Higher Courant-Dorfman algebras and associated higher Poisson vertex algebras

早見崚

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A Courant algebroid is a vector bundle equipped with a non-degenerate metric, Courant bracket on the sections of the bundle, and an anchor map to tangent bundle, satisfying a set of compatibility conditions.

Example (Courant 90)

Dorfman bracket on $TM \oplus T^*M$

$$\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \iota_{v_1}\alpha_2 + \iota_{v_1}\alpha_2,$$

$$[(v_1, \alpha_1), (v_2, \alpha_2)] = ([v_1, v_2], L_{v_1}\alpha_2 + \frac{1}{2}d(\iota_{v_2}\alpha_1 - \iota_{v_1}\alpha_2) - L_{v_2}\alpha_1).$$

Courant algebroids are in one-to-one correspondence with degree 2 differential-graded (dg for short) symplectic manifolds.
A Courant-Dorfman algebra is an algebraic generalization of a Courant algebroid.
The relation between Courant-Dorfman algebras and Poisson vertex algebras was found in the context of current algebras.

Current algebras: Poisson algebras consisting of functions on mapping spaces.

Example (Alekseev-Strobl 04)

Current algebras on $C^\infty(T^*LM)$ ($LM = \mathrm{Map}(S^1, M)$)

$$J_{(v,\alpha)}(\sigma) = v^i(x(\sigma))p_i(\sigma) + \alpha_i(x(\sigma))\partial_\sigma x^i(\sigma).$$

$$\{J_{(v,\alpha)}(\sigma), J_{(u,\beta)}(\sigma')\} = J_{[(v,\alpha),(u,\beta)]}(\sigma)\delta(\sigma-\sigma') + \langle (v,\alpha),(u,\beta)\rangle(\sigma)\delta'(\sigma-\sigma'),$$

u, v is a vector field on M, α, β is a 1-form on M, $[(v,\alpha),(u,\beta)] = ([v,u], L_v\beta - \iota_u d\alpha)$ is the Dorfman bracket on the generalized tangent bundle $TM \oplus T^*M$ and
$\langle (v,\alpha),(u,\beta)\rangle = \iota_u\alpha + \iota_v\beta$.

Ekstrand and Zabzine studied the algebraic structure underlying more general current algebras on loop spaces, They found that a weak notion of Courant-Dorfman algebras (weak Courant-Dorfman algebras) appears when we consider the Poisson bracket of currents.
Ekstrand derived weak CD algebras and CD algebras using the language of Lie conformal algebras (LCA for short) and Poisson vertex algebras (PVA for short).
Ekstrand proved that Poisson vertex algebras that are graded and generated by elements of degree 0 and 1 are in a one-to-one correspondence with Courant-Dorfman algebras.

Alekseev-Strobl current algebras can be described in the language of dg symplectic geometry.
Poisson algebras on the mapping space
$n-1$ dimensional manifolds→degree n dg symplectic manifolds
were constructed. They are called BFV(Batalin-Fradkin-Vilkovisky) current algebras.

BFV current algebras (Ikeda-Koizumi 11, Ikeda-Xu 13, Arvanitakis 21)

$$J_\epsilon(\alpha)(\phi) = \int_{T[1]\Sigma_{n-1}} \epsilon \cdot \phi^*(a)(\sigma,\theta) d^{n-1}\sigma d^{n-1}\theta,$$

$$\{J_{\epsilon_1}(\alpha), J_{\epsilon_2}(\beta)\} = \int_{T[1]\Sigma_{n-1}} \epsilon_1\epsilon_2\{\{\alpha,\Theta\},\beta\}(\sigma,\theta) d^{n-1}\sigma d^{n-1}\theta + \int_{T[1]\Sigma_{n-1}} (d\epsilon_1)\epsilon_2\{\alpha,\beta\}(\sigma,\theta) d^{n-1}\sigma d^{n-1}\theta.$$

They can be seen as a higher generalization of Alekseev-Strobl current algebras.

We can expect that we can construct higher Poisson vertex algebras and higher Courant-Dorfman algebras which are generalizations of BFV currents. In this thesis, we give a higher analog of the relation between Poisson vertex algebras and Courant-Dorfman algebras.

- We define (extended) higher Courant-Dorfman algebras and give examples.
- We make graded Poisson algebras of degree $-n$ from a non-degenerate higher Courant-Dorfman algebras, generalizing Keller-Waldman Poisson algebras. For higher Courant-Dorfman algebras from finite-dimensional graded vector bundles, this graded Poisson algebra is isomorphic to the algebra of functions of degree n dg symplectic manifolds.
- We define higher analogs of Lie conformal algebras and Poisson vertex algebras. We derive a weak notion of higher Courant-Dorfman algebras from higher Lie conformal algebras, and give the correspondence between higher Poisson vertex algebras and higher Courant-Dorfman algebras. Moreover, we check higher Lie conformal algebras and higher Poisson vertex algebras have LCA-like and PVA-like properties.

