quantum) BRST-BV 形式。

Quantum BV master equation : $\Delta \left(e^{S[\phi]} O_A \right) = 0$

Batalin-Vilkovisky's antifield formalism

Quantum BV 形式： 場の量子論 → *Derived Geometry*

- BRST-BV 形式では、

integrand modulo $\dfrac{\delta}{\delta\phi}(\ldots)$ $\iff$ **integrand modulo Δ-exacts**

- 模型の **BRST-exacts** に加え、**次数 1 の Laplacian Δ** も法となる：

BV Laplacian $\quad \Delta = (-)^{\phi^a} \dfrac{\overleftarrow{\delta}}{\delta\phi^a} \dfrac{\overleftarrow{\delta}}{\delta\phi^\star_a}$

- 作用 $S[\phi]$ と 物理量 $O_A = O_A[\phi]$ について $\Delta\left(e^{S[\phi]}O_A\right) = 0$ を書き下すと . . .

$$\hbar\,\Delta S + \frac{1}{2}\left(S\,,S\right) = 0 \quad , \qquad \hbar\Delta O_A + (O_A\,,S) = 0$$

Batalin-Vilkovisky's antifield formalism

Quantum BV 形式： 場の量子論 → *Derived Geometry*

- 「模型の **BRST op.** $s = (\ ,S)$ と **BV Laplacian** Δ の和 $s + \hbar\Delta$ 」が微分となる：

$$(s + \hbar\Delta)^2 = 0 \qquad \text{(BRST-BV operator)}$$

場の古典論では、Config. sp. M の各点に、classical BV の複体 が付与された

鎖複体 $(\mathscr{F}, s)$ ：

$$\cdots \xrightarrow{s} \mathscr{F}_{-1} \xrightarrow{s} \mathscr{F}_0 \xrightarrow{s} 0$$

場の量子論では、Config. sp. M の各点に、BV 複体 が付与される

鎖複体 $(\mathscr{F}, s + \hbar\Delta)$ ：

$$\cdots \xrightarrow{s+\hbar\Delta} \mathscr{F}_{-1} \xrightarrow{s+\hbar\Delta} \mathscr{F}_0 \xrightarrow{s+\hbar\Delta} 0$$

(摂動的な) 経路積分 = この複体の BRST-BV cohomology への射影 $\mathscr{F} \xrightarrow{P} \mathscr{F}_{\mathrm{Phys}}$

Batalin-Vilkovisky's antifield formalism

Quantum BV 形式： 場の量子論 → *Derived Geometry*

- このような複体 (微分 Q & 字数付きベクトル空間の列 $\mathscr{F}$) 、および、
  BRST-BV コホモロジー $\mathscr{F}_{\text{phys}}$ への射影 p についての**情報**をまとめて、次のように書く：

$$h \circlearrowright (\mathscr{F}, Q) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, Q_{\text{phys}}) \qquad \text{(strong deformation retract)}$$

  i は自然な injection、h はグリーン関数 (s の逆) に対応する。

- Q_{phys} は BRST-BV コホモロジー $\mathscr{F}_{\text{phys}}$ 上での微分。
  (e.o.m. を解いた / 経路積分した場について $Q_{\text{phys}} = 0$, その他の場では $Q_{\text{phys}} \neq 0$.)

→　各データには、およそ、次の対応がある

Batalin-Vilkovisky's antifield formalism

まとめ： *Homology* 的な記述

$$h \circlearrowright (\mathscr{F}, Q) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, Q_{\text{phys}})$$

(strong deformation retract)

複体 $\mathscr{F}$ の与え方 $\iff$ どのような場 (力学変数) を考えているのか

微分 Q の与え方 $\iff$ どのような Lagrangian (模型の作用) を与えたか

射影 p の与え方 $\iff$ どの場について **「 e.o.m.を解く / 経路積分する」** か

逆演算 h の与え方 $\iff$ どの**境界条件の下で解く**のか

BV 鎖複体 $(\mathscr{F}_{\text{phys}}, Q_{\text{phys}})$ $\iff$ 得られる「有効場の理論 (経路積分値) 」の情報

Batalin-Vilkovisky's antifield formalism

Homology 的記述の使用例　(*Hodge* 分解が鍵)

free, classical

$$h \circlearrowright (\mathscr{F}, s_{\rm free}) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\rm phys}, 0)$$

$\xrightarrow{+\hbar\Delta}$

free, quantum

$$H' \circlearrowright (\mathscr{F}, s_{\rm free} + \hbar\Delta) \underset{I'}{\overset{P'}{\rightleftarrows}} (\mathscr{F}_{\rm phys}, 0)$$

$\downarrow +s_{\rm int}$

ホモロジー摂動論

$\downarrow +s_{\rm int}$

interacting, classical

$$h' \circlearrowright (\mathscr{F}, s_{\rm free} + s_{\rm int}) \underset{i'}{\overset{p'}{\rightleftarrows}} (\mathscr{F}_{\rm phys}, s_{\rm tree})$$

$\xrightarrow{+\hbar\Delta}$

interacting, quantum

$$H \circlearrowright (\mathscr{F}, s_{\rm free} + s_{\rm int} + \hbar\Delta) \underset{I}{\overset{P}{\rightleftarrows}} (\mathscr{F}_{\rm phys}, s_{\rm eff} + \hbar\Delta_{\rm eff})$$

初期情報 p, i, h と摂動 $s_{\rm int} + \hbar\Delta$ から、欲しい情報 P, I, H が systematic に作れる！

Batalin-Vilkovisky's antifield formalism

Projection P = the perturbative path-integral

$$\mathrm{P}\Big[A[\phi]\Big] = e^{G_{\mathrm{free}}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}}\Big[e^{S_{\mathrm{int}}[\phi]}A[\phi]\Big]_{\phi=0} = \int \mathscr{D}[\phi]e^{S[\phi]}\Big[A[\phi]\Big]$$

- 計算の概略は、後半に説明：
  2. QFT calculations via BV Lagrangian's homotopy algebra

- 詳しいBVでの計算 および homotopy代数での計算は、次の論文中に

PTEP 2022 113 B04　(arXiv 2003.05021 hep-th)

Outline

0. What is homotopy algebra ?

1. BV formalism

2. BV Lagrangian’s homotopy algebra

互いに書き換え可能：

BV形式 と ホモトピー代数 の等価性

Homology/Homotopy代数による記述方法

BRST-BV 形式 と *Homotopy* 代数 の書き換え

- 多重線型形式は「成分表示」と「微分形式表示」の **書き換え** ができた：

例えば...

Maxwell 方程式：	$\partial_\mu F^{\mu\nu} = j^\nu$	$\Longleftrightarrow$	$d \star F = j$
カレント保存：	$\partial_\mu j^\mu \approx 0$	$\Longleftrightarrow$	$d j \approx 0$
ベクトル場などの模型：	$\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$	$\Longleftrightarrow$	$\mathrm{tr}\, F \wedge F$

- 実は「BRST-BV 形式による記述」と「Homotopy代数による記述」も同様の関係：

BV 方程式 $\Delta\, e^S = 0$ $\Longleftrightarrow$ quantum A_∞ / L_∞ 関係式 $(\hbar\, \Delta + \mu)^2 = 0$

- **Example: Classical Scalar field (free theory)**

the BV master action : $S[\phi] = \frac{1}{2}\varphi\,(\partial^2 - m^2)\,\varphi$

Since $S[\phi]$ has no ϕ^* dependence, Q acts only non-trivially on ϕ^* as follows:

$$Q\,\varphi = 0\ , \quad Q\,\varphi^* = (\,S\,,\varphi^*\,) = (\partial^2 - m^2)\phi\ ,$$

which acts on the components : $\underbrace{0}_{\text{ghost}} \xleftarrow{Q} \underbrace{H_\varphi}_{\text{field}} \xleftarrow{Q} \underbrace{H_{\varphi^*}}_{\text{antifield}} \xleftarrow{0} \underbrace{0}_{\text{ghost*}}\ .$

→ We assign the basis $\{e, e_*\}$ and use a “super-field” $\phi = \varphi\, e + \varphi^*\, e_*$.

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→ We assign the basis $\{e, e_*\}$ and use a "super-field" $\phi = \varphi\, e + \varphi^*\, e_*$.

Then, we find A_∞ str $\quad \mu_1(\phi) = (\partial^2 - m^2)\,\varphi\, e_*$,

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- **Example: Classical Scalar field (interacting theory)**

the BV master action : $S[\phi] = \frac{1}{2}\varphi\,(\partial^2 - m^2)\,\varphi + \frac{\kappa}{3!}\varphi^3 + \frac{\lambda}{4!}\varphi^4$

Since $S[\phi]$ has no ϕ^* dependence, Q acts only non-trivially on ϕ^* as follows:

$$Q\,\varphi = 0\,, \qquad Q\,\varphi^* = (\,S\,,\varphi^*\,) = (\partial^2 - m^2)\phi + \frac{\kappa}{2}\varphi^2 + \frac{\lambda}{3!}\varphi^3\,,$$

which acts on the components : $\underbrace{0}_{\text{ghost}} \xleftarrow{Q} \underbrace{H_\varphi}_{\text{field}} \xleftarrow{Q} \underbrace{H_{\varphi^*}}_{\text{antifield}} \xleftarrow{0} \underbrace{0}_{\text{ghost*}}$.

→ We assign the basis $\{e, e_*\}$ and use a “super-field” $\phi = \varphi\, e + \varphi^*\, e_*$.

Then, we find A_∞ str $\quad \mu_1(\phi) = (\partial^2 - m^2)\,\varphi\, e_*\,, \quad \mu_2(\phi,\phi) = \kappa\,\varphi^2\, e_*\,, \quad \mu_3(\phi,\phi,\phi) = \lambda\,\varphi^3\, e_*\,,$

which acts on the basis : $\underbrace{0}_{\text{ghost}} \xrightarrow{0} \underbrace{H_e}_{\text{field}} \xrightarrow{\mu} \underbrace{H_{e_*}}_{\text{antifield}} \xrightarrow{\mu} \underbrace{0}_{\text{ghost*}}$.

- **Example: classical Yang-Mills field**

the classical action $S_{\text{cl}}[A] = -\frac{1}{4}\int dx\, F_{\mu\nu}F^{\mu\nu}$ can be rewritten as the form of

$$S_{\text{cl}}[A] = \frac{1}{2}\langle A, \tilde{\mu}_1 A \rangle + \frac{1}{3}\langle A, \tilde{\mu}_2(A, A) \rangle + \frac{1}{4}\langle A, \tilde{\mu}_3(A, A, A) \rangle$$ where A_∞ products are

$$\tilde{\mu}_1(A) = d \star d\, A$$

$$\tilde{\mu}_2(A_1, A_2) = d \star (A_1 \wedge A_2) - (\star dA_1) \wedge A_2 + A_1 \wedge (\star dA_2)$$

$$\tilde{\mu}_3(A_1, A_2, A_3) = A_1 \wedge (\star (A_2 \wedge A_3)) - (\star (A_1 \wedge A_2)) \wedge A_3$$

- This A_∞ is **incomplete** and a part of the **full** A_∞ of the Yang-Mills theory.

$$\underbrace{0}_{\text{ghost}} \xrightarrow{\mu} \underbrace{H_e}_{\text{field}} \xrightarrow{\mu} \underbrace{H_{e*}}_{\text{antifield}} \xrightarrow{\mu} \underbrace{0}_{\text{ghost*}}$$ is not exact !

- Example: classical Yang-Mills field

- The Yang-Mills field action $S_{\rm cl}[A] = -\frac{1}{4}\int dx\, F_{\mu\nu}F^{\mu\nu}$ is invariant under the gauge transf. $\delta A_\mu = D_\mu \lambda = \partial_\mu \lambda + [\, A_\mu \,, \lambda \,]$.

- We thus need the ghost contribution $(A^*_\mu)\cdot D_\mu\, c$ and must solve **(S , S) = 0** :

  The master action is given by $S = S_{\rm cl}\,[A] + (A^*_\mu)\cdot D_\mu\, c + \frac{1}{2}\, c^* \cdot [\, c\,, c\,]$,

  which enables us to perform the path-integral.

- The BRST transf. $\delta\varphi = (\,\varphi\,, S\,) = \mu_1(\varphi) + \mu_2(\varphi,\varphi) + \cdots$ tells us the full A_∞ structure.

- **Example: classical Yang-Mills field**

- We can find **the full A_∞ structure**, which is given by

$$\mu_1 \begin{pmatrix} c \\ A \\ A^* \\ c^* \end{pmatrix} = \begin{pmatrix} 0 \\ d\,c \\ d \star d\,A \\ d \star\ A \end{pmatrix} \quad \text{with} \quad \underbrace{0}_{\text{higher}} \xrightarrow{\mu} \underbrace{H_c}_{\text{ghost}} \xrightarrow{\mu_1 = d} \underbrace{H_\phi}_{\text{field}} \xrightarrow{\mu_1 = d\star d} \underbrace{H_{\phi^*}}_{\text{antifield}} \xrightarrow{\mu_1 = d} \underbrace{H_{c^*}}_{\text{ghost}^*} \xrightarrow{\mu} \underbrace{0}_{\text{higher}^*} ,$$

$$\mu_2(\phi_1, \phi_2) = \begin{pmatrix} c_1 c_2 \\ c_1 A_2 + A_1 c_2 \\ \tilde{\mu}_2(A_1, A_2) - c_1 A_2^* + A_1^* c_2 \\ c_1 c_2^* - c1^* c_2 - \star\,(A_1 \wedge (\star A_2^*)) + \star\,(A_1^* \wedge \star A_2) \end{pmatrix}, \quad \mu_3(\phi_1, \phi_2, \phi_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\mu}_3(A_1, A_2, A_3) \end{pmatrix}, \quad \phi_i = \begin{pmatrix} c_i \\ A_i \\ A_i^* \\ c_i^* \end{pmatrix} \quad \text{(i=1~3).}$$

- As is known, **the Berends-Giele current recursion relations** can be quickly derived from this A_∞ structure of the Yang-Mills theory.

