Calogero Model from Real Symmetric Φ^4 Matrix Model as a Noncommutative Scalar Field Theory

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Introduction

- Quantum field theories on noncommutative spaces such as Moyal spaces have given a new perspective to matrix models.

Matrix Model on Noncommutative Spaces (Grosse-Wulkenhaar model)

- It corresponds to scalar field theories on noncommutative spaces, which is renormalizable by adding a harmonic oscillator potential to the action.
- Φ^3 matrix model[Grosse-Steinacker ('05), Grosse-Sako-Wulkenhaar ('17)]
- Φ^4 matrix model[Grosse-Sako ('23), arXiv:2308.11523]

  It has recently been shown that the partition function of a certain Hermitian Φ^4-matrix model corresponds to a zero-energy solution of a Schrödinger equation for the Hamiltonian of N-body harmonic oscillator system.

Introduction

Real Symmetric Φ^4 Matrix Model [Grosse-N.K-Sako-Wulkenhaar ('24)]

- We study a real symmetric Φ^4-matrix model whose kinetic term is given by $\mathrm{Tr}(E\Phi^2)$, where E is a positive diagonal matrix without degenerate eigenvalues.
- We show that the partition function of this matrix model corresponds to a zero-energy solution of a Schödinger type equation with Calogero-Moser Hamiltonian.
- A family of differential equations satisfied by the partition function is also obtained from the Virasoro algebra (Witt Algebra).

The discussion [Awata-Matsuo-Odake-Shiraishi ('94)] on an arbitrary polynomial-type potential $V(\Phi) = \sum_{n=0}^{\infty} \eta_n \Phi^n$ with a coupling constant η_n is quite different from the discussion [Grosse-Sako ('23), Grosse-N.K-Sako-Wulkenhaar ('24)] in this study, where the potential $V(\Phi) = \frac{\eta}{4}\Phi^4$ is fixed.

Φ^4 Matrix Model

Definition(Action of Φ^4 Matrix Model)

$$S_E[\Phi] = N \,\mathrm{Tr}\left\{ E\Phi^2 + \frac{\eta}{4}\Phi^4 \right\}$$

- $\Phi = (\Phi_{ij}), \quad i,j = 1,\cdots,N$:
  real symmetric matrix ($\beta = 1$), Hermitian matrix ($\beta = 2$)
- $E = (E_k \delta_{km}), \quad k,m = 1,\cdots,N$: diagonal matrix
- $\eta \in \mathbb{R}$

Definition(Partition Function)

$$Z(E,\eta) := \int d\Phi \; e^{-S_E[\Phi]}$$

Main Theorem
[Grosse-Sako ('23), Grosse-N.K-Sako-Wulkenhaar ('24)]

Let $\Delta(E)$ be the Vandermonde determinant $\Delta(E) := \prod_{k<l}(E_l - E_k)$. Then the function

$$\Psi(E,\eta) := e^{-\frac{N}{\beta\eta}\sum_{i=1}^{N} E_i^2}\Delta(E)^{\frac{\beta}{2}}Z(E,\eta)$$

is a zero-energy solution of the Schrödinger type equation

$$\mathcal{H}\Psi(E,\eta) = 0,$$

- $\mathcal{H}$ is the Hamiltonian for the N-body harmonic oscillator system ($\beta = 2$)
- $\mathcal{H}$ is the Hamiltonian for Calogero-Moser model ($\beta = 1$)

$$\mathcal{H} := \frac{-\eta}{2N}\left(\beta\sum_{i=1}^{N}\frac{\partial^2}{\partial E_i^2} + \frac{2-\beta}{4}\sum_{i\neq j}\frac{1}{(E_i - E_j)^2}\right) + 2\frac{N}{\beta\eta}\sum_{i=1}^{N}E_i^2. \qquad (1)$$

In this sense, this matrix model is a solvable system.