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A graded manifold $\mathcal{M}$ on a smooth manifold M: a locally ringed space whose structure sheaf (we denote it by $C^\infty(\mathcal{M})$) is locally isomorphic to $(U, C^\infty(U) \otimes \mathrm{Sym} V^*)$. ($U \subset \mathbb{R}^n$ is open, V is a finite-dimensional graded vector space)
A graded vector field on a graded manifold $\mathcal{M}$ of degree k: a graded linear map

$$X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})[k],$$

where $W[k]^i = W^{k+i}$, which satisfies the graded Leibniz rule, i.e.

$$X(fg) = X(f)g + (-1)^{k|f|} f X(g)$$

holds for all homogeneous smooth functions $f, g \in C^\infty(\mathcal{M})$. ($f$: the degree of $|f|$)
A cohomological vector field Q is a graded vector field of degree 1 which satisfies $Q^2 = 0$.

A graded symplectic form of degree k on a graded manifold $\mathcal{M}$ is a two-form ω which has the following properties;

- ω is homogeneous of degree k,
- ω is closed with respect to the de-Rham differential

$$d = \sum_i dz^i \frac{\partial}{\partial z^i},$$

- ω is non-degenerate, i.e. the induced morphism,

$$\omega : T\mathcal{M} \to T^*[k]\mathcal{M},$$

is an isomorphism. There $[k]$ means degree shifting the fibres of the vector bundle.

Let ω be a graded symplectic form on a graded manifold $\mathcal{M}$. A vector field X is Hamiltonian if there is a smooth function H such that $\iota_X \omega = dH$.
For a degree k graded symplectic manifold $(\mathcal{M}, \omega)$, we can define a degree $-n$ Poisson bracket $\{-,-\}$ on $C^\infty(\mathcal{M})$ via

$$\{f, g\} := (-1)^{|f|+1} X_f(g)$$

where X_f is the unique graded vector field that satisfies $\iota_{X_f}\omega = df$(a Hamiltonian vector field of f). If the vector field Q is Hamiltonian, one can find a Hamiltonian function S such that

$$Q = \{S, -\}.$$

$Q^2 = 0$(i.e. Q is cohomological) is equivalent to

$$\{S, S\} = 0.$$

This equation is known as the classical master equation.

A cohomological vector field with a Hamiltonian function S such that $Q = \{S, -\}$ is called a symplectic cohomological vector field.
A graded manifold endowed with a graded symplectic form and a symplectic cohomological vector field is called a differential graded symplectic manifold, or dg symplectic manifold for short.
There is a one-to-one correspondence between the isomorphism class of dg symplectic manifolds of degree 2 and isomorphism class of Courant algebroids.

Definition (Liu-Weinstein-Xu 95)

A Courant algebroid is a vector bundle E over a smooth manifold M, with a non-degenerate symmetric bilinear form $\langle , \rangle$, and a bilinear bracket $*$ on $\Gamma(E)$. The form and the bracket must be compatiable, in the meaning defined below, with the vector fields on M. We must have a smooth bundle map, the anchor

$$\pi : E \to TM.$$

These structure satisfy the following five axioms, for all $A, B, C \in \Gamma(E)$ and $f \in C^\infty(M)$.

Axiom.1 : $\pi(A * B) = [\pi(A), \pi(B)]$ (The bracket of the right hand side is the Lie bracket of vector fields).

Axiom.2 : $A * (B * C) = (A * B) * C + B * (A * C)$.

Axiom.3 : $A * (fB) = (\pi(A)f)B + f(A * B)$.

Axiom.4 : $\langle A, B * C + C * B \rangle = \pi(A)\langle B, C \rangle$.

Axiom.5 : $\pi(A)\langle B, C \rangle = \langle A * B, C \rangle + \langle B, A * C \rangle$.

$(\mathcal{M}, \omega, S)$: degree 2 dg symplectic manifold
Denote $C_i(C^\infty(\mathcal{M})) = \{f \in C^\infty(\mathcal{M}) : |f| = i\}$. Then

$$C_0(C^\infty(\mathcal{M})) \simeq C^\infty(M), C_1(C^\infty(\mathcal{M})) \simeq \Gamma(E).$$

For $f \in C_0(C^\infty(\mathcal{M}))$ and $A, B \in C_1(C^\infty(\mathcal{M}))$, we define

$$\{\{A, S\}, B\} = A * B$$
$$\{\{A, S\}, f\} = \pi(A)f = \partial(f)A = \{\{S, f\}, A\}.$$

This computation gives a dg symplectic manifold of degree 2 the structure of a Courant algebroid.