Homology/Homotopy代数による記述方法

BRST-BV 形式 と Homotopy 代数 の書き換え

- どの模型の作用も、次の **DeWitt 縮約表示**で書ける：

$$S[\phi] = \frac{1}{2}\mu_{ab}\,\phi^b\,\phi^a + \sum_n \frac{1}{(n+1)!}\,\mu_{a_0 a_1 \dots a_n}\,\phi^{a_n}\dots\phi^{a_1}\,\phi^{a_0}$$

- DeWitte縮約は、Lorentz・内部自由度だけでなく、「場の種類」も添字で表す：

$O(N)$ スカラー場： $\phi^a = \{\,\varphi_1,\dots,\varphi_N,\,\varphi_1^\star,\dots,\varphi_N^\star\,\}$

QED の場： $\phi^a = \{\,\psi,\bar{\psi},A_\mu,c,\,\psi^\star,\bar{\psi}^\star,A_\mu^\star,c^\star\,\}$

- 自由場の演算子と相互作用項の情報は、場の多項式の "係数" $\{\,\mu_{a_0 a_1\dots a_n}\,\}_{n\in\mathbb{N}}$ に現れる。

Homology/Homotopy代数による記述方法

BRST-BV 形式 と *Homotopy* 代数 の書き換え

- 作用の満たす BV 方程式 $\Delta\, e^{S[\phi]} = 0$ を、この成分表示で書き下す：

$$\sum_{s,t}\left[\frac{\hbar}{2}(-)^{\psi^{a_t}}\,\omega^{a_t a_s}\,\mu_{a_1\cdots a_s\cdots a_t\cdots a_n} + \sum_{m=0}^{n-1}\mu_{a_1\cdots a_s\cdots a_m}\,\omega^{a_s a_t}\,\mu_{a_{m+1}\cdots a_t\cdots a_n}\right] = 0$$

(Antibracket に由来する ω^{ab} は「次数 1 の sympletcic 形式の標準形」の逆行列)

- BRST-BV形式で記述するときの、作用 S の "場の係数" $\{\,\mu_{a_0 a_1 \ldots a_n}\,\}_{n\in\mathbb{N}}$ が満たす関係式。

- 多重線形性より、係数 $\mu_{a_0 a_1 \ldots a_n}$ は「多重線型形式の成分」と同一視できる：

$\mu_{a_0 a_1 \ldots a_n}$ $\iff$ n-形式 $\mu_n : V^{\otimes n} \to V$ の成分 (V : ある線型空間)

Homology/Homotopy代数による記述方法

BRST-BV 形式 と *Homotopy* 代数 の書き換え

- この n-形式 を “ひとまとめ” にすると、テンソル代数 $T(V)$ 上の線型写像となる：

$$\mu \equiv \mu_1 + \mu_2 + \cdots + \mu_n + \cdots \qquad \Longrightarrow \qquad \mu : T(V) \to T(V)$$

- 余代数で書くと、係数 $\{ \mu_{a_0 a_1 \ldots a_n} \}_{n \in \mathbb{N}}$ の満たす $\Delta\, e^S = 0$ の関係式は 微分 として表せる：

$$\text{quantum } A_\infty / L_\infty \text{ 関係式} \qquad (\hbar\, \Delta + \mu)^2 = 0$$

- 特に、$(S, S) = 0$ を満たす古典作用が、量子補正なしで $\Delta\, e^S = 0$ を満たすとき、

$$A_\infty / L_\infty \text{ 関係式} \qquad \mu^2 = 0 \qquad \& \qquad \Delta\, \mu + \mu\, \Delta = 0$$

Homology/Homotopy代数による記述方法

書き換えの前後で、*Homology* 代数的な情報は等価

- 関係式 $(\Delta + \mu)^2 = 0$ より、$T(V)$ の複体上の **微分** として $\Delta + \mu$ を使える：

$$H_{\circlearrowleft}\left(\mathscr{F}\,,\, s + \hbar\Delta\right) \underset{I}{\overset{P}{\rightleftarrows}} \left(\mathscr{F}_{\rm phys}\,, s_{\rm eff} + \hbar\Delta_{\rm eff}\right) \quad \Longleftrightarrow \quad \mathsf{h}_{\circlearrowleft}\left(T(V)\,,\, \mu + \hbar\Delta\right) \underset{\mathsf{I}}{\overset{\mathsf{P}}{\rightleftarrows}} \left(T(V_{\rm phys})\,,\, \mu_{\rm eff} + \hbar\Delta_{\rm eff}\right)$$

- BRST-BV 形式での計算は、Homotopy 代数による記述に **書き換え** 可能。逆も然り。

(この際に「右作用と左作用が入れ換わる」ことに注意)

Homotopy 代数による経路積分の記述：　$\mathsf{P} = p\,\dfrac{1}{1-(\mu + \hbar\,\Delta)\,h}$

摂動

$$\mathsf{P}\left[A[\phi]\right] = e^{G_{\rm free}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}}\left[e^{S_{\rm int}[\phi]}A[\phi]\right]_{\phi=0} = \frac{1}{Z}\int \mathscr{D}[\phi]e^{S[\phi]}\left[A[\phi]\right]$$

BV 形式・homotopy 代数における、右・左作用の入れ換え

$$H \circlearrowleft \left(\mathscr{F} , s + \hbar\Delta \right) \underset{I}{\overset{P}{\rightleftarrows}} \left(\mathscr{F}_{\text{phys}} , s_{\text{eff}} + \hbar\Delta_{\text{eff}} \right) \quad \Longleftrightarrow \quad \mathsf{h} \circlearrowleft \left(T(V) , \mu + \hbar\Delta \right) \underset{\mathsf{I}}{\overset{\mathsf{P}}{\rightleftarrows}} \left(T(V_{\text{phys}}) , \mu_{\text{eff}} + \hbar\Delta_{\text{eff}} \right)$$

経路積分： $iP(\phi^N) = i\,\mathsf{P}(\phi^N)$ where $P = p \circ \sum_k (\Delta\, h)^k$ and $\mathsf{P} = p \circ \sum_k (\mathsf{h}\,\Delta)^k$

(多重)線形写像 $s_1, \ldots, s_n$ の合成は、引き戻しの合成などのように、$\mu_n \cdots \mu_1$ となる：

$$\mu_1 \cdots \mu_n(\phi) = \left[\mu_1 \cdots \mu_n \begin{pmatrix} e & e_* \end{pmatrix} \right] \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix} \overset{\text{transpose}}{=} \begin{pmatrix} e & e_* \end{pmatrix} \left[s_n \cdots s_1 \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix} \right] = s_n \cdots s_1 \, (\phi) \, .$$

The super-form $\phi = \varphi\, e + \varphi^* e_*$ with $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ switches $s \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix} = \begin{pmatrix} 0 \\ (\partial^2 - m^2)\phi \end{pmatrix}$ to $\mu_1(\phi) = (\partial^2 - m^2)\,\varphi\, e_* = \begin{pmatrix} e & e_* \end{pmatrix} s \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix}$.

Homology/Homotopy代数による記述方法

このような記述の利点

- Dyson 方程式：　$G(p) = G_{\mathrm{free}}(p) + G_{\mathrm{free}}(p)\,\Sigma(p)\,G(p)$

- この式は、Homology / Homotopy代数による記述から、自然に従う：

$$\mathsf{h} = \mathsf{h}_{\mathrm{free}} + \mathsf{h}\left(\mu_{\mathrm{int}} + \hbar\Delta\right)\mathsf{h}_{\mathrm{free}}$$

(ホモロジー摂動で得られる h)

- Homotopy 代数による記述では、他の項も Dyson 方程式 or その帰結：

例えば . . .　$\mu_{\mathrm{eff}} = \mu_{\mathrm{tree}} \dfrac{1}{1-\Sigma_{\mathrm{h.p.}}}$　($\Sigma_{\mathrm{h.p.}} = \mathsf{h}_{\mathrm{free}}\left(\mu_{\mathrm{int}} + \hbar\Delta\right)$)

- 射影 P， 単射 I ， 逆演算 h ， 微分 $\mu = \mu_1 + \mu_2 + \cdots$ などのデータへ翻訳

これまでに紹介したこと

- (かなり広いクラスの) 場の理論は、Homology 代数的に記述できる。

例えば、(摂動的) 経路積分は コホモロジーへの射影 として書ける。

e.g. Dyson 方程式 $G(p) = G_{\text{free}}(p) + G_{\text{free}}(p)\,\Sigma(p)\,G(p)$ は、homology代数で書くと、

“ホモロジー摂動論” からの自然な帰結として理解できる。

また、相互作用を含む場合であっても類似の式で表せる。

次に紹介すること

- 場の理論のいろいろな操作が、ホモロジー代数的なデータをいじることで書けそう。
- 摂動的な場の理論について、ある程度の理解が得られた。

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2. QFT calculations via BV Lagrangian's Homotopy algebra

Osaka Metropolitan Univ. College of Tech.

Hiroaki Matsunaga

はじめに

どのような模型を扱うのか

- 場の古典論：

自由場 $S_{\text{free}}[\phi] = \frac{1}{2}\phi^a \mu_{ab} \phi^b$ の Hessian が non-degenerate にとれて,

$S[\phi] = \frac{1}{2}\phi^a \mu_{ab} \phi^b + \text{interactions}$ という形の作用を持つ模型 .

- 場の量子論：

自由場の Gaussian integral が規格化できる模型： $\int \mathscr{D}[\phi]\, e^{\frac{1}{2}\phi^a \mu_{ab} \phi^b} = 1$.

E.g. ordinary Lagrangians : ϕ^4 scalar, Yang-Mills, lattice, strings and so on.

前半まとめ： Homology 代数的な記述

場の古典論

- **模型を考える (= 作用をつくる)**

作用の停留点を探す　　運動方程式を解く　　解 (Physical State) をとる

$$\delta S[\phi] = ? \quad \rightarrow \quad \frac{\delta S[\phi]}{\delta \phi} \overset{!}{=} 0 \quad \rightarrow \quad a \quad \text{s.t.} \quad \frac{\delta S}{\delta \phi}[a] = 0$$

- **次数付きベクトル空間と微分 (chain complex) を与える**

resolution を探す　　BRST-BV cohomology への射影を作る　　射影を施す

前半まとめ： Homology 代数的な記述

場の量子論

- **模型を考える (= 分配関数を作用で表す)**

SD eq. を与える　　摂動的に積分する　　積分値 (Physical states) を読む

$$\frac{\delta}{\delta\phi}\Big(\,\text{integrand}\,\Big) \equiv 0 \quad \rightarrow \quad \int \mathscr{D}[\phi]\,\Big(\,\text{integrand}\,\Big) \quad \rightarrow \quad A = \int \mathscr{D}[\phi]\, e^{S[\phi]} \Big(O_A\Big)$$

- **次数付きベクトル空間と微分 (chain complex) を与える**

resolution を探す　　BRST-BV cohomology への射影を作る　　射影を施す

BV Lagrangian → Homotopy algebra → Homological data

$$h \circlearrowright (\mathscr{F}, Q) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\rm phys}, Q_{\rm phys})$$ (strong deformation retract)

複体 $\mathscr{F}$ の与え方 $\Longleftrightarrow$ どのような場 (力学変数) を考えているのか

微分 Q の与え方 $\Longleftrightarrow$ どのような Lagrangian (模型の作用) を与えたか

射影 p の与え方 $\Longleftrightarrow$ どの場について「**e.o.m.を解く / 経路積分する**」か

逆演算 h の与え方 $\Longleftrightarrow$ どの**境界条件の下で解き, green関数を得た**か

BV 鎖複体（$\mathscr{F}_{\rm phys}, Q_{\rm phys}$） $\Longleftrightarrow$ 得られる「有効場の理論 (経路積分値) 」の情報

⟶ **板書： 簡単な例 (古典はわりと簡単, 量子論は非自明)**

How to calculate the following diagram

Homological approach to QFT calculations (Key : Hodge-dec.)

free, classical

$$h \circlearrowright (\mathscr{F}, s_{\text{free}}) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, 0)$$

$\xrightarrow{+\hbar\Delta}$

free, quantum

$$H' \circlearrowright (\mathscr{F}, s_{\text{free}} + \hbar\Delta) \underset{I'}{\overset{P'}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, 0)$$

$\downarrow +s_{\text{int}}$ (left) $\qquad$ **Homological Perturbation** $\qquad$ $\downarrow +s_{\text{int}}$ (right)

interacting, classical

$$h' \circlearrowright (\mathscr{F}, s_{\text{free}} + s_{\text{int}}) \underset{i'}{\overset{p'}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, s_{\text{tree}})$$

$\xrightarrow{+\hbar\Delta}$

interacting, quantum

$$H \circlearrowright (\mathscr{F}, s_{\text{free}} + s_{\text{int}} + \hbar\Delta) \underset{I}{\overset{P}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, s_{\text{eff}} + \hbar\Delta_{\text{eff}})$$

Initial data p, i, h & perturbation $s_{\text{int}} + \hbar\Delta \xrightarrow{\textit{systematic}}$ the wanted data P, I, H

Plan

(i) Path integral as a homological perturbation

(ii) Some examples in QFT calculations

(ii-a) Applications to "perturbative QFT"

(ii-b) Applications to "realization of symmetry"

(ii-c) Application to "non-perturbative effects"

If time permits

Path integral as a homological perturbation

Projection P = the perturbative path-integral

- We consider free field theories $S_{\text{free}}[\phi] = \frac{1}{2}\,\phi^a\,\mu_{ab}\,\phi^b$,

  which satisfies $\Delta\, S_{\text{free}}[\phi] = 0$ with $\Delta = (-)^{|\phi^a|}\frac{\delta}{\delta\phi^a}\frac{\delta}{\delta\phi^+_a}$.

  (The classical BV-BRST cohomology is empty except for $\cdots \xrightarrow{s|_{-1}} \mathscr{F}\,|_0 \xrightarrow{s|_0} 0$.)

- It solves not only the classical master equation $(\,S\,,S\,) = 0$ but also

  the quantum master equation $\hbar\,\Delta(\,e^{S[\phi]}\,) = \left[\,\hbar\,\Delta\,S + \frac{1}{2}(\,S\,,S\,)\,\right]\,e^{S[\phi]} = 0$.

- *The (quantum) BV-BRST operator* $s + \hbar\,\Delta$ *is the differential :* $(\,s + \hbar\,\Delta\,)^2 = 0$.

Path integral as a homological perturbation

Projection P = the perturbative path-integral

- We get a perturbation : $s = \mu_{ab}\phi^b \dfrac{\delta}{\delta\phi_a^+} \;\to\; s + \hbar\,\Delta$ of differentials .

- $(\mu^{-1})^{ab}$ of $h = (\mu^{-1})^{ab}\phi_b^+ \dfrac{\delta}{\delta\phi^a}$ is the Feynman propagator. We have $p = 0$ and $i = \mathrm{Id}$:

  Now, the fields $\phi = \underbrace{\phi_{\text{on shell}}}_{=0} + \phi_{\text{off shell}}$ are fluctuation around “0” and $p(\phi) = \underbrace{\phi_{\text{on shell}}}_{=0}$.

- From the original (classical) data $h \circlearrowright (\mathscr{F}, s) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, 0)$ + the perturbation $\hbar\,\Delta$ of differentials,

  we can construct a new (quantum) data $H \circlearrowright (\mathscr{F}, s + \hbar\,\Delta) \underset{I}{\overset{P}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, \tilde{\mu})$ by solving

  the recursion relations : $P = p + P(\hbar\,\Delta)h$ and $I = i + h(\hbar\,\Delta)I$.

Path integral as a homological perturbation

Projection P = the perturbative path-integral

- The solutions are $P = p \sum_n \left[(\hbar \Delta) h \right]^n$ and $I = \sum_n \left[h (\hbar \Delta) \right]^n i$.

- We can check that $IP = ip \sum_n \left[(\hbar \Delta) h \right]^n$ gives the path-integral.

We notice that p, i, Δ, h act on $a\,\phi^n \in \mathscr{F} \cong \mathrm{Sym}(\mathfrak{F}^*)$ $(a \in \mathbb{R})$ as follows:

$$p\,(a\,\phi^n) = a\,p\,(\phi^n) = a(p\,\phi)^n \,, \qquad i\,(a\,\phi^n) = a\,i\,(\phi^n) = a(i\,\phi)^n \,,$$

$$\Delta\,(a\,\phi^n) = a\,(-)^{|\phi^a|} \frac{\delta}{\delta \phi^a} \frac{\delta}{\delta \phi^+_a} (\phi^n) \,, \qquad h\,(a\,\phi^n) = a\,h\,(\phi^n) = a \sum_{k=1}^{n} \phi^{k-1} (h\,\phi) (i\,p\,\phi)^{n-k} \,.$$

Path integral as a homological perturbation

Projection P = the perturbative path-integral

- Note that $p\,(a\,\phi^n) = a\,(\,p\,\phi\,)^n$ means $p\,(\phi^n) = (\,p\,\phi\,)^n$ and $p\,(a) = a$ for $a \in \mathbb{R}$ and

  that $ip(\phi) = 0$ yields $h\,(\,a\,\phi^n) = a\,h\,(\,\phi^n) = a\,(h\phi)\,\phi^{n-1}$.