Schwinger-Dyson Equation

First, a Schwinger-Dyson equation is derived from

$$\int_{S_N} d\Phi \frac{\partial}{\partial \Phi_{tt}} \left(\Phi_{tt} e^{-S[\Phi]} \right) = 0,$$

which is expressed as

$$Z(E,\eta) - 2N \sum_{i=1}^{N} \langle H_{it} \Phi_{tt} \Phi_{ti} \rangle - \eta N \sum_{k,l=1}^{N} \langle \Phi_{tk} \Phi_{kl} \Phi_{lt} \Phi_{tt} \rangle = 0 \qquad (2)$$

- S_N: the space of real symmetric $N \times N$-matrices
- $H = (H_{ij}), \ \ i,j = 1, \cdots, N$: real symmetric matrix with nondegenerate eigenvalues $\{E_1, E_2, \cdots, E_N \mid E_i \neq E_j \text{ for } i \neq j\}$

Schwinger-Dyson Equation

Similarly, for $p \neq s$, from

$$\int_{S_N} d\Phi \frac{\partial}{\partial \Phi_{ps}} \left(\Phi_{ps} e^{-S[\Phi]} \right) = 0,$$

the following is obtained:

$$Z(E,\eta) - 2N \sum_{i=1}^{N} \left(\langle H_{ip} \Phi_{ps} \Phi_{si} \rangle + \langle H_{si} \Phi_{ip} \Phi_{ps} \rangle \right)$$
$$-2N\eta \sum_{k,l=1}^{N} \langle \Phi_{sk} \Phi_{kl} \Phi_{lp} \Phi_{ps} \rangle = 0. \tag{3}$$

From (2) and (3), after taking sum over the indices t, p, s, we get the follwing:

$$\frac{N(N+1)}{2} Z(E,\eta) - 2N \sum_{i,p,s=1}^{N} H_{ip} \langle \Phi_{is} \Phi_{sp} \rangle - \eta N \sum_{k,l,s,p=1}^{N} \langle \Phi_{ps} \Phi_{sk} \Phi_{kl} \Phi_{lp} \rangle$$

$$= 0.$$

By using

$$\frac{\partial Z(E,\eta)}{\partial H_{ps}} = -2N \sum_{k=1}^{N} \langle \Phi_{pk} \Phi_{ks} \rangle \quad \text{for} \quad p \neq s$$

$$\frac{\partial Z(E,\eta)}{\partial H_{pp}} = -N \sum_{k=1}^{N} \langle \Phi_{pk} \Phi_{kp} \rangle$$

$$\frac{\partial^2 Z(E,\eta)}{\partial H_{ps} \partial H_{tu}} = 4N^2 \sum_{k,l=1}^{N} \langle \Phi_{pk} \Phi_{ks} \Phi_{tl} \Phi_{lu} \rangle \text{ for } \; p \neq s, t \neq u$$

$$\frac{\partial^2 Z(E,\eta)}{\partial H_{pp} \partial H_{pp}} = N^2 \sum_{k,l=1}^{N} \langle \Phi_{pk} \Phi_{kp} \Phi_{pl} \Phi_{lp} \rangle \,,$$

a partial differential equation is obtained:

$$\frac{N(N+1)}{2}Z(E,\eta)+\sum_{i\neq p}H_{ip}\frac{\partial}{\partial H_{ip}}Z(E,\eta)+2\sum_{p=1}^{N}H_{pp}\frac{\partial}{\partial H_{pp}}Z(E,\eta)$$

$$-\frac{\eta}{N}\sum_{s=1}^{N}\frac{\partial^2}{\partial H_{ss}\partial H_{ss}}Z(E,\eta)-\frac{\eta}{4N}\sum_{s\neq l}\frac{\partial^2}{\partial H_{sl}\partial H_{ls}}Z(E,\eta)=0, \tag{4}$$

where we denote $\sum_{p=1}^{N}\sum_{i=1,i\neq p}^{N}$ by $\displaystyle\sum_{i\neq p}$. We define H'_{ij} by $H_{ii}=\sqrt{2}H'_{ii}$ for $i=1,\cdots,N$ and $H_{ij}=H'_{ij}$ for $i,j=1,\cdots,N\ \ (i\neq j)$, and we use an indices set $U=\{(p,s)|\ \ p\leq s,\ \ p,s\in\{1,2,\cdots,N\}\}$, for convenience.