Theorem (Roytenberg 02)

Dg symplectic manifolds of degree 2 are in 1-1 correspondence with Courant algebroids.

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Definition (Roytenberg 09)

A Courant-Dorfman algebra consists of the following data:

- a commutative algebra R,
- an R-module E,
- a symmetric bilinear form $\langle , \rangle : E \otimes E \to R$,
- a derivation $\partial : R \to E$,
- a Dorfman bracket $[,] : E \otimes E \to E$,

which satisfies the following conditions;

$$[e_1, fe_2] = f[e_1, e_2] + \langle e_1, \partial f \rangle e_2, \langle e_1, \partial \langle e_2, e_3 \rangle \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle,$$

$$[e_1, e_2] + [e_2, e_1] = \partial \langle e_1, e_2 \rangle, [e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]],$$

$$[\partial f, e] = 0, \langle \partial f, \partial g \rangle = 0,$$

where $f, g \in R$ and $e_1, e_2, e_3 \in E$.

If $\langle,\rangle$ is non-degenerate, we can make a graded Poisson algebra of degree -2
When $R = C^\infty(M), E = \Gamma(F)$ for a vector bundle $F \to M$, this algebra is isomorphic to the algebra of functions of the associated dg symplectic manifold.

Definition (Zabzine-Ekstrand 09)

A weak Courant-Dorfman algebra $(E, R, \partial, \langle,\rangle, [,])$ is defined by the following data:

- a vector space R,
- a vector space E,
- a symmetric bilinear form $\langle,\rangle : E \otimes E \to R$,
- a map $\partial : R \to E$,
- a Dorfman bracket $[,] : E \otimes E \to E$,

which satisfy the following conditions:

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]], [A, B] + [B, A] = \partial\langle A, B\rangle, [\partial f, A] = 0.$$

Definition

A Lie conformal algebra is a $\mathbb{C}[\partial]$-module W (i.e.∂ acts on elements of W) with a λ-bracket $\{_\lambda\} : W \otimes W \to W[\lambda]$, $\{a_\lambda b\} = \sum_{j \in \mathbb{Z}_+} \lambda^j a_{(j)} b$ which satisfies the conditions.

Sesquilinearity :$\{\partial a_\lambda b\} - \lambda\{a_\lambda b\}, \ \{a_\lambda \partial b\} = (\partial + \lambda)\{a_\lambda b\}$

Skew-symmetry :$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}$

Jacobi-identity :$\{a_\lambda\{b_\mu c\}\} = \{\{a_\lambda b\}_{\mu+\lambda} c\} + \{b_\mu\{a_\lambda c\}\}$

A Poisson vertex algebra is a commutative algebra W with a derivation ∂(i.e.$\partial(ab) = (\partial a)b + a(\partial b)$) and λ-bracket $\{_\lambda\} : W \otimes W \to W[\lambda]$ such that W is a Lie conformal algebra and satisfies the Leibniz rule.

Leibniz rule :$\{a_\lambda b \cdot c\} = \{a_\lambda b\} \cdot c + b \cdot \{a_\lambda c\}$

The relation between current algebras on loop spaces and Lie conformal and Poisson vertex algebras
Denote the coordinates on T^*LM by $u^\alpha(\sigma) = \{X^\alpha(\sigma), P_{\alpha-d}(\sigma)\}^\alpha$, where $\alpha = 1, ..., 2d$ and let $u^{\alpha(n)} = \partial^n u^\alpha$. Fix $N < \infty$ and assume the local functions can be written as polynomials

$$a(u^\alpha, ..., u^{\alpha(N)}).$$

We have a total derivative operator by

$$\partial = u^{\alpha(1)} \frac{\partial}{\partial u^\alpha} + \cdots + u^{\alpha(N+1)} \frac{\partial}{\partial u^{\alpha(N)}}.$$

The algebra of these polynomials with the total derivative is called an algebra of differential equation $\mathcal{V}$. When we integrate functions over S^1, the function of the form $\partial_\sigma(\cdots)$ doesn't contribute. We can take the quotient $\mathcal{V}/\partial\mathcal{V}$.

A local Poisson bracket on the phase space can be described by

$$\{u^\alpha(\sigma), u^\beta(\sigma')\} = H_0^{\alpha\beta}(\sigma')\delta(\sigma-\sigma') + H_1^{\alpha\beta}(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') + \cdots + H_N^{\alpha\beta}(\sigma')\partial_{\sigma'}^N\delta(\sigma-\sigma').$$

For $a, b \in \mathcal{V}$, we have

$$\{a(\sigma), b(\sigma')\} = \sum_{m,n} \frac{\partial a(\sigma)}{\partial u^{\alpha(m)}} \frac{\partial b(\sigma')}{\partial u^{\beta(n)}} \partial_\sigma^m \partial_{\sigma'}^n \{u^\alpha(\sigma), u^\beta(\sigma')\}.$$