- We find

$$p\left[\,(\,\hbar\,\Delta\,)\,h\,\right]^m(\phi^{2n}) = p\left[\,(\,\hbar\,\Delta\,)\,h\,\right]^{m-1}(2n-1)\Big[\Big((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\phi\Big)\Big(\frac{\delta}{\delta\phi^a}\phi\Big)\Big]\phi^{2(n-1)}$$

$$= p\left[\,(\,\hbar\,\Delta\,)\,h\,\right]^{m-2}(2n-1)(2n-3)\Big[\Big((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\phi\Big)\Big(\frac{\delta}{\delta\phi^a}\phi\Big)\Big]^2\phi^{2(n-2)}$$

$$\vdots$$

$$= p\left[\,(\,\hbar\,\Delta\,)\,h\,\right]^{m-n}(2n-1)!!\Big[\Big((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\phi\Big)\Big(\frac{\delta}{\delta\phi^a}\phi\Big)\Big]^n$$

* It implies that m = n .

Path integral as a homological perturbation

Projection P = the perturbative path-integral

- We can further rewrite it as follows:

$$p\left[(\hbar\Delta)h\right]^{m}(\phi^{2n}) = p\left[(\hbar\Delta)h\right]^{m-n}(2n-1)!!\left[\left((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\phi\right)\left(\frac{\delta}{\delta\phi^a}\phi\right)\right]^n$$

$$= p\left[(\hbar\Delta)h\right]^{m-n}\frac{(2n-1)!!}{2}\left[\left((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\phi\right)\left(\frac{\delta}{\delta\phi^a}\phi\right)\right]^{n-1}\left[\left((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\frac{\delta}{\delta\phi^a}\right]\phi^2$$

$$\vdots$$

$$= p\left[(\hbar\Delta)h\right]^{m-n}\frac{1}{(2n)!!}\left[\left((\mu^{-1})^{ab}\frac{\delta}{\delta\phi^b}\frac{\delta}{\delta\phi^a}\right]^n\phi^{2n}$$

- As a result, we obtain

$$P(\phi^{2n}) = \frac{1}{n!}\left[\frac{1}{2}\frac{\delta}{\delta\phi^b}(\mu^{-1})^{ab}\frac{\delta}{\delta\phi^a}\right]^n\phi^{2n} \quad \text{and} \quad P(\phi^{2n+1}) = 0 \ .$$

Path integral as a homological perturbation

Projection P = the perturbative path-integral

- Projection $P = p + P(\hbar\Delta)h$ reproduces **Wick's theorem** !!

- With $IP = iP$, we get the desired result

$$IP\,(\dots) = e^{\frac{1}{2}\frac{\delta}{\delta\phi^b}(\mu^{-1})^{ab}\frac{\delta}{\delta\phi^a}}\,(\dots)\Big|_{\phi=0} = \int \mathscr{D}[\phi]e^{S_{\mathrm{free}}[\phi]}\left(e^{S_{\mathrm{int}}[\phi]}\dots\right) .$$

- Hence, the projection onto the (quantum) BV-BRST cohomology, "IP" of $H_{\circlearrowleft}(\mathscr{F}, s+\hbar\Delta) \underset{I}{\overset{P}{\rightleftarrows}} (\mathscr{F}_{\mathrm{phys}}, 0)$, indeed reproduces the path-integral.

Plan

(i) Path integral as a homological perturbation

(ii) Some examples in QFT calculations

(ii-a) Applications to “perturbative QFT”

(ii-b) Applications to “realization of symmetry”

(ii-c) Application to “non-perturbative effects”

If time permits

3. Application to perturbative QFT

App.1) Typical Examples of $\phi = \phi' + \phi''$

- Your effective theory $A[\phi']$ has A∞/L∞ corresponding to ***the splitting*** $\phi = \phi' + \phi''$ because it changes ***the propagators*** $(\mu_1'')^{-1}$ *given by* $\mu_1 = \mu_1' + \mu_1''$,

$$A[\phi'] \equiv \ln \int \mathscr{D}\phi'' \, e^{S[\phi'+\phi'']} = \frac{1}{2}\langle \phi', \mu_1' \phi' \rangle + \frac{1}{3}\langle \phi', \mu_2'(\phi', \phi') \rangle + \frac{1}{4}\langle \phi, \mu_3(\phi', \phi', \phi') \rangle + \cdots$$

- Typical examples :

(1) As usual, $\phi = \phi'_{\rm IR} + \phi''_{\rm UV}$ gives Wilsonian with *A∞/L∞ .*

(2) $\phi = \phi'_{\text{on shell}} + \phi''_{\text{off shell}}$ gives the S-matrix $A[\phi']$ as a minimal model of *A∞/L∞ .*

(3) $\phi = \phi'_{\rm massless} + \phi''_{\rm massive}$ gives *A∞/L∞* effective QFT $A[\phi']$ (finite α’ for strings) .

(4) $\phi = \phi'_{\rm phys} + \phi''_{\rm gauge+unphys}$ gives “gauge-removed” QFT with *A∞/L∞ .*

3. Application to perturbative SFT

Straightforward example: String Field Theory

- As is known, ***SFT is a consistent UV finite QFT***, which satisfies $\Delta\, e^{S[\phi]} = 0$.

  For a given master action $S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi)\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi, \phi, \phi)\rangle + \cdots$,

  we can consider $\phi = \phi' + \phi''$ and *the path-integral of* ϕ'' as follows

$$P \;:\; S[\phi' + \phi''] \;\longmapsto\; A[\phi'] = \ln \int \mathscr{D}\phi''\, e^{S[\phi' + \phi'']}$$

- Then, ***thanks to BV***, *the quantum A_∞/L_∞ of your effective action* is *automatic*

$$A[\phi'] = \frac{1}{2}\langle \phi', \mu_1' \phi' \rangle + \frac{1}{3}\langle \phi', \mu_2'(\phi', \phi')\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi', \phi', \phi')\rangle + \cdots$$

3. Application to perturbative SFT

App.2) Light-cone reduction : a special choice of $\phi = \phi'_{\rm phys} + \phi''_{\rm gauge+unphys}$

- For a given covariant SFT, there exists the corresponding light-cone SFT.
- The BRST operator of (super) strings has the similarity transformation, for example,

$$Q = e^{-R} \Big(\overbrace{c_0 L_0}^{\mu'_1} - p^+ \overbrace{\sum_{n \neq 0} c_{-c}\, a_n^+}^{\mu''_1} \Big) e^R \qquad \text{(open strings) .}$$

[Aisaka, Kazama 2004] for bosonic
[Kazama, Yokoi 2011] for super

It induces $\phi_{\rm covariant} = \phi'_{\text{light cone}} + \phi''_{a^\pm,\, b,\, c}$ and the A_∞ / L_∞ light-cone SFT :

$$A[\phi'] = \underbrace{\frac{1}{2} \langle \phi', c_0 L_0^{\rm lc} \phi' \rangle + \sum_n \frac{1}{n+1} \langle \phi', \mu_n(\phi', \dots, \phi') \rangle}_{\textit{same form as the covariant SFT}} + \underbrace{\sum_{g,n} \frac{\hbar^g}{n+1} \langle \phi, \mu'_{n,[g]}(\phi', \dots, \phi') \rangle}_{\textit{effective vertices}}$$

"When effective vertices vanish and it reduces to Kaku-Kikkawa's theory" will be reported by [corroboration with Ted Erler] .

3. Application to EFT

App.3) Realization of symmetry as A∞/L∞

- Recall that *composite operators of symmetry* $\delta_{\mathrm{sym}}\phi = \mathscr{O}_{\mathrm{sym}}[\phi]$ survive along **ERG flows**

$$\int \mathscr{D}\phi \, \mathscr{O}_{\mathrm{sym}}[\phi] \, e^{S[\phi]} = \int \mathscr{D}\phi' \, \mathscr{O}'_{\mathrm{sym}}[\phi'] \, e^{A[\phi']}$$

[Review: Y.Igarashi et al 2009]

  and there is *no loss of symmetry*, although their forms drastically change along flows.

$$\delta_{\mathrm{sym}}\phi = \mathscr{O}_{\mathrm{sym}}[\phi] \longmapsto \delta_{\mathrm{sym}}\phi' = \mathscr{O}'_{\mathrm{sym}}[\phi']$$

- This is also true for our case. The relation between $\mathscr{O}_{\mathrm{sym}}[\phi]$ and $\mathscr{O}'_{\mathrm{sym}}[\phi']$ is explicit. It is given by a morphism of A∞/L∞ :

$$\mathscr{O}'_{\mathrm{sym}}[\phi'] = P \left(\mathscr{O}_{\mathrm{sym}}[\phi] \right)$$

[in preparation]

  So, symmetry of the original QFT also exists in your effective QFT in terms of A∞/L∞ , though it may take some highly-nonlinear form. (e.g. Lorentz generators in light-cone SFT.)

応用例： くりこみ

古き好き “くりこみ” の記述

Free, classical

$$h \circlearrowright (\mathscr{F}, s_{\mathrm{free}}) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\mathrm{phys}}, 0)$$

→

Bare

$$\mathrm{h} \circlearrowright (\mathscr{F}, s_{\mathrm{free}} + s_{\mathrm{int}} + \hbar\Delta) \underset{\mathrm{I}}{\overset{\mathrm{P}}{\rightleftarrows}} (\mathscr{F}_{\mathrm{phys}}, s_{\mathrm{eff}} + \hbar\Delta_{\mathrm{eff}})$$

↓

Renormalized

$$\mathrm{h}_{\mathrm{R}} \circlearrowright (\mathscr{F}, s_{\mathrm{free}} + s_{\mathrm{int}} + \hbar\Delta + s_{c.t.}) \underset{\mathrm{I_R}}{\overset{\mathrm{P_R}}{\rightleftarrows}} (\mathscr{F}_{\mathrm{phys}}, s_{\mathrm{eff,R}} + \hbar\Delta_{\mathrm{eff,R}})$$

- 初期情報 p, i, h ＋ 摂動 $s_{int} + \hbar\Delta$ ＝ Bare action による Feynman 図展開 P
- 更にもう１度、 $s_{c.t.}$ によるホモロジー摂動を実行 → くり込まれた Feynman 図展開 $\mathrm{P_R}$

応用例： 古典解まわり

古典解 a のまわり： $0+\phi_{\text{old}} = a + \phi_{\text{new}}$

Free, classical

$$h \circlearrowright (\mathscr{F}, s_{\text{free}}) \underset{i}{\overset{p}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, 0)$$

→

摂動真空まわり

$$\mathsf{h} \circlearrowright (\mathscr{F}, s_{\text{free}} + s_{\text{int}} + \hbar\Delta) \underset{\mathsf{I}}{\overset{\mathsf{P}}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, s_{\text{eff}} + \hbar\Delta_{\text{eff}})$$

↓

古典解まわり

$$\mathsf{h}_a \circlearrowright (\mathscr{F}, s_{\text{free}} + s_{\text{int}} + \hbar\Delta + s_a) \underset{\mathsf{I}_a}{\overset{\mathsf{P}_a}{\rightleftarrows}} (\mathscr{F}_{\text{phys}}, s_{\text{eff},a} + \hbar\Delta_{\text{eff},a})$$

インスタントン
キンク解
etc.

. 初期情報 p, i, h ＋ 摂動 $s_{\text{int}} + \hbar\Delta$ ＝ 摂動真空 0 周りの Feynman 図展開 P

. 更にもう１度、 s_a によるホモロジー摂動を実行 → 解 a 周りの Feynman 図展開 P_a

応用例： 汎関数くりこみ群 (in progress)

$$\int dp\, \Lambda \frac{\partial \mu_1^{-1}(p)}{\partial \Lambda} \frac{\delta^2}{\delta\phi(-p)\delta\phi(p)}$$

Homology 代数的なデータの関係式となる

- 繰り込み群の方程式の記述： $\mu_1 = \dfrac{\mu_{1,bare}}{K_0 - K}$,　$K = K(p/\Lambda)$,　$P' = \int \mathscr{D}[\phi']\, e^{S[\phi+\phi']}$

例えば . . .
$$\Lambda \frac{d}{d\Lambda} \underbrace{\mathrm{P}'\left(e^{-S_{\mathrm{free}}[\phi]}\right)}_{\exp\, S_{\mathrm{int}}[\phi]_\Lambda} = \underbrace{\Delta\, (\partial_{\ln \Lambda} K)\, \mathrm{h}\, \mathrm{P}'\left(e^{-S_{\mathrm{free}}[\phi]}\right)}_{\Delta\text{-exact}}$$

$$\longrightarrow \quad \Lambda \frac{d}{d\Lambda} S_{\mathrm{int}}[\phi]_\Lambda = \frac{1}{2} \int dp\, \Lambda \frac{\partial \mu_1^{-1}(p)}{\partial \Lambda} \left[\frac{\delta^2 S_{\mathrm{int}}[\phi]_\Lambda}{\delta\phi(-p)\delta\phi(p)} + \frac{\delta S_{\mathrm{int}}[\phi]_\Lambda}{\delta\phi(p)} \frac{\delta S_{\mathrm{int}}[\phi]_\Lambda}{\delta\phi(-p)} \right]$$
(Polchinski eq.)

- これは「 経路積分 $\Longleftrightarrow$ BRST-BV コホモロジーへの射影 P 」と整合的
- くりこみ群は「 BV 方程式 / ホモトピー代数 を満たしつつ 」流れる

$\longrightarrow$ K.Costello, Igarashi-Itoh-Sonoda, Morris らの仕事と整合的

Plan

(i) Path integral as a homological perturbation

(ii) Some examples in QFT calculations

(ii-a) Applications to “perturbative QFT”

(ii-b) Applications to “realization of symmetry” ← If time permits

(ii-c) Application to “non-perturbative effects”

We learned that …

Lagrangian's homotopy algebraic structure : μ_{bv}

- For a given Lagrangian, we can solve the BV master equation $\Delta e^{S_{bv}[\varphi]} = 0$,

  which tells us Lagrangian's homotopy algebra $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

$$S_{bv}[\varphi] = \frac{1}{2}\omega\big(\,\varphi, \mu_1(\varphi)\,\big) + \frac{1}{3\,!}\omega\big(\,\varphi, \mu_2(\varphi, \varphi)\,\big) + \cdots$$

- Homological perturbation lemma describes the Feynman graph expansion.

  Hence, the path-integral *P* preserves the nilpotent property $P\,\mu_{bv} = \mu_{effective}\,P$.

P : homotopy alg. of the original QFT → (loop) homotopy alg. of its effective QFT

Reminder: how to get μ_{bv}

Quick review & notation in this talk

- Consider a master action $S_{bv}[\varphi] = S_{cl}[\phi] + \cdots$, which solves $\Delta\, e^{S_{bv}[\varphi]} = 0$.

- Write φ^a for all of fields and antifields collectively.

  e.g. For QED, $\varphi^a = A_\mu, c, \psi, \bar{\psi}$ (if any, antighosts & auxiliary fields) and their antifields.

