Drinfeld realization of the centrally extended $\mathfrak{psl}(2|2)$ Yangian algebra with the manifest coproducts

Takuya Matsumoto[1]
Based on arXiv:2208.11889 [math.QA]
partly working with Prof. Yoshiyuki Koga[1]

[1]University of Fukui

Tagen MathPhys. Seminar
September 13, 2022 @ Nagoya University

Motivations

- $\mathfrak{sl}(2|2)$ is a distinguished Lie superalgebra.
  - **Math:** $\#$(defect) are two, the Killing form is degenerated, allows two central extensions. **[Iohara,Koga]**
  - **Phys:** Supersymmetries in particles phys, 1-dim Hubbard model in statistical phys. **[Beisert][Shastry]**
- The Yangian algebra $Y(\mathfrak{g})$ assoc. with the Lie alg. $\mathfrak{g}$ is a def. of the UEA $U(\mathfrak{g})$, **[Drinfeld,'85]**
- having the **non-local actions,** called the **coproducts Δ,**

$$\Delta : Y \to Y \otimes Y \quad \text{(alg. hom.)}$$
$$\Delta(\widehat{J}^A) = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \tfrac{\hbar}{2} f^A_{BC} J^B \otimes J^C \,.$$

- There are several realizations of $Y(\mathfrak{g})$; D1, D2, RTT. In particular, the D2 fits for the repr. th. **[Drinfeld,'88]**
- We've shown that the compatibility of Δ with the D2. **[M,'22]**

$$\Delta([x,y]) = [\Delta(x), \Delta(y)] \quad \text{for} \quad x, y \in Y_{\mathsf{D2}}(\mathfrak{sl}(2|2) \oplus \mathbb{C}^2) \,.$$

- This allows us to prove the PBW thm. **[M, in prep.]**

Plan of this talk

Lie Superalgebra $\mathfrak{sl}(2|2)$

Central extensions of $\mathfrak{sl}(2|2)$

Yangian and the Hopf algebraic structures

Quantum affine algebra

Affine Lie Superalgebra $\widehat{\mathfrak{sl}}(2|2)$

Odd reflections

- Weyl group
- Generalized Verma modules

Lie Superalgebra $\mathfrak{sl}(2|2)$

Central extensions of $\mathfrak{sl}(2|2)$

Yangian and the Hopf algebraic structures

Quantum affine algebra

Affine Lie Superalgebra $\widehat{\mathfrak{sl}}(2|2)$

Odd reflections

Lie Superalgebra

- **Lie superalgebra** $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector sp. equipped with the graded commutator $[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$,

$$[A, B] = AB - (-1)^{\bar{A}\bar{B}} BA\,, \qquad \bar{A} = \begin{cases} \bar{0} & \text{(even)} \\ \bar{1} & \text{(odd)} \end{cases}.$$

- **Ex.** $\mathfrak{gl}(m|n)$ is generated by E_{ij} $(i, j = 1, \cdots, m+n)$ and they satisfy the relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} \delta_{il} E_{jk}$$

The parity is defined by $\bar{E_{ij}} = \bar{i} + \bar{j}$ and

$$\bar{i} = \begin{cases} 0 & (i = 1, \ldots, m) \\ 1 & (i = m+1, \cdots, m+n) \end{cases}.$$

i.e. $\mathfrak{gl}(m|n) = \left[\begin{array}{c|c} \mathfrak{gl}_m & \text{Odd} \\ \hline \text{Odd} & \mathfrak{gl}_n \end{array}\right] \supset \mathfrak{gl}_m \oplus \mathfrak{gl}_n$: even subalg.

Lie Superalgebra $\mathfrak{sl}(2|2)$

- Supertrace STr for a supermatrix $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is define by $\mathrm{STr} M = \mathrm{Tr} A - \mathrm{Tr} D$.
- Lie superalgebra $\mathfrak{sl}(2|2)$ is the supertraceless part of $\mathfrak{gl}(2|2)$;

$$\mathfrak{sl}(2|2) = \{ \, x \in \mathfrak{gl}(2|2) \mid \mathrm{STr} x = 0 \, \} \, .$$

  Set $I = \mathrm{diag}(1,1,-1,-1)$, then $\mathbb{C}I \ltimes \mathfrak{sl}(2|2) = \mathfrak{gl}(2|2)$.
- Set $C = \frac{1}{2}\mathrm{diag}(1,1,1,1)$ (center). $\mathfrak{sl}, \mathfrak{pgl}, \mathfrak{psl}$ are related as

$$\begin{aligned} \mathfrak{gl}(2|2) &= \mathbb{C}I \ltimes \mathfrak{sl}(2|2) && \text{(subalg.)} \\ &= \qquad \mathfrak{pgl}(2|2) \ltimes \mathbb{C}C && \text{(projected out)} \\ &= \mathbb{C}I \ltimes \mathfrak{psl}(2|2) \ltimes \mathbb{C}C \, . \end{aligned}$$

c.f. $\mathfrak{psl}(2|2) = A_{1,1}$. $\mathfrak{psl}(2|2) \supset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ (ev. subalg.) .

Central extensions

- $\mathfrak{psl}(2|2)$ has the three-dim. central extensions. **[Iohara,Koga]**

$$\mathfrak{psl}(2|2) \oplus \mathbb{C}^3 := \left[\begin{array}{c|c} \mathfrak{sl}_2 & \text{Odd} \\ \hline \text{Odd} & \mathfrak{sl}_2 \end{array}\right] \oplus \mathbb{C}C \oplus \mathbb{C}P^+ \oplus \mathbb{C}P^-$$
$$= \mathfrak{sl}(2|2) \oplus \mathbb{C}P^+ \oplus \mathbb{C}P^-$$

- It could be obtained from the excep. Lie salg. $D(2,1;\alpha)$;

$$D(2,1;\alpha) \underset{\text{even}}{\supset} \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$$
$$\xrightarrow{\alpha \to 0} \quad \mathfrak{psl}(2|2) \oplus \mathbb{C}^3 \underset{\text{even}}{\supset} \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}^3$$

- Introduce the simple root generators as

$$\left[\begin{array}{cc|cc} h & x_1^+ & & \\ x_1^- & h & x_2^+ & \\ \hline & x_2^- & h & x_3^+ \\ & & x_3^- & h \end{array}\right], \quad \begin{cases} h_1 = E_{11} - E_{22} \\ h_2 = E_{22} + E_{33} \\ h_3 = -E_{33} + E_{44} \end{cases}.$$

c.f. $\frac{1}{2}h_1 + h_2 + \frac{1}{2}h_3 = \sum_{i=1}^4 E_{ii} = C$ is central.