Using a Fourier transformation of this Poisson bracket, we get a Poisson vertex algebra. Define the Fourier transformed bracket by

$$\{a_\lambda b\} = \int_{S^1} e^{\lambda(\sigma-\sigma')}\{a(\sigma), b(\sigma')\}d\sigma.$$

This bracket (called a λ-bracket) with $\mathcal{V}, \partial$ is a Lie conformal algebra. Considering the multiplication of polynomials on $\mathcal{V}$, it is a Poisson vertex algebra. So we can translate the relation between (weak) Courant-Dorfman algebras and currents on the phase space into that between (weak) Courant-Dorfman algebras and Lie conformal algebras and Poisson vertex algebras.

From Lie conformal algebras and Poisson vertex algebras, we can make Lie algebras and Poisson algebras using formal power series. For a Lie conformal algebra W, $W \otimes \mathbb{C}[[t,t^{-1}]]/Im(\partial + \partial_t)$ is a Lie algebra with the Lie bracket

$$[\alpha \otimes t^m, \beta \otimes t^n] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (\alpha_{(j)} \beta) t^{m+n-j}.$$

Moreover, for a Poisson vertex algebra W, $W \otimes \mathbb{C}[[t,t^{-1}]]/Im(\partial + \partial_t) \cdot W \otimes \mathbb{C}[[t,t^{-1}]]$ is a Poisson algebra with the same Lie bracket.

We can derive the properties of a weak Courant-Dorfman algebra from a Lie conformal algebra by comparing the 0th-order terms of λ, of these axioms.

$$[a_\lambda b] = \sum_{j\geq 0} a_{(j)} b \lambda^j,\ \ a_{(0)} b = [a, b],\ \ [a_\lambda b] - [a, b] = \langle a_\lambda b\rangle,$$

$$\langle a, b\rangle = \frac{1}{2}(\langle a_{-\partial} b\rangle + \langle b_{-\partial} a\rangle).$$

$$[\partial a, b] + o(\lambda) = \{\partial a_\lambda b\} = \lambda\{a_\lambda b\} \Rightarrow [\partial a, b] = 0,$$

$$[a, b] + o(\lambda) = \{a_\lambda b\} = -\{b_{-\lambda-\partial} a\} = -[b, a] + \partial\langle b_{-\partial} a\rangle + o(\lambda) \Rightarrow [a, b] + [b, a] = \partial\langle a, b\rangle,$$

$$[a, [b, c]] + o(\lambda) = [[a, b], c] + [b, [a, c]] + o(\lambda) \Rightarrow [a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

The right formulas are the conditions of a weak Courant-Dorfman algebra.

Moreover, a one-to-one correspondence between graded Poisson vertex algebras generated by elements of degree 0 and 1 and Courant-Dorfman algebras is established. In this case, the λ-bracket is of the form

$$\{a_\lambda b\} = [a,b] + \lambda\langle a,b\rangle.$$

Substituting this for the axioms of Poisson vertex algebras, we can get the axioms of Courant-Dorfman algebraic structure.

Theorem (Ekstrand 11)

The Poisson vertex algebras that are graded and generated by elements of degree 0 and 1 are in a one-to-one correspondence with the Courant-Dorfman algebras via

$$W^0 = R,\ W^1 = E,\ \partial = \partial$$

$$[e_\lambda e'] = [e,e'] + \lambda\langle e,e'\rangle,\ [e_\lambda f] = \langle e,\partial f\rangle$$

In the case of $E = TM \oplus T^*M$, the associated Poisson vertex algebra can be seen as the algebraic description of Alekseev-Strobl currents.

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$R = E^0$:a commutative algebra
$E = \oplus_{i=0}^{n-1} E^i$:a graded R-module
$\langle , \rangle : E \otimes E \to R$ a pairing such that $\langle a, b \rangle = 0$ unless $|a| + |b| = n$
Consider the graded-commutative algebra freely generated by E and denote it by $\tilde{\mathcal{E}} = (\mathcal{E}^k)_{k \in \mathbb{Z}}$. We restrict this graded-commutative algebra to the elements of degree $n - 1 \geq k \geq 0$ and denote it by $\mathcal{E} = (\mathcal{E}^k)_{n-1 \geq k \geq 0}$. The pairing $\langle , \rangle$ can be extended to $\mathcal{E}$ by the Leibniz rule

$$\langle a, b \cdot c \rangle = \langle a, b \rangle \cdot c + (-1)^{(|a|-n)|b|} b \cdot \langle a, c \rangle.$$

Definition (Definition 4.1)