Proposition 1 [Grosse-N.K-Sako-Wulkenhaar ('24)]

The partition function $Z(E, \eta)$ satisfies the following partial differential equation:

$$\mathcal{L}_{SD}^{H} Z(E, \eta) = 0.$$

Here, $\mathcal{L}_{SD}^{H}$ is a second order differential operator defined by

$$-\mathcal{L}_{SD}^{H} := \frac{N(N+1)}{2} + 2 \sum_{(p,s)\in U} H_{ps} \frac{\partial}{\partial H_{ps}} - \frac{\eta}{2N} \sum_{(p,s)\in U} \frac{\partial^2}{\partial H'_{ps} \partial H'_{sp}}.$$

We obtain the following.

Theorem 1 [Grosse-N.K-Sako-Wulkenhaar ('24)]

The partition function defined by $Z(E,\eta) := \int_{S_N} d\Phi \exp\left(-S[\Phi]\right)$ satisfies the partial differential equation

$$\mathcal{L}_{SD} Z(E,\eta) = 0,$$

where

$$\mathcal{L}_{SD} := \left\{ \frac{\eta}{2N} \sum_{i=1}^{N} \frac{\partial^2}{\partial E_i^2} + \frac{\eta}{2N} \sum_{l \neq i}^{N} \frac{1}{E_i - E_l} \frac{\partial}{\partial E_i} - 2 \sum_{k=1}^{N} E_k \frac{\partial}{\partial E_k} - \frac{N(N+1)}{2} \right\} \tag{5}$$

Diagonalization of $\mathcal{L}_{SD}$

Proposition 2 [Grosse-N.K-Sako-Wulkenhaar ('24)]

The differential operator $\mathcal{L}_{SD}$ defined in (5) is transformed as

$$e^{-\frac{N}{\eta}\sum_{i=1}^{N}E_i^2}\Delta(E)^{\frac{1}{2}}\mathcal{L}_{SD}\Delta(E)^{-\frac{1}{2}}e^{\frac{N}{\eta}\sum_{i=1}^{N}E_i^2}=-\mathcal{H}_{CM}.$$

Here, we denote the Hamiltonian of the Calogero-Moser model by $\mathcal{H}_{CM}$:

$$\mathcal{H}_{CM}:=-\frac{\eta}{2N}\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial E_i^2}+\frac{1}{4}\sum_{i\neq j}\frac{1}{(E_i-E_j)^2}\right)+2\frac{N}{\eta}\sum_{i=1}^{N}E_i^2. \qquad (6)$$

The Hamiltonian of the Calogero-Moser model is defined as follows:

$$H_{C_\gamma} := \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial y_j^2} + y_j^2\right) + \sum_{j>k}\frac{\gamma(\gamma-1)}{(y_j-y_k)^2}. \qquad (7)$$

After changing variable $\sqrt{\frac{2N}{\eta}}E_i = y_i$, if $\gamma = \frac{1}{2}$, (6) is identified with (7) up to global factor $\frac{1}{2}$:

$$H_{C_{\gamma=\frac{1}{2}}} = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial y_j^2} + y_j^2\right) - \frac{1}{4}\sum_{j>k}\frac{1}{(y_j-y_k)^2} = \frac{1}{2}\mathcal{H}_{CM}.$$

In the following, we consider only the case $\gamma = \frac{1}{2}$.

Virasoro Algebra (Witt Algebra)

Using $y_i = \sqrt{\frac{2N}{\eta}} E_i$, $\mathcal{L}_{SD}$ is expressed as

$$-\frac{1}{2}\mathcal{L}_{SD} = \sum_{k=1}^{N} y_k \frac{\partial}{\partial y_k} - \frac{1}{2}\left\{ \sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2} + \frac{1}{2}\sum_{l \neq i}^{N} \frac{1}{y_i - y_l}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_l}\right) \right\} + \frac{N(N+1)}{4}.$$

The Hamiltonian of Calogero-Moser model with $\gamma = \frac{1}{2}$ is given as

$$H_{C_{\gamma=\frac{1}{2}}} = g\left(-\frac{1}{2}\mathcal{L}_{SD}\right) g^{-1}. \quad (8)$$

Here $g = e^{-\frac{1}{2}\sum_i y_i^2} \prod_{j>k} (y_j - y_k)^{\frac{1}{2}}$.