Defining relations

Def. 1 ($\mathfrak{psl}(2|2) \oplus \mathbb{C}^3$)
The centrally extended Lie superalgebra $\mathfrak{g} = \mathfrak{psl}(2|2) \oplus \mathbb{C}^3$ over $\mathbb{C}$ has the generators $h_{i,0}$, $x^{\pm}_{i,0}$ with $i = 1, 2, 3$ and the central elements $P^{\pm}_0$, and they satisfy the following relations;

$$[h_{i,0}, h_{j,0}] = 0 \qquad (1)$$

$$[h_{i,0}, x^{\pm}_{j,0}] = \pm a_{ij} x^{\pm}_{j,0} \qquad (2)$$

$$[x^{+}_{i,0}, x^{-}_{j,0}] = \delta_{ij} h_{i,0} \qquad (3)$$

$$[x^{\pm}_{2,0}, x^{\pm}_{2,0}] = [x^{\pm}_{1,0}, x^{\pm}_{3,0}] = 0 \qquad (4)$$

$$[x^{\pm}_{i,0}, [x^{\pm}_{i,0}, x^{\pm}_{2,0}]] = 0 \quad \text{for} \quad i = 1, 3 \qquad (5)$$

$$[[x^{\pm}_{1,0}, x^{\pm}_{2,0}], [x^{\pm}_{3,0}, x^{\pm}_{2,0}]] = P^{\pm}_0 . \qquad (6)$$

The Cartan matrix is $(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

Remark:
$\frac{1}{2}h_1 + h_2 + \frac{1}{2}h_3 =: C$ is central in $\mathfrak{sl}(2|2)$ and
$A_{1,1} = \mathfrak{psl}(2|2) = \mathfrak{sl}(2|2)/\mathbb{C}C$.

Representations

Theorem 2 (M-Molev,'14)

A complete list of pairwise non-isomorphic finite-dimensional irreducible representations of $\mathfrak{g}$ where the central elements act by $C \mapsto 0$, $P_0^- \mapsto 0$, $P_0^+ \mapsto 1$, consists of

1. *the Kac modules $K(m,n)$ with $m,n \in \mathbb{Z}_+$ and $m \neq n$,* $\dim K(m,n) = 16\,(m+1)(n+1)$,
2. *the modules S_n with $n \in \mathbb{Z}_+$,* $\dim S_n = 8\,(n+1)(n+2)$.

Note:

- There exists outer automorphism of $\mathfrak{g}$ sending

$$\begin{pmatrix} C & -P_0^- \\ P_0^+ & -C \end{pmatrix} \mapsto \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

- The reps. of the former case coincide with those of $\mathfrak{sl}(2|2)$.
- The reps. of the latter are discussed in the above theorem.

What's the Yangian?

- The Yangian $Y_\hbar(\mathfrak{g})$ is the one-param.($\hbar$) def. of the UEA $U(\mathfrak{g})$ of the Lie alg. $\mathfrak{g}$, introduced by **[Drinfeld,'85]**,
- with the $\mathbb{Z}_{\geq 0}$-degree.

$$\{0\} \subset Y(\mathfrak{g})_0 = U(\mathfrak{g}) \subset Y(\mathfrak{g})_1 \subset Y(\mathfrak{g})_2 \subset \cdots$$

$$Y(\mathfrak{g}) = \bigcup_{n=0}^{\infty} Y(\mathfrak{g})_n \,, \quad Y(\mathfrak{g})_n = \{\, x \in Y(\mathfrak{g}) \mid \deg(x) \leq n \,\} \,.$$

- having the **non-local actions** called the **coproducts,**

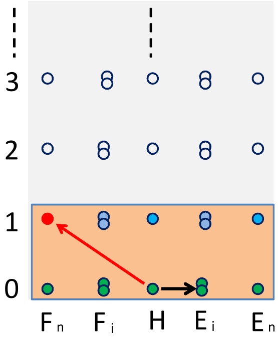
$$\Delta : Y \to Y \otimes Y \quad \text{(alg. hom.)}$$

$$\Delta(\widehat{J}^A) = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \tfrac{\hbar}{2} f^A_{BC} J^B \otimes J^C \,.$$

- From the QIMs point of view, the Yangian arises as the symmetries of the rational R-matrices.

Models	R-matrix	inf. dim. symmetries
XXX	Rational	Yangian
XXZ	Trigonometric	Quantum affine alg.
XYZ	Elliptic	Ell. quan. aff. alg.

How to define the Yangians (1/3)



Drinfeld's first realization

Generators:

J^A(Lie alg.), $\widehat{J}^A$(deg. 1)

$(A = 1, \cdots, \dim \mathfrak{g}\,.)$

Relations:

$$[J^A, J^B] = f^{AB}{}_C J^C$$

$$[\widehat{J}^A, J^B] = f^{AB}{}_C \widehat{J}^C$$

$$[[\widehat{J}^A, \widehat{J}^B], J^C] - [[\widehat{J}^A, J^B], \widehat{J}^C]$$

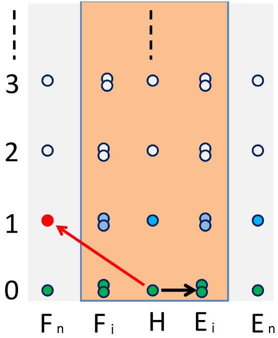
$$= \hbar^2 a^{ABC}_{DEF} J^D J^E J^F$$

(Serre rel.)

Nice: fewer generators, a natural lift of the Lie alg.

Bad: Not suitable for the representation theory.

How to define the Yangians (2/3)



Drinfeld's second realization

Generators:

$E_{i,r}\,, F_{i,r}\,, H_{i,r}\,,$
$(i = 1, \cdots, \mathrm{rank}\mathfrak{g}\,, r = 0, 1, 2, \cdots)$

Relations:

$[H_{i,r}, H_{j,s}] = 0$

$$[H_{i,r+1}, E_{j,s}] - [H_{i,r}, E_{j,s+1}] = \frac{\hbar}{2} a_{ij} \{H_{i,r}, E_{j,s}\}$$

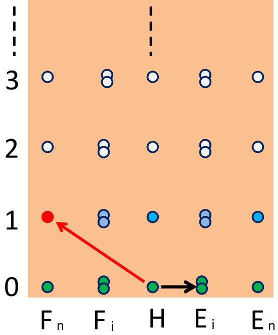
$\cdots$

Nice: Including the Cartan gens. , suitable for repr. theory.

Bad: The coproduct structure is **Not** transparent.

$\Rightarrow$ **Remedy:** Levendorskii's realization

How to define the Yangians (3/3)



RTT formulation

Generators: $t_{ij}^{(r)}$
$(i, j = 1, \cdots, n\,, r = 0, 1, 2, \cdots.)$

Relations:

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

where

$$t_{ij}(u) := \delta_{ij} + \textstyle\sum_{r\geq 1} t_{ij}^{(r)} u^{-r}$$
$$T(u) := e_{ij} \otimes t_{ij}(u)$$
$$R_{12}(u) := 1 - P_{12}u^{-1}$$

Nice: Yangians as sols. of the YBE, the coproducts are obvious, and suitable for repr. theory. See **[Molev san's book]**!