$\mathcal{E} = (\mathcal{E}^k)$ is *a higher Courant-Dorfman algebra* if $\mathcal{E}$ has a differential $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ which satisfies $d^2 = 0$ and $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$ and a bracket $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $1 - n$ which satisfies the following condition:

sesquilinearity :

$$\langle da, b\rangle = -(-1)^{|a|-n}[a,b],\ [da,b] = 0.$$

skew-symmetry :

$$[a,b] + (-1)^{(|a|+1-n)(|b|+1-n)}[b,a] = -(-1)^{|a|}d\langle a,b\rangle,$$
$$\langle a,b\rangle = -(-1)^{(|a|-n)(|b|-n)}\langle b,a\rangle.$$

Jacobi identity :

$$[a,[b,c]] = [[a,b],c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b,[a,c]],$$
$$[a,\langle b,c\rangle] = \langle [a,b],c\rangle + (-1)^{(|a|+1-n)(|b|+1-n)}\langle b,[a,c]\rangle,$$
$$\langle a,\langle b,c\rangle\rangle = \langle\langle a,b\rangle,c\rangle + (-1)^{(|a|-n)(|b|-n)}\langle b,\langle a,c\rangle\rangle.$$

Leibniz rule :

$$[a\cdot b, c] = [a,b]\cdot c + (-1)^{(|a|+1-n)|b|}b\cdot [a,c].$$

We can define a Dirac submodule, like an ordinary Couarnt-Dorfman algebra.

Definition

Suppose $\mathcal{E}$ is a higher Courant-Dorfman algebra. An R-submodule $\mathcal{D} \subset \mathcal{E}$ is said to be a Dirac submodule if $\mathcal{D}$ is isotropic with respect to $\langle , \rangle$ and closed under $[-,-]$.

Example

·n=2: These algebras are Courant-Dorfman algebras
·Given R, let $E^1 = \Omega^1$, $E^{n-1} = \mathfrak{X} = Der(R)$ ($\mathcal{E}^{n-1} = \mathfrak{X} \oplus \Omega^{n-1}$)
It becomes a higher Courant-Dorfman algebra with respect to

$$\langle v, \alpha \rangle = \iota_v \alpha, [v, \alpha] = L_v \alpha, [\alpha, v] = -\iota_v d\alpha, [v_1, v_2] = \iota_{v_1} \iota_{v_2} \omega (\omega \in \Omega^{n+1,cl}).$$

When $R = C^\infty(M)$, $\mathcal{E}^{n-1} = TM \oplus \wedge^{n-1} T^* M$ (higher Dorfman bracket).
In this case, a Dirac submodule is called a higher Dirac structure. [Bursztyn-Alba-Rubio,16]

Example

We can replace $\mathfrak{X}$ by a Lie-Rinehart algebra (R, L) and let $E^{n-1} = L$

$$\langle v, \alpha \rangle = \iota_{\rho(v)}\alpha, [v, \alpha] = L_{\rho(v)}\alpha, [\alpha, v] = -\iota_{\rho(v)}d\alpha, [v_1, v_2] = \iota_{\rho(v_1)}\iota_{\rho(v_2)}\omega(\omega \in \Omega^{n+1,cl}).$$

$\cdot(\mathcal{M}, \omega, \Theta)$: a dg symplectic manifold of degree n
Then, $C^{n-1}(C^\infty(\mathcal{M})) = \{a \in C^\infty(\mathcal{M}) : |a| \leq n-1\}$ is a higher Courant-Dorfman algebra via

$$[a, b] = \{\{a, \Theta\}, b\}, \ \langle a, b \rangle = \{a, b\}.$$

In order to focus on the relation with higher Poisson vertex algebras, we should define extended higher Courant-Dorfman algebras, relaxing the condition of $\langle , \rangle$.

Definition (Definition4.6)

Let $R = E^0$ be a commutative algebra, and $E = E^i (1 \le i \le n-1)$ be a graded R-module. Consider the graded-commutative algebra freely generated by E and denote it by $\tilde{\mathcal{E}} = (\mathcal{E}^k)_{k \in \mathbb{Z}}$. We restrict this graded-commutative algebra to the elements of degree $n-1 \ge k \ge 0$ and denote it by $\mathcal{E} = (\mathcal{E}^k)_{n-1 \ge k \ge 0}$.
$\mathcal{E} = (\mathcal{E}^k)_{n-1 \ge k \ge 0}$ is *an extended higher Courant-Dorfman algebra* of degree n if $\mathcal{E}$ has a differential $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ which satisfies $d^2 = 0$ and $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$, a pairing $\langle , \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $-n$ and a bracket $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ of degree $1-n$ which satisfies the sesquilinearity, skew-symmetry, Jacobi identity, and Leibniz rule.

The difference with a higher Courant-Dorfman algebra is that an extended Courant-Dorfman algebra allows the pairing $\langle , \rangle : E^i \otimes E^j \to E^{i+j-n}$ with $i + j \ge n+1$.