- Rewrite our action into the contracted form :

$$S_{bv}[\varphi] = \sum_n \frac{1}{(n+1)!} \int dx\; \mu_{a_0 a_1 \ldots a_n}\, \varphi^{a_n} \ldots \varphi^{a_1} \varphi^{a_0}$$

BV symplectic form : ω_{ab}

$$= \sum_n \frac{1}{(n+1)!} \int dx\; \varphi^{a_0}\, \omega_{a_0 b} \left(\mu^b{}_{a_1 \ldots a_n}\, \varphi^{a_n} \ldots \varphi^{a_1} \right) .$$

Reminder: how to get μ_{bv}

How to get Lagrangian's L∞

- We can always start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{(n+1)!} \int dx\, \varphi^{a_0}\, \omega_{a_0 b} \left(\mu^b{}_{a_1 \dots a_n}\, \varphi^{a_n} \dots \varphi^{a_1} \right) \quad .$$

- We **assume** that $\mu_{a_0 a_1 \dots a_n}$ is **graded symmetric** $\mu_{\dots a\, b \dots} = (-)^{ab} \mu_{\dots b\, a \dots}$,

  which ensures the "cyclic property" $\mu_{a_0 a_1 \dots a_n} = (-)^{a_0(a_1 + \dots + a_n)} \mu_{a_1 \dots a_n a_0}$.

- Then, the condition $\Delta\, e^{S_{bv}} = 0$ gives **the (quantum) L∞ relations** :

$$\hbar\, \omega^{ab} \mu^c{}_{ab\, \underline{a_n \dots a_1}} + \frac{1}{2} \sum_m \frac{1}{m!(n-m)!}\, \mu^c{}_{\underline{a_n \dots a_{m+1}} b}\, \mu^b{}_{\underline{a_m \dots a_1}} = 0$$

underline denotes the right sum

Reminder: how to get μ_{bv}

You can weaken L_∞ 's assumption :

- We can always start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{(n+1)!} \int dx \, \varphi^{a_0} \, \omega_{a_0 b} \left(\mu^b{}_{a_1 \dots a_n} \, \varphi^{a_n} \dots \varphi^{a_1} \right) \quad .$$

- We **assume** that $\mu_{a_0 a_1 \dots a_n}$ is **graded symmetric** $\mu_{\dots a\, b \dots} = (-)^{ab} \mu_{\dots b\, a \dots}$,

  which ensures the "cyclic property" $\mu_{a_0 a_1 \dots a_n} = (-)^{a_0(a_1 + \dots + a_n)} \mu_{a_1 \dots a_n a_0}$.

- Then, the condition $\Delta \, e^{S_{bv}} = 0$ gives **the (quantum) L_∞ relations**.

→ *When we relax this assumption, we get (quantum) A_∞ .*

Reminder: how to get μ_{bv}

How to get Lagrangian's A∞

- We can start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{n+1} \int dx \; \varphi^{a_0} \, \omega_{a_0 b} \left(\mu^b{}_{a_1 \ldots a_n} \, \varphi^{a_n} \ldots \varphi^{a_1} \right) \quad .$$

- We just **assume** the "cyclic property" $\mu_{a_0 a_1 \ldots a_n} = (-)^{a_0(a_1+\cdots+a_n)} \mu_{a_1 \ldots a_n a_0}$ only.

- Then, the condition $\Delta \, e^{S_{bv}} = 0$ gives **the (quantum) A∞ relations**.

Reminder: how to get μ_{bv}

How to get Lagrangian's A∞

- We can start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{n+1} \int dx\, \varphi^{a_0}\, \omega_{a_0 b} \left(\mu^b{}_{a_1 \ldots a_n}\, \varphi^{a_n} \ldots \varphi^{a_1} \right) .$$

- We just **assume** the "cyclic property" $\mu_{a_0 a_1 \ldots a_n} = (-)^{a_0(a_1 + \cdots + a_n)} \mu_{a_1 \ldots a_n a_0}$ only.

- Then, the condition $\Delta\, e^{S_{bv}} = 0$ gives **the (quantum) A∞ relations**.

→ *Lagrangian's (quantum) A∞ algebra does not need an additional "matrix-like structure" or "space-time non-commutativity".*

But, when $\mu_{\ldots a\, b \ldots} = (-)^{ab} \mu_{\ldots b\, a \ldots}$ *comes from physics, A∞ may be physically redundant.*

Notation

The relation between ${\mu^b}_{a_1 \ldots a_n}$ *and* $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

- We can get the L∞ relation $\sum_m \frac{1}{m!(n-m)!} {\mu^c}_{\underline{a_n \ldots a_{m+1}} b} {\mu^b}_{\underline{a_m \ldots a_1}} = 0$ from $(S_{bv}, S_{bv}) = 0$,

  These give a "component" expression.

- As we can switch from $\partial_\mu j^\mu \approx 0$ to $d j^{D-1} \approx 0$ ($j^{D-1} = j^\mu \star dx^\mu$: (D-1)-form) ,

  we can switch from ${\mu^b}_{a_1 \ldots a_n}$ to $\mu_n : H^{\otimes n} \to H$ (coder $\mu_n : T(H) \to T(H)$) .

  (Now, instead of dx^μ , we need to consider $d\varphi^a$ as bases of *H*.)

- Then, we can obtain Lagrangian's homotopy algebra $(\mu_{bv})^2 = 0$

  where $\mu_{bv} \equiv \mu_1 + \mu_2 + \mu_3 + \cdots$ is a coderivation acting on $T(H)$ or $S(H)$.

These are Lagrangian’s homotopy algebras.

What I would like to tell you is as follows . . .

Today, I would like to tell you …

Symmetry's homotopy algebraic structure : μ_{sym}

1. Homotopy algebras μ_{sym} also appear in realization of given symmetries.

2. We can incorporate symmetry's μ_{sym} into Lagrangian's μ_{bv} and get

$$\big(\underbrace{\mu_{sym} + \mu_{bv} + \cdots}_{\equiv \, \mu_{total}} \big)^2 = 0 \; .$$

3. The Feynman graph expansion $P \equiv \int \mathscr{D}[\phi] e^{S_{free}[\phi]}/Z$ **preserves** this

$\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ in the sense that $P\,\mu_{total} = \mu'_{total}\,P$ with $(\mu'_{total})^2 = 0$.

Today, I would like to tell you …

What we can read from μ_{sym}

4. Homotopy algebraic structure μ_{sym} or $(\mu_{total})^2 = (\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

- tells us how to realize symmetries in every “effective” theory.

- naturally includes 1-form symmetries, etc.

- may explain why symmetry or anomaly remains under the path-integral,
  even if it may break the manifest invariance.

Homotopy alg. in symmetry

We consider a Lagrangian $S[\phi]$ without/with gauge degrees and suppose that $S[\phi]$ is invariant under $\delta\phi = \epsilon^a \cdot \delta_a \phi$ (ϵ^a : constants) .

- We will explain

(i) Homotopy algebra μ_{sym} appears in the realization of symmetry, which can be incorporated into homological data.

(ii) Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ under the path-integral

1. Homotopy algebra μ_{sym} in the realization of symmetries

We consider . . .

- First, we explain μ_{sym} intuitively within the canonical formalism.

  momentum π & the Poisson bracket $\{A, B\} \equiv A\left[\frac{\overleftarrow{\delta}}{\delta\phi^a}\frac{\delta}{\delta\pi_a} - \frac{\overleftarrow{\delta}}{\delta\pi_a}\frac{\delta}{\delta\phi^a}\right]B$

  — we see a homotopy algebra in QFTs :

  what is it / why or how it appears

- Next, we give a rigorous explanation by using the antifield formalism.

1. Homotopy algebra μ_{sym} in the realization of symmetries

In the canonical formalism, we find ...

We consider a Lagrangian $S[\phi]$ without gauge degree:

the canonical form $S[\phi] \longrightarrow S[\phi, \pi] = \int dx \left(\pi \cdot \dot{\phi} - H \right)$.

Suppose that $S[\phi]$ is invariant under $\delta\phi = \epsilon^a \cdot \delta_a \phi$ (ϵ^a : constants) .

- These global symmetries may or may not be linearly realized :

  The Poisson bracket gives $\epsilon^a \cdot \delta_a \phi = \epsilon^a \left\{ S_a[\phi, \pi] , \phi \right\}$.

- This $S_a[\phi, \pi] \sim \int dx \, \pi \cdot \delta_a \phi + \cdots$ is a **realization of symmetry generator**.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Global symmetry's Lie algebra in the canonical formalism

Suppose that a Lie algebra $[\,\hat{T}_a, \hat{T}_b\,] = f_{ab}{}^c\,\hat{T}_c$ is realized on-shell :

$$\{\, S_a[\phi,\pi]\,, S_b[\phi,\pi]\,\} \approx f_{ab}{}^c S_c[\phi,\pi] \qquad \text{(equality up to e.o.m.)}$$

- Notice that the action $S = S[\phi,\pi]$ generates **trivial** transformations

$$\{\, S\,, F[\phi,\pi]\,\} = \left(\frac{d\phi}{dt} - \frac{\delta H}{\delta \pi}\right)\cdot\frac{\delta F}{\delta \phi} + \left(\frac{d\pi}{dt} + \frac{\delta H}{\delta \phi}\right)\cdot\frac{\delta F}{\delta \phi} \approx 0$$

- By using functionals $S_{ab}[\phi,\pi]$, we can get **the off-shell equality** :

$$\{\, S_a[\phi,\pi]\,, S_b[\phi,\pi]\,\} = f_{ab}{}^c\; S_c[\phi,\pi] + \{\, S\;, S_{ab}[\phi,\pi]\,\}$$

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- Take $\{\, S_c\, , \quad \}$ of $\{\, S_a\, , S_b\, \} = f_{ab}{}^{c} S_c + \{\, S\ , S_{ab}\, \}$ and consider the cyclic sum :

$$\Big\{ S_c\ , \{\, S_a\, , S_b\, \} \Big\} + (cyclic) = \Big\{\, S_c\ , f_{ab}{}^{d} S_d + \{\, S\ , S_{ab}\, \} \,\Big\} + (cyclic)$$

- After some calculations, we get

$$\underbrace{S_k[\phi,\pi]\, f_{\underline{la}}{}^{k} f_{\underline{bc}}{}^{l}}_{Jacobi\ id.} = \Big\{\, S\ ,\ f_{\underline{ab}}{}^{k} S_{\underline{kc}}[\varphi] - \{\, S_{\underline{a}}[\phi,\pi]\ , S_{\underline{bc}}[\phi,\pi]\, \} \Big\}$$

- Both sides of this equality vanish separately.

- Notice that the r.h.s. takes the $\{\, S,\quad \}$-exact form.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- By using new functionals $S_{abc}[\phi,\pi]$, this (r.h.s.)=0 implies

$$\{ S_{\underline{a}}[\phi,\pi] , S_{\underline{bc}}[\phi,\pi] \} = f_{\underline{ab}}{}^{k} S_{k\underline{c}}[\phi,\pi] + \frac{1}{3} f_{\underline{abc}}{}^{k} S_{k}[\phi,\pi] + \frac{1}{3} \{ S \, , S_{\underline{abc}}[\phi,\pi] \}$$

- We get higher structure constants $f_{abc}{}^{d}$.

  We can repeat the same calculations by introducing a set of $\{S, S_a, S_{ab}, S_{abc}, \dots\}$:

$$\underbrace{\Big\{ S_{\underline{a_0}}[\phi,\pi], \{ S_{\underline{a_1 \dots}}[\phi,\pi] , S_{\underline{\dots a_k}}[\phi,\pi] \} \Big\}}_{order\ k\ Jacobi\ id.\ off-shell} = \underbrace{\{ S \, , \ \dots \ \}}_{\approx 0}$$

  Again, (l.h.s.) and (r.h.s.) vanish separately.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- We will get a set of structure constants $\{ f_{ab}{}^{c}, f_{abc}{}^{d}, f_{abcd}{}^{e}, \dots \}$, a set of generators $\{ S, S_a, S_{ab}, S_{abc}, \dots \}$, and a set of algebraic relations :

(r.h.s.) $$\sum_k \{ S_{\underline{a_1 \dots a_k}}[\phi, \pi] , S_{\underline{a_{k+1} \dots a_n}}[\phi, \pi] \} = \sum_l f_{\underline{a_1 \dots a_l}}{}^{b} S_{b\underline{a_{l+1} \dots a_n}}[\phi, \pi]$$

(l.h.s.) $$\sum_m \frac{1}{m!(n-m)!} f_{\underline{a_1 \dots a_m}}{}^{b} f_{b\,\underline{a_{m+1} \dots a_n}}{}^{c} = 0 \qquad \cdots \quad L_\infty\text{-relations}$$

- When there is no higher conservation low $\partial_\mu j^{\mu\nu} \approx 0$, higher $f_{abc}{}^{d}, \dots$ cannot occur.

The same goes for the BV formalism.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Symmetry in the BV formalism

- BV defines the odd Poisson bracket as follows.

the BV bracket : $$(A , B) \equiv A \left[\frac{\overleftarrow{\delta}}{\delta \phi^a} \frac{\delta}{\delta \phi^*_a} - \frac{\overleftarrow{\delta}}{\delta \phi^*_a} \frac{\delta}{\delta \phi^a} \right] B \quad .$$

- Odd symmetry generators $S_A[\varphi] \sim \int dx \, \phi^*_a \cdot \delta_A \phi^a + \cdots$ gives $\delta_A \phi = (S_A[\varphi] , \phi)$.

  These $S_A[\varphi]$ generate symmetries of the action: we have $(S_A[\varphi] , S) = 0$ now.

- The action is the trivial symmetry generator : $(S , \phi^*_a) = \dfrac{\overleftarrow{\delta} S[\phi]}{\delta \phi^a}$, $(S , \phi^a) = 0$

  — the action itself generates every on-shell vanishing functions.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Realization of symmetry

We first consider a Lagrangian $S[\phi]$ without gauge degree.

Then, the BV master action is this $S[\phi]$ itself.

Suppose that $S[\phi]$ is invariant under $\delta\phi = \epsilon^A \cdot \delta_A \phi$ (global sym) .

- For these constants ϵ^A , we introduce constant (or global) ghosts ξ^A .

  Then, the action $S[\phi]$ is still invariant under $\delta\phi = \xi^A \cdot \delta_A \phi$.

- We write φ for all $\phi \, , \phi^*$ correctively.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Realization of symmetry

- We can get generators $S_A[\varphi] = S_A[\phi, \phi^*]$ satisfying

$$\delta_A \phi = \big(\, S_A[\varphi] \, , \, \phi \, \big) \qquad \text{with the BV bracket } \big(\, A \, , B \, \big) \equiv A \left[\frac{\overleftarrow{\delta}}{\delta \phi^a} \frac{\delta}{\delta \phi_a^*} - \frac{\overleftarrow{\delta}}{\delta \phi_a^*} \frac{\delta}{\delta \phi^a} \right] B \, .$$

- These $S_A[\varphi]$ generate symmetries of the action: we have $\big(\, S_A[\varphi] \, , \, S \, \big) = 0$ now.

- We couple symmetry generators $S_A[\varphi] \sim \int dx \, \phi_a^* \cdot \delta_A \phi^a + \cdots$ to global ghosts ξ^A :

  The linear combination $S_A[\varphi] \, \xi^A$ is of ghost number zero and

  includes all of symmetry generators $\{ \, S_A \, \}_A$ for a given Lie algebra.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Realization of symmetry

- We can always find functionals $S_{AB}[\varphi]$ giving the off-shell equality :

$$\big(S_A \xi^A , S_B \xi^B \big) + \big(S , S_{AB} \xi^B \xi^A \big) = f_{AB}{}^C \, S_C \xi^B \xi^A \; .$$

- We can repeat the same calculations as before.
  (Every step is precise in BV, which is not intuitive one unlike before.)