Bad: Relations to the other defs. are not clear. "Top down."

Yangian

Def. 3 (Drinfeld realization $Y_D(\mathfrak{sl}(2|2) \oplus \mathbb{C}^3)$,[M])
The Yangian $Y_D(\mathfrak{sl}(2|2) \oplus \mathbb{C}^3)$ is generated by $h_{i,r}, x^{\pm}_{i,r}$ with $i = 1, 2, 3$ and the **central elements** $P^{\pm}_r$ with $r = 0, 1, 2, \cdots$.
They satisfy the following relations,

$$[h_{i,r}, h_{j,s}] = 0\,, \quad [x^+_{i,r}, x^-_{j,s}] = \delta_{ij} h_{i,r+s}\,, \quad [h_{i,0}, x^{\pm}_{j,r}] = \pm a_{ij} x^{\pm}_{j,r}\,, \tag{7}$$

$$[h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = \pm\frac{1}{2} a_{ij} \{h_{i,r}, x^{\pm}_{j,s}\} \quad \text{for } i, j \text{ not both } 2 \tag{8}$$

$$[h_{2,r}, x^{\pm}_{2,s}] = 0 \tag{9}$$

$$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] - [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] = \pm\frac{1}{2} a_{ij} \{x^{\pm}_{i,r}, x^{\pm}_{j,s}\} \quad \text{for } i, j \text{ not both } 2 \tag{10}$$

$$[x^{\pm}_{2,r}, x^{\pm}_{2,s}] = 0 \tag{11}$$

$$\left[x^{\pm}_{j,r}, [x^{\pm}_{j,s}, x^{\pm}_{2,t}]\right] + \left[x^{\pm}_{j,s}, [x^{\pm}_{j,r}, x^{\pm}_{2,t}]\right] = 0 \qquad \text{for} \qquad j = 1, 3 \tag{12}$$

$$\left[[x^{\pm}_{1,r}, x^{\pm}_{2,0}], [x^{\pm}_{3,s}, x^{\pm}_{2,0}]\right] = P^{\pm}_{r+s}\,. \tag{13}$$

The Cartan matrix is $(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

Yangian

Denote $\mathfrak{sl}(2|2) \oplus \mathbb{C}^3$ by $\mathfrak{g}$.

Purpose:

We would like to show that
the Yangian $\mathrm{Y}_D(\mathfrak{g})$ has the **Hopf algebraic structures.**

But, hard to show the **coproducts** Δ for the Drinfeld realizations.

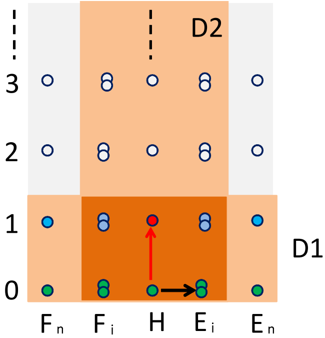
Remedy:

1. Define the truncated system $\mathrm{Y}_L(\mathfrak{g})$, called the **Levendorskii's** realization. **[Levendorskii,'93]**
2. Introduce the Hopf alg. str. for $\mathrm{Y}_L(\mathfrak{g})$.
3. Show the isom. $\mathrm{Y}_D(\mathfrak{g}) \simeq \mathrm{Y}_L(\mathfrak{g})$.

The Hopf alg. str. of $\mathrm{Y}_D(\mathfrak{g})$ are induced from those of $\mathrm{Y}_L(\mathfrak{g})$.

c.f. **[Spill, Torrielli]**

Levendorskii's realization of the Yangians



Levendorskii's realization

[Levendorskii,'93]

Generators:

$E_{i,0}, F_{i,0}, H_{i,0}$ (deg. 0)
$E_{i,1}, F_{i,1}, H_{i,1}$ (deg. 1)

Relations:

Truncation of D2 up to deg. 0 and 1.

$$\widetilde{H}_{i,1} := H_{i,1} - \frac{1}{2}H_{i,0}^2$$
$$\Rightarrow E_{i,r+1} = \frac{1}{a_{ii}}[\widetilde{H}_{i,1}, E_{i,r}]$$

"Boost operator"

Nice: Reduced system of D2. The Hopf alg. str. could be proven (by brute force...).

Bad: Isomorphism with D2 is shown by tedious induction.

Yangian

Def. 4 (Levendorskii's realization $\mathrm{Y}_L(\mathfrak{sl}(2|2) \oplus \mathbb{C}^3)$)
The Yangian $\mathrm{Y}_L(\mathfrak{sl}(2|2) \oplus \mathbb{C}^3)$ generated by $h_{i,0},\ x^{\pm}_{i,0},\ \tilde{h}_{i,1},\ x^{\pm}_{i,1}$ with $i = 1, 2, 3$ and the central elements $P^{\pm}_0,\ P^{\pm}_1$. They satisfy the relations of the Lie algebra (Def.1) and

$$[\tilde{h}_{i,1}, h_{j,0}] = 0\,, \quad [\tilde{h}_{i,1}, \tilde{h}_{j,1}] = 0 \qquad \text{(degree two rels.)}\,, \tag{14}$$

$$[\tilde{h}_{i,1}, x^{\pm}_{j,0}] = \pm a_{ij} x^{\pm}_{j,1}\,, \quad [x^{+}_{i,1}, x^{-}_{j,0}] = \delta_{ij} h_{i,1} \tag{15}$$

$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] - [x^{\pm}_{i,0}, x^{\pm}_{j,1}] = \pm\frac{1}{2} a_{ij} \{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} \tag{16}$$

$$[x^{\pm}_{2,1}, x^{\pm}_{2,0}] = 0 \tag{17}$$

$$[\tilde{h}_{j,1}, [x^{+}_{j,1}, x^{-}_{j,1}]] = 0 \qquad \text{for} \qquad j = 1, 3 \tag{18}$$

$$[\tilde{h}_{1,1}, [x^{+}_{2,1}, x^{-}_{2,1}]] = 0 \qquad \text{(degree three rels.)} \tag{19}$$

$$[[x^{\pm}_{1,1}, x^{\pm}_{2,0}], [x^{\pm}_{3,0}, x^{\pm}_{2,0}]] = P^{\pm}_1 \tag{20}$$

where $h_{i,1}$ in (15) is defined by $h_{i,1} = \tilde{h}_{i,1} + \frac{1}{2}(h_{i,0})^2$.
The Cartan matrix is the same as before.