We consider the case that $\langle , \rangle$ is non-degenerate, and study the relationship between the algebras and functions of degree n dg symplectic manifolds. We construct a graded Poisson algebra of degree $-n$, generalizing the Keller-Waldman Poisson algebras.

Definition (Definition4.3)

The bilinear form $\langle , \rangle$ gives rise to a map

$$(-)^{\flat} : E^i \to (E^{n-i})^{\vee} = Hom_R(E^{n-i}, R)$$

defined by

$$e^{\flat}(e') = \langle e, e' \rangle.$$

$\langle , \rangle$ is non-degenerate if $(-)^{\flat}$ is an isomorphism, and a higher Courant-Dorfman algebra is non-degenerate if $\langle , \rangle$ is non-degenerate.

$r \geq n$. $\mathcal{C}^r(\mathcal{E}) \subset \oplus_{1\leq j\leq n-1} \oplus_{1\leq k\leq r-j} \oplus_{\sum_{t=1}^{k} i_t = r-j} \mathrm{Hom}_K(E^{n-i_1} \otimes \cdots \otimes E^{n-i_k}, E^j)$ consists of elements C for which there exists a K-multilinear map

$$\sigma_C \in \oplus_{1\leq k\leq r-j} \oplus_{\sum_{t'=1}^{k-1} i_{t'} = r-n} \mathrm{Hom}_K(E^{n-i_1} \otimes \cdots \otimes E^{n-i_{k-1}}, \mathfrak{X}),$$

satisfying the following conditions:

(1)For all $x_1, ..., x_{k-1}, u, w \in E$, we have

$$\sigma_C(x_1, ..., x_{k-1})\langle u, w\rangle = \langle C(x_1, ..., x_{k-1}, u), w\rangle + \langle u, C(x_1, ..., x_{k-1}, w)\rangle.$$

(2)For all $x_1, ..., x_k, u \in E$, we have

$$\langle C(x_1, ...x_i, x_{i+1}, ..., x_k) - (-1)^{(|x_i|-n)(|x_{i+1}|-n)} C(x_1, ..., x_{i+1}, x_i, ..., x_k), u\rangle$$
$$= \sigma_C(x_1, ..., x_{i-1}, x_{i+2}, ..., x_k, u)\langle x_i, x_{i+1}\rangle.$$

Furthermore, $\mathcal{C}^0(\mathcal{E}) = R, \mathcal{C}^i(\mathcal{E}) = \mathcal{E}^i$ for $1 \leq i \leq n-1$ and define

$$\mathcal{C}^\bullet(\mathcal{E}) = \oplus_{r\geq 0}\mathcal{C}^r(\mathcal{E}).$$

We call σ_C the symbol of C.

$$[,]:\mathcal{C}^r(\mathcal{E})\otimes\mathcal{C}^s(\mathcal{E})\to\mathcal{C}^{r+s-n}(\mathcal{E})$$

is defined by

$$[a,b]=0,[a,x]=0=[x,a],[x,y]=\langle x,y\rangle,[D,a]=\sigma_D a=-[a,D],$$
$$[C,x]=\iota_x C=-(-1)^{(r+n)(|x|+n)}[x,C]$$

for elements $a,b\in R,x,y\in E,D\in\mathcal{C}^n(\mathcal{E}),C\in\mathcal{C}^r(\mathcal{E})$ for $r\geq n$, and by the recursion

$$\iota_x[C_1,C_2]=[[C_1,C_2],x]=[C_1,[C_2,x]]-(-1)^{(|C_1|+n)(|C_2+n|)}[C_2,[C_1,x]]$$

A product is defined by

$$a\wedge b=ab=b\wedge a,a\wedge x=ax=x\wedge a$$

for $a,b\in R$ and $x\in E$ and by the recursion rule

$$[C_1\wedge C_2,x]=[C_1,C_2]\wedge x+(-1)^{(|C_1|-n)|C_2|}C_2\wedge[C_1,x]$$

$(\mathcal{C}^\bullet(\mathcal{E}),[,],\wedge)$ is a graded Poisson algebra of degree $-n$.