We get $\displaystyle\sum_k \big(S_{\underline{A_1 \dots A_k}} \xi^{A_k} \dots \xi^{A_1} , S_{\underline{A_{k+1} \dots A_n}} \xi^{A_{k+1}} \dots \xi^{A_n} \big) = \sum_l f_{\underline{A_1 \dots A_l}}{}^B \, S_{B \, \underline{A_{l+1} \dots A_n}} \xi^{A_n} \dots \xi^{A_1}$

and L∞ relations $\displaystyle\sum_m \frac{1}{m!(n-m)!} f_{\underline{A_1 \dots A_m}}{}^B \, f_{B \, \underline{A_{m+1} \dots A_n}}{}^c \xi^{A_n} \dots \xi^{A_1} = 0$.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Realization of symmetry

- We can simplify the relation $\sum_m \frac{1}{m!(n-m)!} f_{\underline{A_1 \dots A_m}}{}^{B} f_{B\, \underline{A_{m+1} \dots A_n}}{}^{c}\, \xi^{A_n} \dots \xi^{A_1} = 0$.

- Let us consider the generating function of structure constants and the new bracket :

$$S_{sym}[\xi] = \sum_n \frac{1}{(n+1)!}\, \xi^*_B\, f^B{}_{A_1 \dots A_n}\, \xi^{A_n} \dots \xi^{A_1} \quad \text{and} \quad (\ ,\ )_\xi \equiv \frac{\overleftarrow{\partial}}{\partial \xi^A} \frac{\partial}{\partial \xi^*_A} - \frac{\overleftarrow{\partial}}{\partial \xi^*_A} \frac{\partial}{\partial \xi^A} \quad .$$

- Then, we find that $(\, S_{sym}[\xi]\, ,\, S_{sym}[\xi]\,)_\xi = 0$ is the same as the L∞ relations.

- BV tells us . . .

  the nilpotency of given $Q_\xi \equiv (\, S_{sym}[\xi]\, ,\quad)_\xi$ = homotopy algebras.

Comment :

A set of $\sum_m \frac{1}{m!(n-m)!} f_{\underline{a_1 \ldots a_m}}{}^{b} \, f_{b\,\underline{a_{m+1} \ldots a_n}}{}^{c} = 0$ *gives a homotopy algebra.*

— *This gives a kind of "component expression".*

Comment :

- We got the L∞-relations μ_{sym} & μ_{total} . What are these inputs?

  Q. What is the vector space H_ξ on which μ_{sym} acts ?

  A. The vector space of constants ghosts $\xi = \xi^A \cdot e_A$ (e_A are "bases")

  or its (symmetrized) tensor algebra $S(H_\xi)$.

- The structure constants $f_{A_1 \ldots A_n}{}^B$ are components of multilinear maps :

$$\mu_{sym}(e_{A_1}, \ldots, e_{A_n}) \equiv f_{A_1 \ldots A_n}{}^B e_B$$

- Then, the relations can be cast as $(\mu_{sym})^2 = 0$, which acts on the Fock sp $S(H_\xi)$.

Comment :

We got a homotopy algebra of given symmetries $\sum_m \frac{1}{m!(n-m)!} f_{\underline{a_1 \dots a_m}}{}^b \, f_{b\, \underline{a_{m+1} \dots a_n}}{}^c = 0$

— *This is invariant under the path-integral.*

This is another homotopy algebra that

is preserved under the path-integral.

1. Homotopy algebra μ_{sym} in the realization of symmetries

The BV master equation is now modified

- We also got $\sum_k \left(S_{\underline{A_1 \dots A_k}}[\varphi] \, , \, S_{\underline{A_{k+1} \dots A_n}}[\varphi] \right) = \sum_l f_{\underline{A_1 \dots A_l}}{}^{B} \, S_{B \underline{A_{l+1} \dots A_n}}[\varphi]$, which provides

  the action $S_{bv}[\varphi]$ and source terms $S_{source}[\varphi, \xi] \equiv \sum_k \frac{1}{k!} S_{A_1 \dots A_k}[\varphi] \, \xi^{A_k} \dots \xi^{A_1}$ satisfy

$$\frac{1}{2} \left(S_{bv}[\varphi] + S_{source}[\varphi, \xi] \, , \, S_{bv}[\varphi] + S_{source}[\varphi, \xi] \right) = - \sum_l \frac{1}{k!} \frac{\partial S_{source}[\varphi, \xi]}{\partial \xi^B} f^B{}_{A_1 \dots A_k} \xi^{A_k} \dots \xi^{A_1}$$

- Now, (S + ⋯ , S+ ⋯) = 0 is obstructed by given global symmetries.

1. Homotopy algebra μ_{sym} in the realization of symmetries

We can resolve it and get $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

- We consider the sum

$$S_{total}[\varphi,\xi] \equiv S_{bv}[\varphi] + \underbrace{\sum_k \frac{1}{k!} S_{A_1 \ldots A_k}[\varphi]\, \xi^{A_k} \ldots \xi^{A_1}}_{S_{source}[\varphi,\xi]} + \underbrace{\sum_n \frac{1}{(n+1)!}\, \xi^*_B\, f^B{}_{A_1 \ldots A_n}\, \xi^{A_n} \ldots \xi^{A_1}}_{S_{sym}[\xi]} \quad .$$

- We also consider the sum of the anti-brackets

$$(\quad , \quad)_{\varphi,\xi} \equiv \left[\frac{\overleftarrow{\delta}}{\delta \phi^a} \frac{\delta}{\delta \phi^*_a} - \frac{\overleftarrow{\delta}}{\delta \phi^*_a} \frac{\delta}{\delta \phi^a} \right] + \left[\frac{\overleftarrow{\partial}}{\partial \xi^A} \frac{\partial}{\partial \xi^*_A} - \frac{\overleftarrow{\partial}}{\partial \xi^*_A} \frac{\partial}{\partial \xi^A} \right] \quad .$$

- Then, we obtain $\big(S_{total}[\varphi,\xi] \,,\, S_{total}[\varphi,\xi] \big)_{\varphi,\xi} = 0$,

  which gives a homotopy algebra $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$.

Comment :

The relation to conservation lows

- We introduced constant ghosts ξ^A for $\delta\phi = \epsilon^A \cdot \delta_A \phi$.

  These ξ^A come from usual conservation lows $\partial_A j^A \approx 0$.

  In many cases, these ξ^A have ghost number "1".

- If there exist higher conservation lows $\partial_{\mu_1} j^{\mu_1 \mu_2 \cdots \mu_n} \approx 0$,

  constant ghosts which have ghost number "n" may appear.

So, when QFT has a 1-form symmetry, constant ghosts ξ^A which have ghost number "1" or "2" naturally appear in the above procedure.

Comment :

For QFT with gauge redundancy

- If your QFT has any gauge degree, first of all, you must solve the BV master equation and get a solution $S_{bv}[\varphi] = S[\phi] + \phi^* \delta\phi + \cdots$.

- You can apply the same calculations to $S_{bv}[\varphi]$, instead of $S[\phi]$.

  Then, you can see symmetries of gauge invariant QFTs.

- If you want to consider symmetries of a gauge-fixed theory $S_{BRS}[\phi]$,

  it is the same as QFTs without gauge degrees.

Summary of the above result:

- We can consider $\langle \ldots \rangle = \int [\mathscr{D}\phi]\,(\ldots)\; e^{S_{total}}/Z$ with

$$S_{total}[\varphi,\xi] \equiv S_{bv}[\varphi] + \underbrace{\sum_k \frac{1}{k!} S_{A_1 \ldots A_k}[\varphi]\, \xi^{A_k} \ldots \xi^{A_1}}_{S_{source}[\varphi,\xi]} + \underbrace{\sum_n \frac{1}{(n+1)!}\; \xi^*_B\; f^B{}_{A_1 \ldots A_n}\; \xi^{A_n} \ldots \xi^{A_1}}_{S_{sym}[\xi]} \;.$$

- $S_{total}[\varphi,\xi]$ gives $\mu_{total} = \mu_{bv} + \mu_{source} + \mu_{sym}$, where we have $(\mu_{total})^2 = 0$ and $(\mu_{bv})^2 = (\mu_{sym})^2 = 0.$

  One can show that the path-integral P satisfies

$$P\,(\mu_{bv} + \mu_{source} + \mu_{sym}) = (\mu'_{bv} + \mu'_{source} + \mu_{sym})\,P$$

  and P is given by a simple recursive relation in terms of these $\mu_{bv}, \mu_{source}, \mu_{sym}$.

Example: one possible μ_{total} in the Maxwell theory

- We consider the Maxwell theory : $S_{bv}[\varphi] = \int dx \Big[\frac{-1}{4} F_{\mu\nu}F^{\mu\nu} + A^{*\,\mu}\partial_\mu C \Big]$.

- Let us consider translations $\delta A_\mu = \epsilon^\nu \partial_\nu A_\mu$ and shifts $\delta A_\mu = \epsilon_{\mu\nu} x^\nu$ with $\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$.
  (The commutator is the gauge transformation with $\epsilon^\mu \epsilon_{\mu\nu} x^\nu$.)

  Usual currents → constant ghosts ξ_μ , $\xi_{\mu\nu}$ which have ghost # 1 appear.

- The Maxwell theory has higher order currents $\epsilon\, \partial_\mu F^{\mu\nu} \approx 0$, which gives constant shifts.
  → a constant ghost η which has ghost # 2 appears.

- $S_{sym}[\xi] = \int dx \Big[\underbrace{-\eta^* \xi^\mu \xi^\nu \xi_{\mu\nu}}_{f^d{}_{abc}} \Big]$ and $S_{source}[\varphi, \xi] = \int dx \Big[A^{*\mu}\big(\partial_\nu A_\mu\, \xi^\nu + x^\nu\, \xi_{\mu\nu}\big) + C^*\big(\partial_\mu C\, \xi^\mu + \underbrace{x^\mu\, \xi_{\mu\nu}\xi^\mu}_{f^c{}_{ab}} + \eta\big) \Big]$

(Likewise, 2-form abelian gauge theory $\int dx\, F_{\mu\nu\rho}F^{\mu\nu\rho}$ gives more interesting result.)

(i) Homotopy algebra μ_{sym} in the realization of symmetries

& How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

(ii) Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ under the path-integral

Comments:

We learned that $(S_{bv}, S_{bv}) = 0 \;\Leftrightarrow\; (\mu_{bv})^2 = 0$

- BV Lagrangian gives a homotopy algebra:

  $(S_{bv}, S_{bv}) = 0$ is $\sum_m \frac{1}{m!(n-m)!}\, \mu^c{}_{\underline{a_n \ldots a_{m+1}} b}\, \mu^b{}_{\underline{a_m \ldots a_1}} = 0$, which is $(\mu_{bv})^2 = 0$.

- Likewise,

  $(S_{sym}[\xi], S_{sym}[\xi])_\xi = 0$ is $\sum_m \frac{1}{m!(n-m)!} f_{\underline{A_1 \ldots A_m}}{}^B\, f_{B \underline{A_{m+1} \ldots A_n}}{}^c = 0$, which is $(\mu_{sym})^2 = 0$.

- Action + source + sym.-generator $S_{total} = S_{bv} + S_{source} + S_{sym}$ does :

  $(S_{total}, S_{total})_{\varphi,\xi} = 0$ gives $(\mu_{total})^2 = (\mu_{bv} + \mu_{source} + \mu_{sym})^2 = 0$.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

$(\mu_{total})^2 = 0$ *in "effective" theories*

- We first split $S_{bv}[\varphi]$ into the kinetic part $S_{free}[\varphi]$ and interacting part $S_{int}[\varphi]$:

$$S_{bv}[\varphi] = S_{free}[\varphi] + S_{int}[\varphi] \ , \ \text{which provides} \quad \mu_{bv} = \mu_1 + \overbrace{\mu_2 + \ldots}^{\mu_{int}} \ .$$

- We split fields $\phi = \phi' + \phi''$ and define a generic "effective" action by integrating out ϕ'' ,

$$P : \ S[\phi' + \phi''] \longmapsto A[\phi'] \equiv \ln \int D[\phi''] \, e^{S[\phi' + \phi'']} \ .$$

- Homological perturbation lemma guarantees that
  an effective one $\hbar \Delta' + (A[\phi'], \ )'$ is nilpotent, which gives $(\mu_{effective})^2 = 0$.

→ We can obtain $(\mu'_{sym})^2 = 0$ and $(\mu'_{total})^2 = 0$ recursively, as $(\mu'_{bv})^2 = 0$.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

$(\mu_{total})^2 = 0$ is preserved under the path-integral

- We know

  $(\, S_{free}[\varphi]\,, \quad)$ is nilpotent , which is $(\mu_1)^2 = 0$.

  $(\, S_{bv}[\varphi]\,, \quad) = (\, S_{free}[\varphi] + S_{int}[\varphi]\,, \quad)$ is nilpotent , which is $(\mu_1 + \mu_{int})^2 = 0$.

- Now, we got

  $(\, S_{total}[\varphi, \xi]\,, \quad)_{\varphi,\xi} = (\, S_{bv}[\varphi]\,, \quad) + (\, S_{source}[\varphi, \xi] + S_{sym}[\xi]\,, \quad)_{\varphi,\xi}$ is nilpotent,

  which is $(\mu_{total})^2 = (\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

$(\mu_{total})^2 = 0$ is preserved under the path-integral

- We also know

$\hbar\Delta + (S_{free}[\varphi], \;)$ is nilpotent , which is $(\hbar\Delta + \mu_1)^2 = 0$.

$\hbar\Delta + (S_{bv}[\varphi], \;) = \hbar\Delta + (S_{free}[\varphi] + S_{int}[\varphi], \;)$ is nilpotent , which is $(\hbar\Delta + \mu_{bv})^2 = 0$.

- As long as symmetries $\delta\phi$ are not anomalous, $\int D[\phi]\, e^{S[\phi]} = \int D[\phi + \delta\phi]\, e^{S[\phi+\delta\phi]}$, we may get

$\hbar\Delta + (S_{total}[\varphi, \xi], \;)_{\varphi,\xi} = \hbar\Delta + (S_{bv}[\varphi], \;) + (S_{source}[\varphi, \xi] + S_{sym}[\xi], \;)_{\varphi,\xi}$ is nilpotent,

which is $(\hbar\Delta + \mu_{total})^2 = (\mu_{sym} + \hbar\Delta + \mu_{bv} + \cdots)^2 = 0$.

→ We consider the Homological Perturbation connecting these.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

Free theories give the Gaussians, which fixes the ambiguity

- Since we can solve free QFTs, we start from a deformation retract of free theories :

$$h'' \circlearrowright \Big(\text{state space}\,,\ \underbrace{(S_{\text{free}}\,,\quad)}_{\mu_1'+\mu_1''} \Big) \underset{i''}{\overset{p''}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{(A_{free},\ )}_{\mu_1'} \Big)$$

  where a BV propagator h'' gives a Hodge decomposition : $\mu_1'' h'' + h'' \mu_1'' = 1 - i'' p''$.

- Even if the path-integral of ϕ'' breaks the manifest invariance,
  we can read (non-linear) realization of symmetries in effective theories.