Yangian

▸ Yangian limit

Proposition 5 (M)

The Yangian $\mathrm{Y}_L(\mathfrak{g})$ *has the Hopf algebra structures with the coproducts* $\Delta : \mathrm{Y}_L(\mathfrak{g}) \to \mathrm{Y}_L(\mathfrak{g}) \otimes \mathrm{Y}_L(\mathfrak{g})$ *given by*

$$\begin{aligned}
\Delta(X) &= X \otimes 1 + 1 \otimes X \quad \textit{for} \quad X \in U(\mathfrak{g}) \\
\Delta(x^+_{2,1}) &= x^+_{2,1} \otimes 1 + 1 \otimes x^+_{2,1} \\
&\quad + h_{2,0} \otimes x^+_{2,0} + E_{12} \otimes E_{31} + E_{34} \otimes E_{42} - E_{14} \otimes P^+_0 \\
&\cdots \quad \textit{(fairly complicated)} \\
\Delta(P^+_1) &= P^+_1 \otimes 1 + 1 \otimes P^+_1 - 2C_0 \otimes P^+_0 \\
\Delta(P^-_1) &= P^-_1 \otimes 1 + 1 \otimes P^-_1 - 2P^-_0 \otimes C_0
\end{aligned}$$

the counits $\epsilon : \mathrm{Y}_L(\mathfrak{g}) \to \mathbb{C}$ *,* $\epsilon(X) = 0$ *for* $X \in \mathrm{Y}_L(\mathfrak{g})$ *, and the antipodes* $\mathrm{S} : \mathrm{Y}_L(\mathfrak{g}) \to \mathrm{Y}_L(\mathfrak{g})$ *satisfying the antipode rels.*

c.f. **[Beisert, 2004]**

Yangian

Introduce the higher degree gens. for $r \in \mathbb{Z}_{\geq 0}$ in $\mathrm{Y}_L(\mathfrak{g})$ by

$$x^{\pm}_{1,r+1} = \pm\frac{1}{2}\,[\tilde{h}_{1,1}, x^{\pm}_{1,r}]\,, \quad x^{\pm}_{2,r+1} = \mp[\tilde{h}_{1,1}, x^{\pm}_{2,r}]\,, \quad x^{\pm}_{3,r+1} = \mp\frac{1}{2}\,[\tilde{h}_{3,1}, x^{\pm}_{3,r}]\,,$$
$$h_{i,r} = [x^{+}_{i,r}, x^{-}_{i,0}] \quad (\ i = 1,2,3\ )\,, \quad P^{\pm}_{r} = [[x^{\pm}_{1,r}, x^{\pm}_{2,0}], [x^{\pm}_{3,0}, x^{\pm}_{2,0}]]\,. \tag{21}$$

Theorem 6 (M)

The Yangian $\mathrm{Y}_D(\mathfrak{g})$ *is isomorphic to* $\mathrm{Y}_L(\mathfrak{g})$ *. The isomorphism* $\phi : \mathrm{Y}_D(\mathfrak{g}) \to \mathrm{Y}_L(\mathfrak{g})$ *is given by*

$$h_{i,r} \mapsto h_{i,r}\,, \qquad x^{\pm}_{i,r} \mapsto x^{\pm}_{i,r}\,, \qquad P^{\pm}_{r} \mapsto P^{\pm}_{r}\,,$$

where the image of ϕ *is defined in* (21) *.*

Yangian

Theorem 6 allows us to induce the **Hopf algebra structures** to $\mathrm{Y}_D(\mathfrak{g})$ from $\mathrm{Y}_L(\mathfrak{g})$ via the following commutative diagrams,

$$\begin{array}{ccc} \mathrm{Y}_D(\mathfrak{g}) & \xrightarrow{\phi} & \mathrm{Y}_L(\mathfrak{g}) \\ \Delta_D \big\downarrow & & \big\downarrow \Delta \\ \mathrm{Y}_D(\mathfrak{g}) \otimes \mathrm{Y}_D(\mathfrak{g}) & \xrightarrow[\phi\otimes\phi]{} & \mathrm{Y}_L(\mathfrak{g}) \otimes \mathrm{Y}_L(\mathfrak{g}) \end{array}$$

$$\begin{array}{ccc} \mathrm{Y}_D(\mathfrak{g}) & \xrightarrow{\phi} & \mathrm{Y}_L(\mathfrak{g}) \\ \mathrm{S}_D \big\downarrow & & \big\downarrow \mathrm{S} \\ \mathrm{Y}_D(\mathfrak{g}) & \xrightarrow[\phi]{} & \mathrm{Y}_L(\mathfrak{g}) \end{array} , \qquad \begin{array}{ccc} \mathrm{Y}_D(\mathfrak{g}) & \xrightarrow{\phi} & \mathrm{Y}_L(\mathfrak{g}) \\ \epsilon_D \big\downarrow & & \big\downarrow \epsilon \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

Δ_D: coproduct, S_D: antipode, and ϵ_D: counit in $\mathrm{Y}_D(\mathfrak{g})$.

Corollary 7

$\mathrm{Y}_D(\mathfrak{g})$ *has the Hopf alg. structures induced from those of* $\mathrm{Y}_L(\mathfrak{g})$ *.*

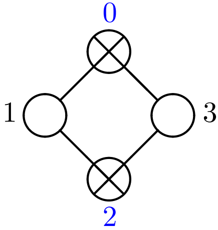
Quantum affine algebra $\mathrm{U}_{g,q}(\hat{\mathfrak{g}})$ [Beisert,Galleas,M,'12]

- Generators: $\{K_i\,, E_i\,, F_i\}_{i=0,1,2,3}$ and
  $\{U_k\,, V_k\}_{k=0,2}$ (centers)
- Parity $p : \mathrm{U}_{g,q} \to \mathbb{Z}_2$,

$$p(E_k) = p(F_k) = 1 \quad (k = 0, 2)\,, \qquad p(\text{others}) = 0\,.$$

- Two deformation parameters ??g and q
- GCM, Normalized matrix, Dynkin diagram ;

$$(b_{ij}) = \left(\begin{array}{c|ccc} 0 & -1 & 0 & 1 \\ \hline -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{array}\right)$$



$$(d_i) = \mathrm{diag}(-1, -1, -1, 1)$$

Defining relations

- Defining relations : c.f. [Jimbo],[Drinfeld]

$$\mathrm{K}_1^{-1}\mathrm{K}_k^{-2}\mathrm{K}_3^{-1} = \mathrm{V}_k^2 \qquad (k = 0, 2)\,, \tag{22}$$

$$K_i E_j K_i^{-1} = q^{b_{ij}} E_j\,, \qquad K_i F_j K_i^{-1} = q^{-b_{ij}} F_j\,, \tag{23}$$

$$[E_j, F_j] = d_j \frac{K_j - K_j^{-1}}{q - q^{-1}}\,, \tag{24}$$

$$[E_i, F_j] = 0 \quad \text{for} \quad i \neq j, \quad (i,j) \neq (0,2)\,, (2,0)\,. \tag{25}$$