For $r \geq 1$ the subspace $\Omega^r_{\mathcal{C}}(\mathcal{E}) \subset \oplus_{1\leq k\leq r} \oplus_{\sum_{t=1}^{k} i_t = r} \mathrm{Hom}_K(E^{n-i_1} \otimes \cdots \otimes E^{n-i_k}, R)$ consists of elements ω satisfying the following conditions;
(1)

$$\omega(x_1, ..., ax_k) = a\omega(x_1, ..., x_k),$$

for all $a \in R$.
(2)For $r \geq n$, there exists a multilinear map,

$$\sigma_\omega \in \oplus_{1\leq k\leq r} \oplus_{\sum_{t'=1}^{k-1} i_{t'} = r-n} \mathrm{Hom}_K(E^{n-i_1} \otimes \cdots \otimes E^{n-i_{k-1}}, \mathfrak{X}),$$

such that

$$\omega(x_1, ...x_i, x_{i+1}, ..., x_k) - (-1)^{(|x_i|-n)(|x_{i+1}|-n)}\omega(x_1, ..., x_{i+1}, x_i, ..., x_k)$$
$$= \sigma_\omega(x_1, ...\overset{i}{\wedge}...\overset{i+1}{\wedge}, x_k)\langle x_i, x_{i+1}\rangle.$$

By the non-degeneracy of $\langle , \rangle$, there is an isomorphism of graded R-modules

$$\mathcal{C}^\bullet(\mathcal{E}) \to \Omega^\bullet_{\mathcal{C}}(\mathcal{E}),$$

given by

$$\omega(x_1, ..., x_t) = \langle C(x_1, ..., x_{t-1}), x_t \rangle.$$

We can construct $\omega_\phi \in \Omega^r_C(\mathcal{E}) \simeq \mathcal{C}^r(\mathcal{E})$ from the map $\phi : \mathcal{E}^{i_1} \otimes \mathcal{E}^{i_2} \otimes \cdots \otimes \mathcal{E}^{i_m} \to \mathcal{E}^{i_1+\cdots+i_m-mn+r}$ by

$$\omega_\phi(e_1, e_2, ..., e_k) = \langle \cdots \langle \phi(e_1, ..., e_m), e_{m+1} \rangle \cdots \rangle, e_k \rangle.$$

Let ϕ be the bracket of the higher Courant-Dorfman algebra. Then, ω_ϕ satisfies $|\omega_\phi| = n + 1$ and $[\omega_\phi, \omega_\phi] = 0$ and the map $[\omega_\phi, -]$ is degree 1 and squares to 0, so it defines a differential on $\mathcal{C}^\bullet(\mathcal{E})$.

Higher Rothstein algebra

A connection ∇ for the graded module $E = (E^i)$: $\nabla : \mathfrak{X} \times E \to E$ of degree 0 such that

$$\nabla_{rD} x = r \nabla_D x$$

$$\nabla_D (rx) = r \nabla_D x + D(r) x$$

for all $x, y \in E$ and $D \in \mathfrak{X}$. ∇ is called metric if in addition

$$D\langle x.y \rangle = \langle \nabla_D x, y \rangle + \langle x, \nabla_D y \rangle$$

for all $x, y \in E$ and $D \in \mathfrak{X}$.

Curvature of ∇

$$R(D_1, D_2)\xi = \nabla_{D_1} \nabla_{D_2} \xi - \nabla_{D_2} \nabla_{D_1} \xi - \nabla_{[D_1, D_2]} \xi$$

for $D_i \in \mathfrak{X}$ and $\xi \in Sym(E)$

Define $r(D_1, D_2) \in \mathrm{Sym}^2 E|_{deg=n}$ by

$$R(D_1, D_2) x = \langle r(D_1, D_2), x \rangle.$$

$$\mathcal{R}^\bullet(\mathcal{E}) = \mathrm{Sym}(\oplus_{1\leq i\leq n-1} E^i[-i] \oplus \mathfrak{X}[-n]).$$

Let ∇ be a metric connection on E. Then there exists a unique graded Poisson structure $\{-,-\}_R$ on $\mathcal{R}^\bullet(\mathcal{E})$ of degree $-n$ such that

$$\begin{aligned}
\{a,b\}_R &= 0 = \{a,x\}_R \\
\{x,y\}_R &= \langle x,y\rangle \\
\{D,a\}_R &= -D(a) \\
\{D,x\}_R &= -\nabla_D x \\
\{D_1,D_2\}_R &= -[D_1,D_2] - r(D_1,D_2).
\end{aligned}$$

for $a,b\in R$, $x,y\in E$ and $D_1,D_2\in\mathfrak{X}$.

the relation between $\mathcal{R}^\bullet(\mathcal{E})$ and $\mathcal{C}^\bullet(\mathcal{E})$
Let the R-linear map $\mathcal{J}:\mathcal{R}^\bullet(\mathcal{E})\to\mathcal{C}^\bullet(\mathcal{E})$ be defined on generators by

$$\mathcal{J}(a)=a, \mathcal{J}(x)=x, \mathcal{J}(D)=-\nabla_D$$

for $a\in R, x\in E$ and $D\in\mathfrak{X}$ and extend to all degrees as homomorphism.
(1) The map $\mathcal{J}$ is a homomorphism of Poisson algebras.
(2) Let $\phi\in\mathcal{R}^\bullet(\mathcal{E})$ with $r\geq n$, then

$$\mathcal{J}(\phi)(x_1,...,x_k)=\{\{...\{\phi,x_1\}_R,...\}_R,x_k\}_R$$

Let $\hat{\mathcal{C}}^\bullet(\mathcal{E})$ be the subalgebra of $\mathcal{C}^\bullet(\mathcal{E})$ generated by R, E and $\mathcal{C}^n(\mathcal{E})$. Then $\hat{\mathcal{C}}^\bullet(\mathcal{E})$ is closed under the bracket $[,]$ and $\mathcal{J}$ is an isomorphism of Poisson algebras

$$\mathcal{J}:\mathcal{R}^\bullet(\mathcal{E})\to\hat{\mathcal{C}}^\bullet(\mathcal{E}).$$