→ HPL tells us recursive relations

Tree part : realization of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ in effective theories

$$h'' \circlearrowleft \Big(\text{state space}\,,\ \underbrace{(S_{\text{free}}\,,\quad)}_{\mu_1'+\mu_1''} \Big) \underset{i''}{\overset{p''}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{(A_{free},\ )}_{\mu_1'} \Big)$$

perturbation : $(S_{\text{free}}\,,\quad) \longmapsto (S_{\text{bv}}\,,\quad) \equiv (S_{\text{free}}\,,\quad) + (S_{\text{int}}\,,\quad)$ gives the tree graph expansion

$$h_{tree} \circlearrowleft \Big(\text{state space}\,,\ \underbrace{(S_{\text{bv}}\,,\quad)}_{\mu_1'+\mu_1''} \Big) \underset{i_{tree}}{\overset{P_{tree}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{(A[\phi'],\ )}_{\mu_{bv}'} \Big)$$

perturbation : $(S_{\text{bv}}\,,\quad) \longmapsto (S_{total}\,,\quad)_{\phi,\xi} \equiv (S_{bv}\,,\quad) + (S_{source} + S_{sym}\,,\quad)_{\phi,\xi}$

$$\tilde{h}_{tree} \circlearrowleft \Big(\text{state space}\,,\ \underbrace{(S_{total}\,,\quad)_{\phi,\xi}}_{\mu_{total}} \Big) \underset{\tilde{i}_{tree}}{\overset{\tilde{P}_{tree}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{(A_{total}[\phi',\xi],\ )_{\phi',\xi}}_{\mu_{total}'} \Big)$$

As the BG-current relation in a generic QFT,
we can get $\mu_{total}' \equiv \mu_{sym}' + \mu_{bv}' + \cdots$ from recursive relations.

Tree + loop : realization of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ in effective theories

$$h'' \circlearrowright \Big(\text{state space}\,,\ \underbrace{(\,S_{\rm free}\,,\quad)}_{\mu'_1+\mu''_1} \Big) \underset{i''}{\overset{p''}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1},\ \underbrace{(A_{free},\ )}_{\mu'_1} \Big)$$

perturbation : $(S_{\rm free}\,,\quad) \longmapsto \hbar\,\Delta + (S_{\rm free}\,,\quad)$ gives the Wick theorem

$$h_{\rm Wick} \circlearrowright \Big(\text{state space}\,,\ \underbrace{\hbar\Delta + (\,S_{\rm free}\,,\quad)}_{\hbar\Delta'+\mu'_1+\hbar\Delta''+\mu''_1} \Big) \underset{I_{\rm Wick}}{\overset{P_{\rm Wick}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1},\ \underbrace{\hbar\Delta' + (A[\phi'],\ )}_{\hbar\Delta'+\mu'_1} \Big)$$

perturbation to obtain $\hbar\Delta + (S_{total}\,,\quad)_{\phi,\xi}$

$$\tilde{h}_{\rm Wick} \circlearrowright \Big(\text{state space}\,,\ \underbrace{\hbar\Delta + (\,S_{total}\,,\quad)_{\phi,\xi}}_{\hbar\Delta+\mu_{total}} \Big) \underset{\tilde{i}_{\rm Wick}}{\overset{\tilde{P}_{\rm Wick}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1},\ \underbrace{\hbar\Delta' + (A_{total}[\phi',\xi],\ )_{\phi',\xi}}_{\hbar\Delta'+\mu'_{q-total}} \Big)$$

We can get $\mu'_{q-total} \equiv \mu'_{sym} + \mu'_{bv} + \cdots$, which includes $\hbar$, from recursive relations.

Example: Lorentz sym of light-cone SFT

- We consider Witten's open SFT : $S_{bv}[\varphi] = \frac{1}{2}\omega\big(\varphi, Q_{BRST}\varphi\big) + \frac{1}{3}\omega\big(\varphi, \mu_2(\varphi,\varphi)\big)$.
- This is manifestly Lorentz covariant : $\delta\varphi = \epsilon_{\mu\nu}\int d\sigma\, X^{\mu}(\sigma)\frac{\delta}{\delta X^{\nu}(\sigma)}\varphi$, which gives $S_{total}[\varphi,\xi]$.
- BRST operator has a similarity transformation $Q_{BRST} = e^{-R}\Big(\overbrace{c_0\, L_0^{lightcone}}^{\mu_1'} - p^{+}\overbrace{\sum_{n\neq 0} c_{-c}\, a_n^{+}}^{\mu_1''} \Big) e^{R}$.
- This gives Kato-Ogawa's no-ghost theorem:

$$h^{long} \circlearrowright \big(\text{covariant states}\,,\ \underbrace{Q_{BRST}}_{\mu_1'+\mu_1''}\big) \underset{i_{long}}{\overset{p^{long}}{\rightleftarrows}} \big(\underbrace{\text{lightcone states}}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{c_0\, L_0^{lightcone}}_{\mu_1'}\big)$$

Example: Lorentz sym of light-cone SFT

- As a result of the perturbation,

$$\tilde{h}^{long} \circlearrowleft \Big(\text{covariant states}\,,\ \underbrace{Q_{BRST} + \mu_2}_{\mu_{bv}} + \mu_{source+sym} \Big) \ \underset{\tilde{i}_{long}}{\overset{\tilde{p}^{long}}{\rightleftarrows}} \ \Big(\underbrace{\text{lightcone states}}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{c_0\, L_0^{lightcone} + \mu_{int}^{lightcone}}_{\mu_{lightcone}} + \mu_{source+sym}^{lightcone} \Big)\,,$$

  we obtain a Witten-type light-cone SFT with nonlinear Lorentz invariance.

- Classical light-cone action :

$$S_{lightcone}[\varphi_{phys}] = \frac{1}{2}\omega\big(\varphi_{phys}, c_0\, L_0^{lightcone}\varphi_{phys}\big) + \sum_{n=2}^{\infty} \frac{1}{n+1}\omega\big(\varphi_{phys}, \mu_n^{lightcone}(\varphi_{phys}, \ldots, \varphi_{phys})\big)$$

- Nonlinear Lorentz transformation :

$$\delta\,\varphi_{phys} = \mu_{Lorentz}[\varphi_{phys}, \xi] \equiv p^{long} \frac{1}{1 - (\mu_{total} - c_0 L_0^{lc}\,)\, h^{long}}\, i_{long}\, \delta\varphi^{cov}$$

- Lorentz symmetry follows from $[\,\mu_{lc}\,, \mu_{Lorentz}\,] = 0$ and cyclic property of $\mu_{Lorentz}$.

Summary

Symmetry's homotopy algebraic structure : μ_{sym}

1. Homotopy algebras μ_{sym} also appear in realization of given symmetries.

2. The Feynman graph expansion $P \equiv \int \mathscr{D}[\phi] e^{S_{free}[\phi]} / Z$ **preserves** this

$$\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$$ in the sense that $P\,\mu_{total} = \mu'_{total}\,P$ with $(\mu'_{total})^2 = 0$.

3. We can incorporate symmetry's μ_{sym} into Lagrangian's μ_{bv} and get

$$\big(\underbrace{\mu_{sym} + \mu_{bv} + \cdots}_{\equiv\, \mu_{total}}\big)^2 = 0 \ .$$

Summary

What we can read from μ_{sym}

4. Homotopy algebraic structure μ_{sym} or $(\mu_{total})^2 = (\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

- tells us how to realize symmetries in every “effective” theory.

- naturally includes 1-form symmetries, etc.

- may explain why symmetry or anomaly remains under the path-integral, even if it may break the manifest invariance.

Plan

(i) Path integral as a homological perturbation

(ii) Some examples in QFT calculations

(ii-a) Applications to “perturbative QFT”

(ii-b) Applications to “realization of symmetry”

(ii-c) Application to “non-perturbative effects”

⟶ Blackboard (if time permits)

全体のまとめ

- 場の理論でよくやる 基本的な操作 は、BRST-BV 形式 / ホモトピー代数 による記述を経由すると、**ホモロジー代数的なデータ**として計算できる。
- 既知の結果にも別の視点： 例えば、ダイソン方程式は **ホモロジー摂動の帰結** そのもの！
- “くりこみ” などの操作も、およそ、ホモロジー摂動で書ける。
- 場の理論の対称性は、homotopy 代数でよく書け、また、よく追えそう。

> 原理的には「BRST-BV 形式が適用できる場の理論を使って実行できる操作」は、すべて、ホモロジー代数・ホモトピー代数 の操作として書けるはず。

- Instanton effect も書けそう。

関連する先行研究

- 数学者による BRST-BV形式の研究 [1992頃~]
- H.Kajiura [2001] など (その他、近年の後発研究)
  BRST-BV形式/Homotopy代数と場の理論の関係や利用方法
- K.Costello の本 [2011, 16, 21]
  BV形式で くりこみ群と有効場の理論
  Factorization Algebra による「摂動的な場の理論の数学的定式化」
- Y.Igarashi, K.Itoh, H.Sonoda の研究 [~2009], T.Morrisらの研究[2019~]
  BRST-BV形式で、汎関数くりこみ群を扱う。

Thank you for your attention !

ご清聴 どうも ありがとうございました。

Supplement 1

BV & A_∞ in QFT

BV and A_∞ in QFT

Set up

- Let us consider a quantum field theory without gauge degree:

$$S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi)\rangle + \cdots$$

- As usual, the path-integral of free QFT can be performed as the Gaussian

$$\int \mathscr{D}\phi \; e^{\frac{1}{2}\langle \phi, \mu_1 \phi \rangle} = (\text{const.})\sqrt{\pi}\, \det \mu_1^{-\frac{1}{2}}$$

- For that, your QFT must has a regular Hessian

$$\det \mu_1 \neq 0 \quad \text{where} \quad (\mu_1)_{ab} = \frac{\partial^2 S[\phi]}{\partial \phi_a \partial \phi_b}$$

BV and A_∞ in QFT

Review of BV & A_∞

- The path-integral condensates “your field configurations” onto “your on-shell”.
- Classically, the on-shell is given by solving your e.o.m. , which is in **the kernel** of the operator $\frac{\partial S[\phi]}{\partial \phi} = \mu_1(\phi) + \mu_2(\phi, \phi) + \cdots = 0$
- **Any** function $F_{\text{e.o.m.}}[\phi]$ proportional to the **e.o.m.** vanish on shell.
  Hence, the field transformation $\delta\phi = F_{\text{e.o.m.}}[\phi]$ gives (trivial) gauge degrees, which must be removed in the path-integral.
- Such gauge degrees can be killed **by adding extra fields** — Koszul-Tate.

BV and A_∞ in QFT

Review of BV & A_∞

- Let us add extra fields ϕ^* and consider the operator Q defined by S :

$$Q \equiv (S, \ \ ) = \frac{\partial_r S}{\partial \phi}\frac{\partial}{\partial \phi^*} - \frac{\partial_r S}{\partial \phi^*}\frac{\partial}{\partial \phi} \quad \longrightarrow \quad Q\,\phi^* = (\mathrm{e\,.\,o\,.\,m.})$$

- Thanks to extra fields, any $F_{\mathrm{e.o.m.}}[\phi]$ becomes Q-exacts.

- As a result, by extending your field configuration to $\{\phi, \phi^*\}$,
  the on-shell is given by the Q-cohomology.

- We can remove trivial gauge degrees by setting $\phi^* = 0$ in the path-integral

$$Z = \int \mathscr{D}\phi \, e^{S[\phi]} = \int \mathscr{D}[\phi, \phi^*] \; \delta(\phi^*) \, e^{S[\phi]}$$

BV and A_∞ in QFT

Review of BV & A_∞

- The consistency condition $QS[\phi] = 0$ is called the classical master equation :

$$QS = \frac{\partial_r S}{\partial \phi}\frac{\partial S}{\partial \phi^*} - \frac{\partial_r S}{\partial \phi^*}\frac{\partial S}{\partial \phi} \equiv (S, S) = 0$$

- The solution of $(S, S) = 0$ is call the master action — Recall that it has A_∞ !!

- If your QFT has no gauge degree, this S is the same as a given classical action.

- As is known, scalar, Dirac, Maxwell or Yang-Mills QFT can be described by BV. So, ordinary QFT has A_∞ — In particular, we can switch from A to L .

- **Example: Classical Scalar field**

the BV master action : $S[\phi] = \frac{1}{2}\varphi\,(\partial^2 - m^2)\,\varphi + \frac{\kappa}{3\,!}\varphi^3 + \frac{\lambda}{4\,!}\varphi^4$

Since $S[\phi]$ has no ϕ^* dependence, Q acts only non-trivially on ϕ^* as follows:

$$Q\,\varphi = 0\ , \qquad Q\,\varphi^* = (\,S\,,\varphi^*\,) = (\partial^2 - m^2)\phi + \frac{\kappa}{2}\varphi^2 + \frac{\lambda}{3\,!}\varphi^3\ ,$$

which acts on the components : $\underbrace{0}_{\text{ghost}} \xleftarrow{Q} \underbrace{H_\varphi}_{\text{field}} \xleftarrow{Q} \underbrace{H_{\varphi^*}}_{\text{antifield}} \xleftarrow{0} \underbrace{0}_{\text{ghost*}}$.

→ We assign the basis $\{e, e_*\}$ and use a "super-field" $\phi = \varphi\,e + \varphi^*\,e_*$.

Then, we find A_∞ str $\mu_1(\phi) = (\partial^2 - m^2)\,\varphi\,e_*\,,\quad \mu_2(\phi,\phi) = \kappa\,\varphi^2\,e_*\,,\quad \mu_3(\phi,\phi,\phi) = \lambda\,\varphi^3\,e_*$,

which acts on the basis : $\underbrace{0}_{\text{ghost}} \xrightarrow{0} \underbrace{H_e}_{\text{field}} \xrightarrow{\mu} \underbrace{H_{e_*}}_{\text{antifield}} \xrightarrow{\mu} \underbrace{0}_{\text{ghost*}}$.

BV and A_∞ in QFT

Review of BV & the path-integral

- **The bare action for scalar fields** might be too trivial to see it.
  But, anyway, such ordinary theories indeed have A_∞ structure.
  So, you can apply **all technique** that we see later to such QFTs.

- As we see later, A_∞ or L_∞ becomes slightly non-trivial and will be helpful when we consider effective field theory or RG flows.

- **The bare action for gauge theory** will give a good example.

BV and A_∞ in QFT

Review of BV & the path-integral

- What will happen if your quantum field theory has **gauge degrees**?

  Your Lagrangian $S_{\rm cl}[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi)\rangle + \cdots$ is invariant under

  the gauge transformation $\delta \phi_a(x) = \int dy\, R_a{}^b(\phi; x, y)\, \phi_b(y)$

- This implies that your e.o.m. and thus Hessian are degenerate

$$\delta S_{\rm cl} = \underbrace{\frac{\partial S_{\rm cl}[\phi]}{\partial \phi_a}}_{\text{e.o.m.}} \delta \phi_a = \underbrace{\left[\frac{\partial S_{\rm cl}[\phi]}{\partial \phi_a} \cdot R_a{}^b \right]}_{=0} \phi_b = 0 \qquad \longrightarrow \qquad \det \frac{\partial^2 S_{\rm cl}[\phi]}{\partial \phi_a \partial \phi_b} \cong 0$$

- It is algebraically problematic, even for the Gaussian.

BV and A_∞ in QFT

Review of BV & the path-integral

- To remedy this kind of degeneracy, you need ghosts

$$\delta S_{\rm cl} = \underbrace{\frac{\partial S_{\rm cl}[\phi]}{\partial \phi_a}}_{\rm e.o.m.} \delta\phi_a = \underbrace{\left[\frac{\partial S_{\rm cl}[\phi]}{\partial \phi_a} \cdot R_a{}^b\right]}_{=0} \phi_b = 0 \qquad \longrightarrow \qquad S = S_{\rm cl}[\phi] + (\phi^*)^a R_a{}^b\, c_b$$

- If your gauge symmetry is **NOT redundant**, the Hessian can be regular in your extended field contents $\varphi = \{\, \phi, c \,\}$ — as Lagrange multipliers.