The **centrally extended** relations ;

$$[E_2, F_0] = -\tilde{g}\left(K_0 - U_2 U_0^{-1} K_2^{-1}\right)\,, \tag{26}$$

$$[E_0, F_2] = +\tilde{g}\left(K_2 - U_0 U_2^{-1} K_0^{-1}\right)\,, \tag{27}$$

where we denote $\tilde{g} := g/\sqrt{1 - g^2(q - q^{-1})^2}\,.$

Defining relations

- The Serre relations (similar for F's):

$$[\mathrm{E}_1, \mathrm{E}_3] = \mathrm{E}_2\mathrm{E}_2 = \mathrm{E}_0\mathrm{E}_0 = [\mathrm{E}_2, \mathrm{E}_0] = 0$$
$$\big[\mathrm{E}_j, [\mathrm{E}_j, \mathrm{E}_k]\big] - (q - 2 + q^{-1})\mathrm{E}_j\mathrm{E}_k\mathrm{E}_j = 0$$

- **The extended Serre relations:**

$$\big[[\mathrm{E}_1, \mathrm{E}_k], [\mathrm{E}_3, \mathrm{E}_k]\big] - (q - 2 + q^{-1})\mathrm{E}_k\mathrm{E}_1\mathrm{E}_3\mathrm{E}_k = g(1 - \mathrm{V}_k^2\mathrm{U}_k^2)$$
$$\big[[\mathrm{F}_1, \mathrm{F}_k], [\mathrm{F}_3, \mathrm{F}_k]\big] - (q - 2 + q^{-1})\mathrm{F}_k\mathrm{F}_1\mathrm{F}_3\mathrm{F}_k = g(\mathrm{V}_k^{-2} - \mathrm{U}_k^{-2})$$

where $j = 1, 3$, $k = 0, 2$.

Note: Introduced U instead of P_0^+ and P_0^-

$\rightarrow$ More restricted algebra is considered.

$$q^C = V_2$$
$$P_0^+ = +g(1 - \mathrm{V}_2^2\mathrm{U}_2^2)$$
$$P_0^- = -g(\mathrm{V}_2^{-2} - \mathrm{U}_2^{-2})$$

Limits of the deformation parameters

$\mathrm{U}_{g,q}(\hat{\mathfrak{g}})$ has two deformation parameters ;

g : " coupling const. " and q : q-deformation.

- $g \to 0$ limit: dropping the central extensions.

$$\lim_{g\to 0} \mathrm{U}_{g,q}(\hat{\mathfrak{g}}) \simeq \mathrm{U}_q(\hat{\mathfrak{sl}}(2|2))$$

- $q \to 1$ limit: Yangian limit, degenerates XXZ to XXX.

$$\lim_{q\to 1} \mathrm{U}_{g,q}(\hat{\mathfrak{g}}) \simeq \mathrm{Y}_\hbar(\mathfrak{g}) \,, \qquad q = \mathrm{e}^\hbar$$

Difficult to see the degeneration of the relations directly.

c.f. [Guay-Ma]

($\because$) Singular limit : Rescale by $1/(1-q)$, then take $q \to 1$

♢ Checked the degeneracy for
the fundamental repr. and the coproducts.

Yanian limit of $\mathrm{U}_{g,q}(\hat{\mathfrak{g}})$

Obs.: Considering the limit $q \to 1$ of fund. rep., we see that

$$F_0 \to [[E_3, E_2], E_1] =: -E_{321}\,, \qquad E_0 \to [[F_3, F_2], F_1] =: F_{321}\,.$$

Then their differences divided by $(q-1)$ could give something finite. In fact, we found the level-1 Yangian as

$$\lim_{q\to 1} \frac{-F_0 - E_{321}}{2ig(q-1)} = \widehat{E}_{321} + \frac{i}{2}(1+U^2)F_2$$

$$\lim_{q\to 1} \frac{E_0 - F_{321}}{2ig(q-1)} = -\widehat{F}_{321} + \frac{i}{2}(1+U^{-2})E_2$$

1. **Evaluation rep.:** $\widehat{E}_{321} \simeq uE_{321}$

$$\lim_{q\to 1} \frac{-F_0 - E_{321}}{2ig(q-1)} \simeq uE_{321} + \frac{i}{2}(1+U^2)F_2$$

2. **Coproduct Δ:** ▸ Yangian Δ

$$\lim_{q\to 1} \frac{-\Delta(F_0) - \Delta(E_{321})}{2ig(q-1)} = \Delta\Big(\widehat{E}_{321} + \frac{i}{2}(1+U^2)F_2\Big)$$

Lie Superalgebra $\mathfrak{sl}(2|2)$

Central extensions of $\mathfrak{sl}(2|2)$

Yangian and the Hopf algebraic structures

Quantum affine algebra

Affine Lie Superalgebra $\widehat{\mathfrak{sl}}(2|2)$

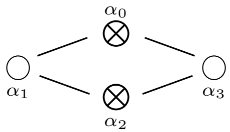
Odd reflections

Affine Lie Superalgebra $\widehat{\mathfrak{sl}}(2|2)$

- **Cartan datum**
  - $I = \{0,1,2,3\} = I_{\bar{0}} \sqcup I_{\bar{1}}$: index set of simple roots with $I_{\bar{0}} = \{1,3\}$ (even) and $I_{\bar{1}} = \{0,2\}$ (odd).
  - $A = (a_{ij})_{i,j\in I}$: Cartan matrix, $\operatorname{rank} A = 2$,

$$A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix},$$

  where $B = DA$ (symmetrized) with $D = \operatorname{diag}(1,1,1,-1)$.
- **Root datum**
  - Cartan subalgebra $\mathfrak{h}$, $\dim \mathfrak{h} = 6$.
  - Simple roots and coroots : $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, $\Pi^\vee = \{h_i\}_{i\in I} \subset \mathfrak{h}$, s.t. $\langle h_i, \alpha_j\rangle = \alpha_j(h_i) = a_{ij}$.
  - Dyunkin diagram



Affine Lie Superalgebra $\widehat{\mathfrak{sl}}(2|2)$

- **Root datum**
  - Root system : $\Delta = \{ \beta + m\delta \, , n\delta \mid \beta \in \bar{\Delta} \, , m, n \in \mathbb{Z} \, , n \neq 0 \}$ where $\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ (imaginary root) and $\bar{\Delta}$ is the root system of $\mathfrak{sl}(2|2)$.
  - $\Delta_{\bar{0}} \, , \Delta_{\bar{1}}$: sets of even and odd roots, resp. In particular, $\bar{\Delta} = \bar{\Delta}_{\bar{0}} \sqcup \bar{\Delta}_{\bar{1}}$ with $\bar{\Delta}_{\bar{0}} = \{ \pm\alpha_1 \, , \pm\alpha_3 \}$, $\bar{\Delta}_{\bar{1}} = \{ \pm\alpha_2 \, , \pm(\alpha_1 + \alpha_2) \, , \pm(\alpha_2 + \alpha_3) \, , \pm(\alpha_1 + \alpha_2 + \alpha_3) \}$
  - Root vectors: $e_\beta \in \mathfrak{g}^\beta \, , f_\beta \in \mathfrak{g}^{-\beta}$, where $\mathfrak{g}^\gamma = \{ x \in \mathfrak{g} \mid [h, x] = \gamma(h)x$ for ${}^\forall h \in \mathfrak{h} \}$ (root subsp. corresponding to γ)

Remark 8

In $\mathfrak{sl}(2|2)$ case, any $\tau \in \Delta_{\bar{1}}$ is isotropic, i.e., $(\tau, \tau) = 0$.