In the following, we will proceed with the discussion with reference to [E. Bergshoeff and M. Vasiliev (1994)]. We define the creation, annihilation operators $a_i^\dagger, a_i$, and the coordinate swapping operator $K_{ij} \quad (i, j = 1, ..., N)$ obeying the following relations:

$$
\begin{aligned}
[a_i, a_j] &= [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = A_{ij} := \delta_{ij}\left(1 + \gamma \sum_{l=1}^{N} K_{il}\right) - \gamma K_{ij}, \\
K_{ij}K_{jl} &= K_{jl}K_{il} = K_{il}K_{ij}, \quad \text{for all } i \neq j, i \neq l, j \neq l, \\
(K_{ij})^2 &= I\,, \qquad K_{ij} = K_{ji}, \\
K_{ij}K_{mn} &= K_{mn}K_{ij}, \quad \text{if all indices } i, j, m, n \text{ are different}, \\
K_{ij}a_j^{(\dagger)} &= a_i^{(\dagger)}K_{ij}.
\end{aligned}
$$

In our case $\gamma = \dfrac{1}{2}$,

- K_{ij} :elementary permutation operators of the symmetric group $\mathfrak{S}_N$
- K_{ij} means the replacement of coordinates as $K_{ij}y_i = y_j$.

To make contact with the Calogero-Moser model, we chose these operators as

$$a_i = \frac{1}{\sqrt{2}}(y_i + D_i)\,, \quad a_i^\dagger = \frac{1}{\sqrt{2}}(y_i - D_i)\,,$$

with Dunkl derivatives

$$D_i = \frac{\partial}{\partial y_i} + \gamma \sum_{j=1, j\neq i}^{N} (y_i - y_j)^{-1}(1 - K_{ij})\,.$$

Dunkl derivatives satisfy the following commutation relations:

$$[y_i, y_j] = [D_i, D_j] = 0, \quad [D_i, y_j] = A_{ij}.$$

Let us introduce the following Hamiltonian like a harmonic oscillator system:

$$H = \frac{1}{2}\sum_{i=1}^{N}\{a_i\,,a_i^{\dagger}\}\,.$$

This Hamiltonian and $H_{C_{\gamma=\frac{1}{2}}}$ are related as

$$\begin{aligned}\mathrm{Res}(H) &= \prod_{j>k}(y_j - y_k)^{-\frac{1}{2}} \cdot H_{C_{\gamma=\frac{1}{2}}} \cdot \prod_{j>k}(y_j - y_k)^{\frac{1}{2}} \\ &= \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial y_j^2} + y_j^2\right) - \frac{1}{4}\sum_{j\neq k}\frac{1}{y_j - y_k}\left(\frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_k}\right),\end{aligned}$$

- $\mathrm{Res}(H)$ means that operator H acts on symmetric function space.
- It is possible to represent any differential operator D including K_{ij}'s as placing the elements of S_n at the right end, i.e. $D = \sum_{\omega\in S_N} D_\omega \omega$.
- Res is defined as $\mathrm{Res}\left(\sum_{\omega\in S_N} D_\omega\omega\right) = \sum_{\omega\in S_N} D_\omega$.

Definition(Representation of Virasoro Generators using Dunkl Operators)

$$L_{-n} = \sum_{i=1}^{N} \left(\alpha (a_i^\dagger)^{n+1} a_i + (1-\alpha) a_i (a_i^\dagger)^{n+1} + \left(\lambda - \frac{1}{2} \right) (n+1)(a_i^\dagger)^n \right)$$

- α, λ : arbitrary parameters

For simplicity, we chose $\lambda = \dfrac{1}{2}$. Especially if $L_{-n} = \sum_{i=1}^{N} \left(\alpha (a_i^\dagger)^{n+1} a_i + (1-\alpha) a_i (a_i^\dagger)^{n+1} \right)$, their commutators are given by the ones of the Virasoro algebra (Witt Algebra) with its central charge $c = 0$:

$$[L_n, L_m] = (n-m) L_{n+m}.$$

Virasoro Algebra Representation for Real Symmetric Φ^4 Matrix Model

From $H = L_0 - \left(\frac{1}{2} - \alpha\right)N + \frac{1}{2}\left(\alpha - \frac{1}{2}\right)\sum_{i \neq j} K_{ij}$, the commutator $[H\ ,\ L_{-m}]$ is obtained as

$$[H\ ,\ L_{-m}] = mL_{-m}.$$

From (8),

$$-\frac{1}{2}\mathcal{L}_{SD} = e^{\frac{1}{2}\sum_j y_j^2}\mathrm{Res}(H)e^{-\frac{1}{2}\sum_j y_j^2}.$$