- If your gauge symmetry is **redundant**, you need "ghosts for ghosts"

$$R_a{}^b\, \delta\phi_b = \underbrace{R_a{}^b \cdot T_b{}^c}_{=0}\, \phi_c = 0 \qquad \longrightarrow \qquad S_{\rm cl}[\phi] + (\phi^*)^a R_a{}^b c_b + (c^*)^a T_a{}^b \eta_b$$

BV and A_∞ in QFT

Review of BV & the path-integral

- The fields $\varphi = \{\phi, c, \dots\}$ are determined by studying the gauge algebra.

- The antifields $\varphi^* = \{\phi^*, c^*, \dots\}$ must be added to kill each **trivial** gauge degree.

- **The extended action** $S[\varphi, \varphi^*]$ is given by solving **the consistency equation**

$$(S, S) = 0 \quad \text{where} \quad (A, B) \equiv \frac{\partial_r A}{\partial \varphi} \cdot \frac{\partial B}{\partial \varphi^*} - \frac{\partial_r A}{\partial \varphi^*} \cdot \frac{\partial B}{\partial \varphi}$$

- Then, you can define **the path-integral** by fixing gauge degrees $\varphi^* = F[\phi]$

$$Z = \int \mathscr{D}[\varphi, \varphi^*]\, \delta(\varphi^* - F[\varphi])\, e^{S[\varphi, \varphi^*]}$$

BV and A_∞ in QFT

Review of BV & the path-integral

- **Example: Yang-Mills field**

  the classical action $S_{\rm cl}[A] = -\frac{1}{4}\int dx\, F_{\mu\nu}F^{\mu\nu}$ can be rewritten as the form of

  $S_{\rm cl}[A] = \frac{1}{2}\langle A, \tilde{\mu}_1 A \rangle + \frac{1}{3}\langle A, \tilde{\mu}_2(A,A) \rangle + \frac{1}{4}\langle A, \tilde{\mu}_3(A,A,A) \rangle$ where A_∞ products are

$$\tilde{\mu}_1(A) = d \star d\, A$$

$$\tilde{\mu}_2(A_1, A_2) = d \star (A_1 \wedge A_2) - (\star dA_1) \wedge A_2 + A_1 \wedge (\star dA_2)$$

$$\tilde{\mu}_3(A_1, A_2, A_3) = A_1 \wedge (\star (A_2 \wedge A_3)) - (\star (A_1 \wedge A_2)) \wedge A_3$$

- This A_∞ is **incomplete** and a part of the **full** A_∞ of the Yang-Mills theory.

BV and A_∞ in QFT

Review of BV & the path-integral

- The Yang-Mills field action $S_{\rm cl}[A] = -\frac{1}{4}\int dx\, F_{\mu\nu}F^{\mu\nu}$ is invariant under the gauge transf. $\delta A_\mu = D_\mu \lambda = \partial_\mu \lambda + [\, A_\mu \,, \lambda \,]$.

- We thus need the ghost contribution $(A^*_\mu) \cdot D_\mu\, c$ and must solve **(S , S) = 0** :

  The master action is given by $S = S_{\rm cl}\,[A] + (A^*_\mu) \cdot D_\mu\, c + \frac{1}{2} c^* \cdot [\, c \,, c \,]$,

  which enables us to perform the path-integral.

- The BRST transf. $\delta\varphi = (\, \varphi \,, S \,) = \mu_1(\varphi) + \mu_2(\varphi,\varphi) + \cdots$ tells us the full A_∞ structure.

BV and A_∞ in QFT

Review of BV & the path-integral

- We can find **the full A_∞ structure**, which is given by

$$\mu_1 \begin{pmatrix} c \\ A \\ A* \\ c* \end{pmatrix} = \begin{pmatrix} 0 \\ d\,c \\ d\star d\,A \\ d\star\ A \end{pmatrix} \quad \text{with} \quad \underbrace{H_c}_{\text{ghost}} \xrightarrow{\mu_1=d} \underbrace{H_\phi}_{\text{field}} \xrightarrow{\mu_1=d\star d} \underbrace{H_{\phi*}}_{\text{antifield}} \xrightarrow{\mu_1=d} \underbrace{H_{c*}}_{\text{ghost*}} \quad ,$$

$$\mu_2\left(\phi_1,\phi_2\right) = \begin{pmatrix} c_1c_2 \\ c_1A_2 + A_1c_2 \\ \tilde{\mu}_2(A_1,A_2) - c_1A_2^* + A_1^*c_2 \\ c_1c_2^* - c1^*c_2 - \star\left(A_1\wedge(\star A_2^*)\right) + \star\left(A_1^*\wedge\star A_2\right) \end{pmatrix}, \quad \mu_3\left(\phi_1,\phi_2,\phi_3\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\mu}_3(A_1,A_2,A_3) \end{pmatrix}, \quad \phi_i = \begin{pmatrix} c_i \\ A_i \\ A_i^* \\ c_i^* \end{pmatrix} \quad \text{(i=1~3).}$$

- As is known, **the Berends-Giele current recursion relations** can be quickly derived from this A_∞ structure of the Yang-Mills theory.

Comments on quantum BV

- We reviewed around **the consistency equation**, which is nothing but A_∞ ,

$$(S, S) = 0 \quad \text{where} \quad (A, B) \equiv \frac{\partial_r A}{\partial \varphi} \cdot \frac{\partial B}{\partial \varphi^*} - \frac{\partial_r A}{\partial \varphi^*} \cdot \frac{\partial B}{\partial \varphi}$$

- It is the classical part of **the quantum consistency equation**

$$\Delta e^S = \left[\Delta S + \frac{1}{2}(S, S) \right] e^S = 0 \quad \text{where} \quad \Delta \equiv \frac{\partial_r}{\partial \varphi} \cdot \frac{\partial}{\partial \varphi^*}$$ is the odd Laplacian.

- Unless your symmetry is **anomalous**, $\Delta S = 0$ holds and $(S, S) = 0$ is enough.

- The quantum consistency equation $\Delta e^S = 0$ is nothing but the quantum A_∞

Lagrangian's L_∞ *algebra*

BV and L_∞ in QFT

Our Lagrangian's homotopy algebra

- We consider a QFT without gauge degree: $S[\varphi] = \sum_n \frac{1}{(n+1)!} \int dx \, (\varphi^a \omega_{ab}) \, \mu_{a_0 a_1 \ldots a_n}{}^b \, \varphi^{a_n} \ldots \varphi^{a_1} \varphi^{a_0}$

  (Or assume that we could perform the Legendre transformation / gauge-fixing
  and could obtain 1PI action / path-integrable gauge-fixed action : $S_{1PI} \, / \, S_{BRS}$.
  Then, vertices of $S_{1PI} \, / \, S_{BRS}$ may or may not have explicit $\hbar$ dependence.)

- In this case, we can find $(S_{bv} , S_{bv}) = 0$ (and $\Delta \, S_{bv} = 0$ with $\Delta \equiv (-)^{|\phi^a|} \frac{\delta}{\delta \phi^a} \frac{\delta}{\delta \phi^*_a}$) .

- The classical BV master equation $(S_{bv} , S_{bv}) = 0$ gives

  the (cyclic) L∞ relations $\sum_m \frac{1}{m!(n-m)!} \, \mu^c{}_{\underline{a_n \ldots a_{m+1}} b} \, \mu^b{}_{\underline{a_m \ldots a_1}} = 0$.

BV and L_∞ in QFT

The relation between $\mu^b{}_{a_1 \dots a_n}$ *and* $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

- The relation $\sum_m \frac{1}{m!(n-m)!} \mu^c{}_{\underline{a_n \dots a_{m+1}} b} \, \mu^b{}_{\underline{a_m \dots a_1}} = 0$ is a "component" expression.

- As $\partial_\mu j^\mu \approx 0$ and $d j^{D-1} \approx 0$,

  we can switch from $\mu^b{}_{a_1 \dots a_n}$ to $\mu_n : H^{\otimes n} \to H$ (coder $\mu_n : T(H) \to T(H)$) .

  (Now, instead of dx^μ , we need to consider $d\varphi^a$ as bases of *H*.)

- So, we can get $(\mu_{bv})^2 = (\mu_1 + \mu_2 + \cdots)^2 = 0$ from $(S_{bv}, S_{bv}) = 0$.

- Then, your action takes $S[\varphi] = \frac{1}{2}\omega\big(\varphi, \mu_1(\varphi)\big) + \frac{1}{3!}\omega\big(\varphi, \mu_2(\varphi,\varphi)\big) + \cdots$

BV and L_∞ in QFT

Lagrangian's homotopy algebraic structure : μ_{bv}

- For a given Lagrangian, we can solve the BV master equation $\Delta S + \frac{1}{2}(S,S) = 0$,

  which tells us Lagrangian's homotopy algebra $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

$$S_{bv}[\varphi] = \frac{1}{2}\omega\big(\,\varphi, \mu_1(\varphi)\,\big) + \frac{1}{3\,!}\omega\big(\,\varphi, \mu_2(\varphi,\varphi)\,\big) + \cdots$$

- Homological perturbation lemma describes the Feynman graph expansion.

  Hence, the path-integral *P* preserves the nilpotent property $P\,\mu_{bv} = \mu_{effective}\,P$.

P : homotopy alg. of the original QFT → (loop) homotopy alg. of its effective QFT

Examples:
Lagrangian’s homotopy algebras

BV and L_∞ in QFT

The BV master equation tells us the following 3 steps :

- Identify H and a complete set of fields $\varphi = \{\phi^a, c^b, \dots ; \phi_a^*, c_b^*, \dots\} \in H$
- Identify its *graded symplectic form* $\langle \varphi^a, \varphi^b \rangle = \omega^{ab}$ and *the kinetic operator* $\boldsymbol{\mu_1}$

  — Your free action takes the form of $S_{\text{free}}[\varphi] = \frac{1}{2} \langle \varphi, \mu_1 \varphi \rangle$

- Identify the multilinear maps $\{\boldsymbol{\mu_n}\}_{n=2}^{\infty}$ from interacting terms.

  — *Your vertices are given by* $S_{\text{int}}[\varphi] = \sum_n \frac{1}{n+1} \langle \varphi, \mu_n(\varphi, \dots, \varphi) \rangle$

Example 1A. Chern-Simons-type QFTs

3d Chern-Simons theory

- The state space takes $\underbrace{H_c}_{\text{ghost}} \xrightarrow{\mu_1=d} \underbrace{H_A}_{\text{field}} \xrightarrow{\mu_1=-d} \underbrace{H_{A*}}_{\text{antifield}} \xrightarrow{\mu_1=d} \underbrace{c*}_{\text{ghost*}}$ and thus $H = \underbrace{H_c}_{1} \oplus \underbrace{H_A}_{0} \oplus \underbrace{H_{A*}}_{-1} \oplus \underbrace{H_{c*}}_{-2}$

- Field variable is $\varphi = \{A, c\,; A*, c*\} \in H$ *and the symplectic form is* $\langle \varphi_1, \varphi_2 \rangle = \langle A_1, A_2^* \rangle - \langle c_1, c_2^* \rangle$

- The kinetic operator and multilinear maps can be read as follows :

$$\mu_1 \begin{pmatrix} c \\ A \\ A* \\ c* \end{pmatrix} = \begin{pmatrix} 0 \\ d\,c \\ -d\,A \\ d\,A* \end{pmatrix} , \quad \mu_2 \begin{pmatrix} c_1 & c_2 \\ A_1 & A_2 \\ A_1^* \,, & A_2^* \\ c_1^* & c_2^* \end{pmatrix} = \begin{pmatrix} -c_1 \wedge c_2 \\ c_1 \wedge A_2 + A_1 \wedge c_2 \\ -A_1 \wedge A_2 - c_1 A_2^* + A_1^* c_2 \\ A_1 \wedge A_2^* - A_1^* \wedge A_2 - c_1 \wedge c_2^* + c1^* \wedge c_2 \end{pmatrix} , \quad \mu_{n\geq 3} = 0\,.$$

— *We find* $S_{BV}[\varphi] = \frac{1}{2} \langle \varphi, \, \mu_1 \varphi \rangle + \frac{1}{3} \langle \varphi, \, \mu_n(\varphi, \varphi) \rangle$

Example 1B. Chern-Simons-type QFTs

Open string field theory

- *The state space :* $\cdots \xrightarrow{\mu_1=Q} \underbrace{H_{\phi_1}}_{\text{ghost}} \xrightarrow{\mu_1=Q} \underbrace{H_{\phi}}_{\text{field}} \xrightarrow{\mu_1=Q} \underbrace{H_{\phi^*}}_{\text{antifield}} \xrightarrow{\mu_1=Q} \underbrace{H_{\phi_1^*}}_{\text{ghost*}} \xrightarrow{\mu_1=Q} \cdots$ and $H = \bigoplus_{g\in\mathbb{N}} \underbrace{H_{\phi_g}}_{g} \oplus \underbrace{H_{\phi}}_{0} \oplus \underbrace{H_{\phi_g^*}}_{-g}$

- *The field* $\varphi = \{\phi, \phi_1, \ldots; \phi^*, \phi_1^*, \ldots\} \in H$ *and the symplectic form* $\langle \varphi_1, \varphi_2 \rangle = \sum_g (-)^g \langle \phi_g, \phi_g^* \rangle$

- The homotopy algebraic structure : $\mu_1 \begin{pmatrix} \vdots \\ \phi_1 \\ \phi \\ \phi^* \\ \phi_1^* \\ \text{\textbackslash vdots} \end{pmatrix} = \begin{pmatrix} \vdots \\ Q\,\phi_2 \\ Q\,\phi_1 \\ Q\,\phi \\ Q\,\phi^* \\ \vdots \end{pmatrix}$, $\mu_2 \begin{pmatrix} \vdots & \vdots \\ \phi_1 & \tilde{\phi}_1 \\ \phi_1 & \tilde{\phi}_2 \\ \phi_1^* , & \tilde{\phi}^* \\ \phi_1^* & \tilde{\phi}_1^* \\ \vdots & \vdots \end{pmatrix} = \sum_g \begin{pmatrix} \vdots \\ \phi_g \star \phi_{1-g} \\ \phi_g \star \phi_{-g} \\ \phi_{g-1} \star \phi_{-g} \\ \phi_{g-2} \star \phi_{-g} \\ \vdots \end{pmatrix}$, $\mu_{n\geq 3} = 0$.

— *We find* $S_{BV}[\varphi] = \frac{1}{2}\langle \varphi, \mu_1 \varphi \rangle + \frac{1}{3}\langle \varphi, \mu_n(\varphi, \varphi) \rangle$

Example 2. scalar fields $S[\phi] = \frac{1}{2}\phi\,(\partial^2 - m^2)\,\phi + \frac{\lambda}{4}\phi^4$

This QFT has no gauge degree

- The state space : $\underbrace{0}_{\text{ghost}} \xrightarrow{\mu_1=0} \underbrace{H_\phi}_{\text{field}} \xrightarrow{\mu_1=\partial^2-m^2} \underbrace{H_{\phi^*}}_{\text{antifield}} \xrightarrow{\mu_1=0} \underbrace{0}_{\text{ghost*}}$ and $H = \underbrace{H_\phi}_{0} \oplus \underbrace{H_{\phi^*}}_{-1}$

- Field is $\varphi = \{\phi ; \phi^*\}$ and the symplectic form is $\langle \varphi, \varphi \rangle = \langle \phi, \phi^* \rangle$

- Homotopy algebraic structure :

$$\mu_1 \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} = \begin{pmatrix} 0 \\ (\partial^2 - m^2)\,\phi \end{pmatrix} \;,\; \mu_2 = 0 \;,\; \mu_3 \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1^*, & \phi_2^*, & \phi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda\,\phi_1\,\phi_2\,\phi_3 \end{pmatrix} \;,\; \mu_{n\geq 4} = 0$$

- $S_{BV}[\varphi]$ has no ϕ^* dependence, which is nothing but the classical action.