Motivations for the odd reflections

- Believe that taking into account all bases $\Pi \in \mathbb{B}$ equally is a **democratic attitude** for the Lie superalg.
- We want to construct the BGG-type resolution for the reprs. of the affine $\widehat{\mathfrak{sl}}(2|2)$, **[Bernstein,Gel'fand,Gel'fand,'71,'75,'76]**

$$\cdots \to N_3(\mathfrak{g}) \to N_2(\mathfrak{g}) \to N_1(\mathfrak{g}) \to L_\Lambda(\mathfrak{g}) \to 0\,,$$

  which explains the character formula.
- expecting that $N_i(\mathfrak{g})$ are the generalized Verma modules , so as the affine Lie superalgebra $\widehat{\mathfrak{sl}}(2|1)$ case. **[Koga]**

Base and odd reflection

Def. 9 (Base)

$\Sigma \subset \Delta$ (lin. indep.) is called a **base** if ${}^{\exists}e_\beta\,,f_\beta(\beta \in \Sigma)$ satisfying
(1) $\{e_\beta\,,f_\beta\,,h\,|\,\beta \in \Sigma\,,h \in \mathfrak{h}\}$ generate $\mathfrak{g}$,
(2) $[e_\beta, f_\gamma] = 0$ if $\beta \neq \gamma$.

Denote the set of bases by $\mathbb{B}$.

Def. 10 (Odd reflection)

Let Σ be a base and $\tau \in \Sigma_{\bar{1}}$. For each $\beta \in \Sigma$, the **odd reflection** $r_\tau(\beta)$ is defined by

$$r_\tau(\beta) = \begin{cases} -\beta & \beta = \tau \\ \beta + \tau & \beta \neq \tau \wedge (\beta, \tau) \neq 0 \\ \beta & \beta \neq \tau \wedge (\beta, \tau) = 0 \end{cases}$$

Note: $r_\tau(\Sigma)$ is also a base for any $\tau \in \Sigma_{\bar{1}}$. Hence, the odd refs. define transformations btw. bases.

Odd reflections: ex. $\mathrm{osp}(3|2) = B_{1,1}$

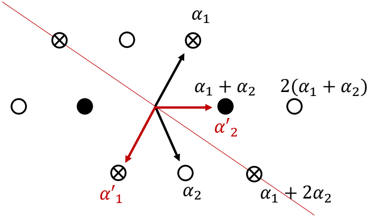
$$\otimes \Longrightarrow \circ\,, \quad \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

$$\Pi = \{\alpha_1, \alpha_2\}$$
$$\Delta^+_{\bar{0}} = \{\alpha_2, 2(\alpha_1 + \alpha_2)\}$$
$$\Delta^+_{\bar{1}} = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$$

$$\otimes \Longrightarrow \bullet\,, \quad \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$$

$$\Pi' = \{\alpha'_1, \alpha'_2\}$$
$$\Delta'^+_{\bar{0}} = \{\alpha'_1 + \alpha'_2, 2\alpha'_2\}$$
$$\Delta'^+_{\bar{1}} = \{\alpha'_1, \alpha'_2, \alpha'_1 + 2\alpha'_2\}$$

$$r_{\alpha_1}(\alpha_1) = -\alpha_1,$$
$$r_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2,$$
But,
$$r_{\alpha_1}(2(\alpha_1 + \alpha_2)) = 2\alpha_2 \notin \Delta\,.$$



Odd ref. is **Not** in $\mathrm{GL}(\mathfrak{h}^*)$.

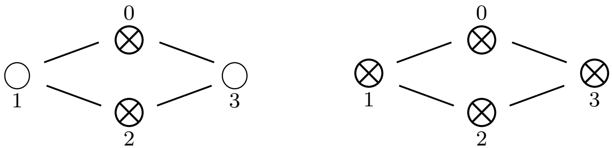
Categorical int. seems more natural. $r_{\alpha_1} \in \mathrm{Hom}(\Pi, \Pi')$

"Weyl groupoid" **[Heckenberger,Yamane]**

Odd reflections for $\widehat{\mathfrak{sl}}(2|2)$

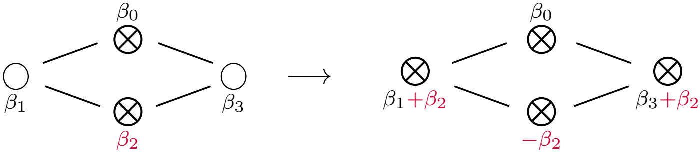
gen. Verma

Two typical types of Dynkins: $\Sigma =$ "XOXO" and $\Xi =$ "XXXX"

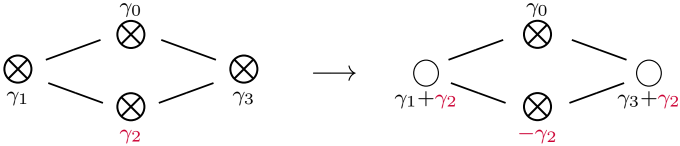


Odd reflections interchange Σ and Ξ.

- $r_{\beta_2} : \Sigma \to \Xi$



- $r_{\gamma_2} : \Xi \to \Sigma$



Odd reflections for $\widehat{\mathfrak{sl}}(2|2)$

Complete list of Bases for $\widehat{\mathfrak{sl}}(2|2)$ $(\delta = \sum_{i=0}^{3} \alpha_i \,, m \in \mathbb{Z})$

$$\Sigma_1^m = \{\ \alpha_0 + m\delta \,, \alpha_1 \,, \alpha_2 - m\delta \,, \alpha_3\ \} \quad \text{(c.f. } \Pi = \Sigma_1^0\text{)}$$
$$\Sigma_2^m = \{\ \alpha_1 \,, -\alpha_0 - \alpha_1 - m\delta \,, \alpha_0 + \alpha_1 + \alpha_2 \,, \alpha_0 + \alpha_3 + m\delta\ \}$$
$$\Sigma_3^m = \{\ \alpha_3 \,, \alpha_0 + \alpha_1 + m\delta \,, \alpha_0 + \alpha_2 + \alpha_3 \,, -\alpha_0 - \alpha_3 - m\delta\ \}$$
$$\Sigma_4^m = \{\ -\alpha_0 - m\delta \,, \alpha_0 + \alpha_1 + \alpha_2 \,, -\alpha_2 + m\delta \,, \alpha_0 + \alpha_2 + \alpha_3\ \}$$
$$\Xi_1^m = \{\ -\alpha_0 - m\delta \,, \alpha_0 + \alpha_1 + m\delta \,, \alpha_2 - m\delta \,, \alpha_0 + \alpha_3 + m\delta\ \}$$
$$\Xi_2^m = \{\ \alpha_0 + m\delta \,, \alpha_1 + \alpha_2 - m\delta \,, -\alpha_2 + m\delta \,, \alpha_2 + \alpha_3 - m\delta\ \}$$