- $\widetilde{L}_{-m} := e^{\frac{1}{2}\sum_j y_j^2} L_{-m} e^{-\frac{1}{2}\sum_j y_j^2}.$

The following is automatically satisfied:

$$[\widetilde{L}_n\ ,\ \widetilde{L}_m] = (n-m)\widetilde{L}_{n+m}.$$

$$D_i^E := \frac{\partial}{\partial E_i} + \frac{1}{2}\sum_{j=1, j\neq i}^{N} \frac{1}{(E_i - E_j)}(1 - K_{ij}) = \sqrt{\frac{2N}{\eta}} D_i.$$

- $[D_i^E, E_j] = A_{ij}$
- $[D_i^E, D_j^E] = 0$

Using this D_i^E, the operators $\widetilde{a}_i, \widetilde{a}_i^\dagger$ and $\widetilde{L}_{-n}$ are written as

$$\widetilde{a}_i = \frac{1}{2}\sqrt{\frac{\eta}{N}} D_i^E, \qquad \widetilde{a}_i^\dagger = 2\sqrt{\frac{N}{\eta}} E_i - \frac{1}{2}\sqrt{\frac{\eta}{N}} D_i^E,$$

$$\widetilde{L}_{-n} = \sum_{i=1}^{N} \left\{ \alpha \left(2\sqrt{\frac{N}{\eta}} E_i - \frac{1}{2}\sqrt{\frac{\eta}{N}} D_i^E \right)^{n+1} \frac{1}{2}\sqrt{\frac{\eta}{N}} D_i^E \right.$$
$$\left. + (1-\alpha)\frac{1}{2}\sqrt{\frac{\eta}{N}} D_i^E \left(2\sqrt{\frac{N}{\eta}} E_i - \frac{1}{2}\sqrt{\frac{\eta}{N}} D_i^E \right)^{n+1} \right\}.$$

Recall $\mathcal{L}_{SD} = -2e^{\frac{1}{2}\sum_j y_j^2}\mathrm{Res}(H)e^{-\frac{1}{2}\sum_j y_j^2}$, then

$$\begin{aligned}\left[\mathcal{L}_{SD}\ ,\ \widetilde{L}_{-m}\right] &= -\,2e^{\frac{1}{2}\sum_j y_j^2}[\mathrm{Res}(H)\ ,\ L_{-m}]e^{-\frac{1}{2}\sum_j y_j^2}\\ &= -\,2e^{\frac{1}{2}\sum_j y_j^2}[L_0\ ,\ L_{-m}]e^{-\frac{1}{2}\sum_j y_j^2} = -2m\widetilde{L}_{-m}.\end{aligned} \tag{9}$$

From Theorem 1 and (9), finally we get the following theorem.

Theorem 2 [Grosse-N.K-Sako-Wulkenhaar ('24)]

The partition function defined by (4) satisfies

$$\mathcal{L}_{SD}(\widetilde{L}_{-m}Z(E,\eta)) = -\,2m(\widetilde{L}_{-m}Z(E,\eta)).$$

This means that $\widetilde{L}_{-m}Z(E,\eta)$ is an eigenfunction of $\mathcal{L}_{SD}$ with the eigenvalue $-2m$.

Overall Summary

- We studied a real symmetric Φ^4-matrix model whose kinetic term is given by $\mathrm{Tr}(E\Phi^2)$, where E is a positive diagonal matrix without degenerate eigenvalues.
- We showed that the partition function of this matrix model corresponds to a zero-energy solution of a Schödinger type equation with Calogero-Moser Hamiltonian.
- A family of differential equations satisfied by the partition function was also obtained from the Virasoro algebra (Witt Algebra).