— We find $S_{BV}[\varphi] = \frac{1}{2}\langle \varphi, \mu_1\,\varphi \rangle + \frac{1}{n+1}\langle \varphi, \mu_n(\varphi, \ldots, \varphi) \rangle = S[\phi]$

Example 3. YangMills-type QFTs

We also find $S_{BV}[\varphi] = \frac{1}{2}\langle\, \varphi,\, \mu_1\,\varphi\,\rangle + \frac{1}{3}\langle\, \varphi,\, \mu_n(\varphi,\varphi)\,\rangle + \frac{1}{4}\langle\, \varphi,\, \mu_n(\varphi,\varphi,\varphi)\,\rangle$

- Field is $\varphi = \{A^\mu, c\,; A^*_\mu, c^*\}$ & *Homotopy algebraic structure* is

$$\mu_1 \begin{pmatrix} c \\ A \\ A^* \\ c^* \end{pmatrix} = \begin{pmatrix} 0 \\ d\,c \\ d\star d\,A \\ d\star\,A \end{pmatrix} \quad \text{with} \quad \underbrace{H_c}_{\text{ghost}} \xrightarrow{\mu_1=d} \underbrace{H_\phi}_{\text{field}} \xrightarrow{\mu_1=d\star d} \underbrace{H_{\phi^*}}_{\text{antifield}} \xrightarrow{\mu_1=d} \underbrace{H_{c^*}}_{\text{ghost}^*} \quad ,$$

$$\mu_2 \begin{pmatrix} c_1 & c_2 \\ A_1 & A_2 \\ A_1^*\,, & A_2^* \\ c_1^* & c_2^* \end{pmatrix} = \begin{pmatrix} c_1 c_2 \\ c_1 A_2 + A_1 c_2 \\ \tilde{\mu}_2(A_1, A_2) - c_1 A_2^* + A_1^* c_2 \\ c_1 c_2^* - c1^* c_2 - \star\,(A_1 \wedge (\star A_2^*)) + \star\,(A_1^* \wedge \star A_2) \end{pmatrix} \quad , \quad \mu_3 \begin{pmatrix} c_1 & c_2 & c_3 \\ A_1 & A_2 & A_3 \\ A_1^*\,, & A_2^*\,, & A_3^* \\ c_1^* & c_2^* & c_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\mu}_3(A_1, A_2, A_3) \end{pmatrix}$$

- As is known, **the Berends-Giele current recursion relations** can be quickly derived from this A_∞ structure of the Yang-Mills theory.

Supplement 2

1. Why every path-integrable QFT have a homotopy algebra

2. Why the path-integral preserves such a homotopy algebra

(In particular, one can obtain Wick's theorem in QFT exactly by using this method.)

1. Why every path-integrable QFT have A_∞ / L_∞

I told you that..

- Each quantum field theory that has the path-integral description

$$\text{correlation fnc.} \; \langle \dots \rangle = \int \nu_\phi (\dots) \qquad \text{e.g.} \;\; \nu_\phi = \mathscr{D}\phi \, e^{S[\phi]}$$

always has own (quantum) A_∞ structure ν.

- Let us explain the meanings of ***"consistent QFT, path-integrable QFT, or QFT that has the path-integral description"*** in this talk.

That is ***"QFT solving the Batalin-Vilkovisky master equation"***.

1. Why every path-integrable QFT have A∞ / L∞

What was BV ?

- BV is a powerful and general formalism that enables us **to perform the path-integral, *even for gauge theory***. It is the geometry of the BV odd Laplacian **Δ** with (**Δ**)^2=0 .

  To define $\int \mathscr{D}\phi \, (\dots)$, **Δ**-exacts vanish $\int \mathscr{D}\phi \, (\Delta \, \text{exact}\,) = 0$ and the integrand must be

  Δ-closed : $\Delta \, (\, \text{integrand}\,) = 0$.

- Then, for each QFT, this consistency condition gives ***"the BV master equation".***

$$\Delta \, e^{S[\phi]} = 0 \iff \hbar \, \Delta \, S + \frac{1}{2} (\, S \, , S \,) = 0$$

(The BV bracket is defined by $(-)^A (A, B) \equiv \Delta(AB) - (\Delta A)B - (-)^A A(\Delta B)$).

1. Why every path-integrable QFT have A∞ / L∞

So, we consider QFT solving BV eq.

- The solution $S[\phi]$ of the BV master equation has the following form:

$$S[\phi] = \underbrace{S_{\rm cl}[\phi_{\rm cl}]}_{\text{classical action}} + \underbrace{\phi^* \, (S[\phi], c\,)}_{\text{for gauge degrees}} + \underbrace{c^* \, (S[\phi], \text{ghosts for ghosts})}_{\text{for redundancy of gauge degrees}} + \cdots$$

- This BV action $S[\phi]$ gives a set of "vertices" $\mu = \{\mu_n\}_{n>1}$ as follows

$$S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi)\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi, \phi)\rangle + \cdots$$

For a given QFT, this BV master action $S[\phi]$ is unique in some sense.
Actually, these multi-linear maps $\{\mu_1, \mu_n\}_{n>1}$ satisfy the (quantum) A∞/L∞ relations.

1. Why every path-integrable QFT have A∞ / L∞

Equivalent rep. of solving BV eq.

- We consider the operator $\hbar\,\Delta_S \equiv \hbar\,\Delta + (\,S,\;\,)$ with $\Delta \equiv (-)^{\phi}\dfrac{\partial^2}{\partial\phi\partial\phi^*}$, which gives

$$\hbar\,\Delta_S\,\phi = -\frac{\partial S[\phi]}{\partial\phi} = \mu_1\phi + \mu_2(\phi,\phi) + \mu_3(\phi,\phi,\phi) + \cdots$$

- Note that *"solving BV eq."* equals to *"requiring $(\,\hbar\,\Delta_S\,)^2 = 0$"* because of

$$(\,\hbar\,\Delta_S\,)^2 = \Big(\,S\,,\hbar\,\Delta\,S + \frac{1}{2}(\,S\,,S\,)\Big)\ .$$

- Actually, as we see, $(\,\hbar\,\Delta_S\,)^2 = 0$ is nothing but the quantum A∞/L∞ .

1. Why every path-integrable QFT have A∞ / L∞

Solving BV eq. = requiring quantum A∞/L∞

- We can expand $(\hbar \, \Delta_S)^2 = 0$ acting on $\phi = \sum \phi_g + \sum \phi_g^*$ as follows

$$(\hbar \, \Delta_S)^2 \phi = \hbar \, \Delta_S \Big[\mu_1 \phi + \mu_2(\phi,\phi) + \mu_3(\phi,\phi,\phi) + \cdots \Big]$$

$$= \sum_n \Big[\hbar \sum_g (-)^g \frac{\partial^2}{\partial \phi_g \partial \phi_g^*} \mu_{n+2}(\phi,\ldots,\phi) \; + \sum_{l+k=n} (-)^{\mathrm{sign}} \mu_{l+1}\big(\ldots,\phi,\mu_k(\phi,\ldots,\phi),\phi,\ldots\big) \Big]$$

- These are nothing but the A∞/L∞ relations, which may become more explicit if we use the symbols mimicking "complete basis of the inner product", $e_{-g} \equiv \dfrac{\partial \phi}{\partial \phi_g}$, $e_{1+g} \equiv \dfrac{\partial \phi}{\partial \phi_g^*}$, and expand each μ_n with respect to $\hbar$, such as $\mu_n = \mu_{n,[0]} + \hbar \, \mu_{n,[1]} + \hbar^2 \mu_{n,[2]} + \hbar^3 \mu_{n,[3]} + \cdots$.

1. Why every path-integrable QFT have A∞ / L∞

In summary..

- To have the path-integral, QFT must solve the BV master equation.

- "Solving BV eq. $\Delta\, e^{S[\phi]} = 0$" is the same as imposing the quantum A∞ on vertices $\mu = \{\mu_1, \mu_n\}_{n>1}$ of your BV master action,

$$S[\phi] = \frac{1}{2}\langle \phi, \mu_1 \phi \rangle + \frac{1}{3}\langle \phi, \mu_2(\phi, \phi)\rangle + \frac{1}{4}\langle \phi, \mu_3(\phi, \phi)\rangle + \cdots$$

- ***So, each QFT has own intrinsic quantum A∞/L∞ arising from BV eq. .***

- This A∞/L∞ structure is **unique**, as is the proper BV action.

2. Why the path-integral preserves A_∞ / L_∞

Next topic..

- We have noticed that ***every QFT have quantum A_∞ / L_∞ .***

- But, why does the path-integral preserve it ??

  — That is also because of BV.

- It might be ***trivial***, as long as you can split $\phi = \phi' + \phi''$ and $\Delta = \Delta' + \Delta''$.

2. Why the path-integral preserves A_∞ / L_∞

As is well known..

- Any effective action $A[\phi']$ for "a given QFT $S[\phi'+\phi'']$ solving BV eq."

$$P: \; S[\phi'+\phi''] \longmapsto A[\phi'] \equiv \ln \int \mathscr{D}\phi''\, e^{S[\phi'+\phi'']}$$

also solves the BV master equation : you quickly find $\int \mathscr{D}\phi'' \Delta''(\dots) = 0$ and

$$\Delta' e^{A[\phi']} = \int \mathscr{D}\phi'' (\Delta' + \Delta'') e^{S[\phi'+\phi'']} = 0 \; .$$

- Hence, your BV effective QFT also has **own (quantum) A_∞/L_∞** , $\mu' = \{\mu'_n\}_n$.

  The path-integral P preserves it in this sense : $P\,\mu = \mu' P$.

2. Why the path-integral preserves A_∞ / L_∞

Actually, these properties have been well used by experts.

- Flows of exact renormalization group with BV. [K.Costello 2007, R.Zucchini 2018]
- Realization of symmetry in ERG with BV. [Y.Igarashi, K.Itoh, H.Sonoda 2009]
- Combing BV and ERG. [T.Morris 2018, Y.Igarashi, K.Itoh, T.Morris 2019, P.Lavrov 2019]

— And there are other many earlier works..

Also, there are some works based on the A_∞/L_∞ side of BV

- BG recursion relations of gluon, scattering amplitudes by using A_∞/L_∞.
  [M.Doubek et al 2017, B.Jurco, et al 2018, LT.Macrelli, et al 2019, A.S.Arvanitakis 2019, B.Jurco et al 2019]

Last year, the speaker studied **the classical (tree graphs) part** of the above result. He proposed how to reduce a given "covariant SFT" to corresponding "light-cone SFT". [HM. JHEP04(2019)143]

2. Why the path-integral preserves A_∞ / L_∞

Now, you may notice that effective quantum A_∞/L_∞ is trivial.

- We have learned that the path-integral preserves BV, and thus A_∞/L_∞ .
- So, as long as your original QFT is path-integrable,
  quantum A_∞/L_∞ structure of your effective QFT is automatic.
- But, is there any “explicit” construction of such a morphism ?

— We have it.

2. Why the path-integral preserves A_∞ / L_∞

"Explicit" construction of such P

- **The perturbative path-integral** gives such a morphism very **explicitly**.

  In other words, the Feynman graph expansion preserves quantum A_∞/L_∞ !!

- We noticed that the (non-perturbative) path-integral gives a morphism of BV,

$$P: \; S[\phi' + \phi''] \; \longmapsto \; A[\phi'] = \ln \int \mathscr{D}\phi'' \, e^{S[\phi'+\phi'']} \quad ,$$

  which is often called a ERG transformation, and thus A_∞/L_∞ is automatic.

  The Feynman graph expansion of this **P** also gives a morphism.

2. Why the path-integral preserves A∞ / L∞

Comments :

- We showed that the Feynman graph expansion preserves quantum A_∞ / L_∞ .

- Recall that the "non-perturbative" path-integral preserves quantum A_∞ / L_∞ ,

$$P : \; S[\phi' + \phi''] \;\longmapsto\; A[\phi'] = \ln \int \mathcal{D}\phi''\, e^{S[\phi'+\phi'']} \quad ,$$

  which is often called a ERG transformation.

- So does "non-perturbative path-integral" - "perturbative path-integral" !!

2. Why the path-integral preserves A∞ / L∞

Can we obtain P directly in terms of A_∞/L_∞?

- We obtained some results in terms of the BV master action $S[\phi] = S_{\rm free}[\phi] + S_{\rm int}[\phi]$.

  We can also obtain corresponding results in terms of A∞/L∞ more directly.

- It is given by "the homological perturbation $\mu_1 \mapsto \mu_1 + \mu_{\rm int} + \hbar\,\Delta$". By using coalgebra description, we get the effective quantum A∞/L∞ and morphism $P\,\mu = \mu'\,P$ directly.

  effective A∞/L∞ $\mu' = \mu_1' + P\,\mu_{\rm int}\,i$ & morphism $P = p\dfrac{1}{1+\mu_1^{-1}(\mu_{\rm int}+\hbar\,\Delta)}$

  It is the same as the Feynman graph expansion, or applying Wick's theorem, for

  the ERG transformation $P:\ S[\phi'+\phi''] \longmapsto A[\phi'] = \ln \int \mathscr{D}\phi''\, e^{S[\phi'+\phi'']}$.

  (We skip "what the homological perturbation was" now, which is in appendix.)

2. Why the path-integral preserves A∞ / L∞

Comments on path-integral by homological perturbation

- The perturbative path-integral, or the Feynman graph expansion, can be obtained as a result of the homological perturbation of $\hbar\,\Delta_{S_{\rm int}}$, and thus it preserves BV eq. and A∞/L∞ .

Following perturbations give the Wick's theorem and the perturbative path-integral :

$$(S_{\rm free}\,,\quad) \longmapsto \underbrace{\overbrace{\hbar\,\Delta}^{\text{perturbation}} + (S_{\rm free},\quad)}_{\text{Wick's theorem}} \longmapsto \hbar\,\Delta_S \equiv \underbrace{\overbrace{\hbar\,\Delta + (S_{\rm int},\quad)}^{\text{full perturbation}} + (S_{\rm free},\quad)}_{\text{Perturbative path integral}}$$

- It is the same as the Feynman graph but can give explicit constructions of some quantities.

(We may give a brief review of these facts later, if you want to know.)

2. Why the path-integral preserves A_∞ / L_∞

Some remarks

- We considered algebraic aspects only, but now it will be trivial to you.
  Please note that all physically important informations are in your *concrete construction of "regular" propagators*. (You might learn it from D-instanton.)

- In general, it may be a challenging problem to solve the BV master equation for QFT with finite cut-off, if QFT is not UV finite.

  Solving BV for QFT without cut-off is not difficult, but "regular propagators" will require cut-off dependence.
  As I know, even for Yang-Mills, we know a 1-loop level BV master action only. [Y.Igarashi, K.Itoh, T.Morris 2019]

- In general, your BV Laplacian may have cut-off dependence and then **ERG flows** are given by BV canonical transformations, or morphisms of quantum A_∞ / L_∞.
  ERG flows shift the cut-off dependence of your BV Laplacian.

- Application to SFT is easier than other UV divergent QFTs and it is exact.