Periodicity: $\pm\delta$-shift is obtained by **4 steps .**

$$\cdots\ \Xi_1^{m-1} \begin{array}{c} \rightleftharpoons \Sigma_2^{m-1} \rightleftharpoons \\ \rightleftharpoons \Sigma_3^{m-1} \rightleftharpoons \end{array} \Xi_2^{m} \begin{array}{c} \rightleftharpoons \Sigma_1^{m} \rightleftharpoons \\ \rightleftharpoons \Sigma_4^{m} \rightleftharpoons \end{array} \Xi_1^{m} \begin{array}{c} \rightleftharpoons \Sigma_2^{m} \rightleftharpoons \\ \rightleftharpoons \Sigma_3^{m} \rightleftharpoons \end{array} \Xi_2^{m+1}\ \cdots$$

Algebraic structures assoc. with Bases

Three hierarchies assoc. with a base Σ:

Root Σ $\rightarrow$ **Algebra** $\mathfrak{g}^+_\Sigma$ $\rightarrow$ **Module** M_Σ .

- **Triangular decomposition** assoc. with a base Σ:

$$\mathfrak{g} = \mathfrak{g}^+_\Sigma \oplus \mathfrak{h} \oplus \mathfrak{g}^-_\Sigma$$

where

$$\mathfrak{g}^\pm_\Sigma = \bigoplus_{\gamma \in \pm\Delta^+_\Sigma} \mathfrak{g}^\gamma \,, \quad \Delta^+_\Sigma = \Delta \cap Q^+_\Sigma \,, \quad Q^+_\Sigma = \sum_{\gamma\in\Sigma} \mathbb{Z}_{\geq 0}\gamma$$

- **Cartan matrices** assoc. w/ $\Sigma =$ XOXO and $\Xi =$ XXXX

$$A_\Sigma = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \qquad A_\Xi = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

can be symm. by $D_\Sigma = \mathrm{diag}(1,1,1,-1)$ and $D_\Xi = \mathrm{diag}(1,1,-1,-1)$, resp.

Algebraic structures assoc. with Bases

- **Weyl vector** ρ_Σ assoc. with a base Σ.
  1. For $\Pi \in \mathbb{B}$, take $\rho_\Pi \in \mathfrak{h}^*$ s.t. $(\rho_\Pi, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)\,, (i \in I)$.
  2. For any $\Sigma \in \mathbb{B}$, set $\rho_\Sigma := \rho_\Pi + \sum_{\beta \in \Delta_\Pi^+ \cap \Delta_\Sigma^-} \beta$.

Note:

- when $\Sigma' = r_\tau(\Sigma) \in \mathbb{B}$ with $\tau \in \Sigma_{\bar{1}}$, $\rho_{\Sigma'} = \rho_\Sigma + \tau$.
- holds that $(\rho_\Sigma, \gamma) = \frac{1}{2}(\gamma, \gamma) \quad ({}^\forall \gamma \in \Sigma)$.
- For $\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \in \mathbb{B}$, we can take ρ_Π as

$$\rho_\Pi = \frac{1}{2}(\alpha_1 + \alpha_3)\,.$$

Principal roots

Def. 11 (Principal root)

An even root $\gamma \in \Delta$ is called a **principal root** if there exists a base $\Sigma \in \mathbb{B}$ obtained from Π by odd reflections s.t. $\gamma \in \Sigma$.

- Principal rts. $=$ all even rts. in $\{\Sigma \in r(\Pi) | r : \text{Odd ref.}\}$.
- The list of bases for $\widehat{\mathfrak{sl}}(2|2)$,

$$\Sigma_1^m = \{\ \alpha_0 + m\delta\,, \alpha_1\,, \alpha_2 - m\delta\,, \alpha_3\ \} \quad (\text{c.f. } \Pi = \Sigma_1^0)$$
$$\Sigma_2^m = \{\ \alpha_1\,, -\alpha_0 - \alpha_1 - m\delta\,, \alpha_0 + \alpha_1 + \alpha_2\,, \alpha_0 + \alpha_3 + m\delta\ \}$$
$$\Sigma_3^m = \{\ \alpha_3\,, \alpha_0 + \alpha_1 + m\delta\,, \alpha_0 + \alpha_2 + \alpha_3\,, -\alpha_0 - \alpha_3 - m\delta\ \}$$
$$\Sigma_4^m = \{\ -\alpha_0 - m\delta\,, \alpha_0 + \alpha_1 + \alpha_2\,, -\alpha_2 + m\delta\,, \alpha_0 + \alpha_2 + \alpha_3\ \}$$
$$\Xi_1^m = \{\ -\alpha_0 - m\delta\,, \alpha_0 + \alpha_1 + m\delta\,, \alpha_2 - m\delta\,, \alpha_0 + \alpha_3 + m\delta\ \}$$
$$\Xi_2^m = \{\ \alpha_0 + m\delta\,, \alpha_1 + \alpha_2 - m\delta\,, -\alpha_2 + m\delta\,, \alpha_2 + \alpha_3 - m\delta\ \},$$

tells us that the principal roots are

$$\{\ \alpha_1\,, \underbrace{\alpha_0 + \alpha_2 + \alpha_3}_{\delta - \alpha_1}\,, \alpha_3\,, \underbrace{\alpha_0 + \alpha_1 + \alpha_2}_{\delta - \alpha_3}\ \}.$$

Weyl group

Def. 12 (Weyl group)

The **Weyl group** W is defined to the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the even reflections r_β for the **principal root** β .

- In $\widehat{\mathfrak{sl}}(2|2)$ case, the principal roots are

$$\{\ \alpha_1\, ,\underbrace{\alpha_0+\alpha_2+\alpha_3}_{\delta-\alpha_1}, \alpha_3\, , \underbrace{\alpha_0+\alpha_1+\alpha_2}_{\delta-\alpha_3}\ \}\, ,$$

- The Weyl group W for $\widehat{\mathfrak{sl}}(2|2)$ coincides with that of $\widehat{\mathfrak{sl}}_2 \oplus \widehat{\mathfrak{sl}}_2$.

Generalized Verma modules

Three hierarchies assoc. with a base Σ:

Root Σ $\rightarrow$ **Algebra** $\mathfrak{g}_\Sigma^+$ $\rightarrow$ **Module** M_Σ .

Notations: For a base $\Sigma \in \mathbb{B}$,

- $\mathfrak{b}_\Sigma = \mathfrak{g}_\Sigma^+ \oplus \mathfrak{h}$: Borel subalg. assoc. with Σ .
- $M_\Sigma(\Lambda)$: Verma mod. with h.w. $\Lambda \in \mathfrak{h}^*$ defined from $\mathfrak{b}_\Sigma$.
- $L_\Sigma(\Lambda)$: the irreducible quotient of $M_\Sigma(\Lambda)$.
- $\mathfrak{p}_{\Sigma;\tau} = \mathfrak{b}_\Sigma \oplus \mathfrak{g}^{-\tau}$: parabolic subalg. for $\tau \in \Sigma_{\bar{1}}$.
- $\mathbb{C}\mathbf{1}_\Lambda$: 1-dim $\mathfrak{p}_{\Sigma;\tau}$-mod. with $\Lambda \in \mathfrak{h}^*$ s.t. $(\Lambda, \tau) = 0$, defined by

$$h.\mathbf{1}_\Lambda = \Lambda(h)\mathbf{1}_\Lambda \ (h \in \mathfrak{h})\,, \quad \mathfrak{g}_\Sigma^+\mathbf{1}_\Lambda = \{0\}\,, \quad \mathfrak{g}^{-\tau}\mathbf{1}_\Lambda = \{0\}\,.$$

Def. 13 (Generalized Verma module with (Σ, τ, Λ))

$$N_\Sigma(\Lambda;\tau) \equiv U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\Sigma;\tau})} \mathbb{C}\mathbf{1}_\Lambda$$

Note: $f_\tau\mathbf{1}_\Lambda \in M_\Sigma(\Lambda)$ is a singular vec. if $(\Lambda, \tau) = 0$. Hence,

$$M_\Sigma(\Lambda)/U(\mathfrak{g})f_\tau\mathbf{1}_\Lambda \simeq N_\Sigma(\Lambda;\tau)\,.$$

Generalized Verma modules: $\mathfrak{sl}(2|2)$ case

Difficulty:

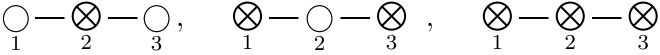
▸ Dynkin

There are **two** orthogonal isotropic odd roots **(defects)**.

c.f. $\widehat{\mathfrak{sl}}(2|2)$ is the defect$=1$ case. **[Koga]**

We are starting with the **non-affine** case $\mathfrak{sl}(2|2)$.

- **Three** types of Dynkins: $\Pi =$ OXO, $\Sigma =$ XOX, and $\Xi =$ XXX.



- List of bases for $\mathfrak{sl}(2|2)$. Principal roots are $\{\alpha_1, \alpha_3\}$.

$$\Pi_1 = \{ \alpha_1, \alpha_2, \alpha_3 \}$$
$$\Pi_2 = \{ \alpha_3, -\alpha_1 - \alpha_2 - \alpha_3, \alpha_1 \}$$
$$\Sigma_1 = \{ -\alpha_1 - \alpha_2, \alpha_1, \alpha_2 + \alpha_3 \}$$
$$\Sigma_2 = \{ \alpha_1 + \alpha_2, \alpha_3, -\alpha_2 - \alpha_3 \}$$
$$\Xi_1 = \{ \alpha_1 + \alpha_2, -\alpha_2, \alpha_2 + \alpha_3 \}$$
$$\Xi_2 = \{ -\alpha_1 - \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, -\alpha_2 - \alpha_3 \}$$

Generalized Verma modules: $\mathfrak{sl}(2|2)$ case

- Weyl group: $W = W_{\mathfrak{sl}_2} \times W_{\mathfrak{sl}_2}$.
- **Odd reflections:**

$$\Pi_1 \rightleftharpoons \Xi_1 \begin{matrix} \rightleftharpoons & \Sigma_1 & \rightleftharpoons \\ & & \\ \rightleftharpoons & \Sigma_2 & \rightleftharpoons \end{matrix} \Xi_2 \rightleftharpoons \Pi_2$$

- We are expecting the gen. Verma mods. are
  - $M_\Pi(\Lambda)/U(\mathfrak{g})f_{\alpha_2}\mathbf{1}_\Lambda$
  - $M_\Sigma(\Lambda)/(U(\mathfrak{g})f_{\alpha_1+\alpha_2}\mathbf{1}_\Lambda + U(\mathfrak{g})f_{\alpha_2+\alpha_3}\mathbf{1}_\Lambda)$

- needs more precise calculations, comparison with **[M, Molev]**

- We hope to report the complete answers **in the near future!**

References

References

The incomplete list of the references.

- K. Iohara and Y. Koga, *Central extensions of Lie superalgebras*, Comment. Math. Helv. **76** (2001), 110–154.
- N. Beisert, *The* $\mathfrak{su}(2|2)$ *dynamic* S*-matrix*, Adv. Theor. Math. Phys. **12** (2008), 945–979.
- M and A. Molev, *Representations of centrally extended Lie superalgebra* $\mathfrak{psl}(2|2)$, J. Math. Phys. **55** (2014) 091704 [arXiv:1405.3420 [math.RT]].
- M, *Drinfeld realization of the centrally extended* $\mathfrak{psl}(2|2)$ *Yangian algebra with the manifest coproducts*, `arXiv:2208.11889[math.QA]`
- S. Z. Levendorskii, *On generators and defining relations of Yangians*, Journal of Geometry and Physics, Volume 12, Issue 1, 1993, Pages 1-11, ISSN 0393-0440.
- F. Spill and A. Torrielli, *On Drinfeld's second realization of the AdS/CFT su(2|2) Yangian*, J. Geom. Phys. **59** (2009) 489 [arXiv:0803.3194 [hep-th]].

References

N. Beisert, W. Galleas and M, *A Quantum Affine Algebra for the Deformed Hubbard Chain*, J. Phys. A **45** (2012), 365206 [arXiv:1102.5700 [math-ph]].

I. Heckenberger, F. Spill, A. Torrielli and H. Yamane, *Drinfeld second realization of the quantum affine superalgebras of* $D^{(1)}(2,1;x)$ *via the Weyl groupoid*, RIMS Kokyuroku Bessatsu B **8** (2008), 171 [arXiv:0705.1071 [math.QA]].

I. Heckenberger, H. Yamane, *A generalization of Coxeter groups, root systems, and Matsumoto's theorem*, Mathematishe Zeitschrift, 259 (2008), 255-276, math.QA/0610823.