$R = C^\infty(M), E^i = \Gamma(F^i)$ for a graded vector bundle $F \to M$
Then we can construct a graded symplectic manifold $(\mathcal{M}, \omega)$ of degree n. (The Poisson bracket $\{-,-\}$ is an extension of $\langle -,-\rangle$.)
$C^\infty(\mathcal{M})$ is isomorphic to $\mathcal{R}^\bullet(\mathcal{E})$.
ω_ϕ corresponds to the Hamiltonian Θ of the graded symplectic manifold.
Locally,

$$\Theta = \sum_{\sum i_t = n+1} \phi(q) \xi^{a_1(i_1)} \cdots \xi^{a_m i_m}$$

$$\phi(q) = \langle \cdots \langle [e^{a_1(n-i_1)}, e^{a_2(n-i_2)}], e^{a_3(n-i_3)} \rangle, \cdots, e^{a_m(n-i_m)} \rangle.$$

(Darboux chart $(\xi^{a(k)}) = (q^{a(l)}, p^{a(n-l)})(1 \leq k \leq n, 1 \leq l \leq \lfloor \frac{n}{2} \rfloor)$, corresponding to a chart (x_i) on M and a local basis $e^{a(k)}$ of sections of E^k such that $\langle e^{a(k)}, e^{b(n-k)} \rangle = \delta^{ab}$

Theorem (Theorem5.3)

Let $(R, E^i(1 \leq i \leq n-1), \langle , \rangle, d, [-,-])$ *be a higher Courant-Dorfman algebra. Suppose* $R = C^\infty(M)$ *for a smooth manifold* M*, and each* $E^i = \Gamma(F^i)$ *for a graded vector bundle* F^i *over M. Degree* n *dg symplectic manifolds are in one-to-one correspondence with higher Courant-Dorfman algebras of these types.*

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Definition (Definition6.1)

A higher Lie conformal algebra is a graded $\mathbb{C}[d]$-module $W = W^m$(i.e. d acts on elements of W) with $|d| = 1$. It has a degree $1-n$ map which we call Λ-bracket $[_\Lambda] : W \otimes W \to W[\Lambda]$ with $|\Lambda| = 1$ which satisfy the conditions.

Sesquilinearity

$$[da_\Lambda b] = -(-1)^{-n}\Lambda[a_\Lambda b], [a_\Lambda db] = -(-1)^{|a|-n}(d+\Lambda)[a_\Lambda b]$$

Skewsymmetry

$$[a_\Lambda b] = -(-1)^{(|a|+1-n)(|b|+1-n)}[b_{-\Lambda-d}a]$$

Jacobi identity

$$[a_\Lambda[b_\Gamma c]] = [[a_\Lambda b]_{\Lambda+\Gamma}c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b_\Gamma[a_\Lambda c]].$$

The Λ-bracket is of the form

$$[a_\Lambda b] = \sum_{j\geq 0} \Lambda^j a_{(j)} b.$$

Let

$$[a, b] = a_{(0)} b, \ \langle a_\Lambda b\rangle = \sum_{j\geq 1} \Lambda^j a_{(j)} b.$$

$$\langle a, b\rangle = \langle a_{-d} b\rangle.$$

We derive the properties of a higher weak Courant-Dorfman algebra via sesquilinearity:

$$[da, b] + o(\Lambda) = \{da_\Lambda b\} = (-1)^{-n}\Lambda\{a_\Lambda b\} \Rightarrow [da, b] = 0.$$

skew-symmetry:

$$[a, b] + o(\Lambda) = \{a_\Lambda b\} = -(-1)^{(|a|+1-n)(|b|+1-n)}\{b_{-\Lambda-d}a\}$$
$$= -(-1)^{(|a|+1-n)(|b|+1-n)}([b, a] + d\langle b_{-d}a\rangle) + o(\Lambda)$$
$$\Rightarrow [a, b] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, a] = (-1)^{(|a|+1-n)(|b|+1-n)}d\langle b, a\rangle.$$

Jacobi-identity:

$$[a, [b, c]] + o(\Lambda) = [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]] + o(\Lambda)$$
$$\Rightarrow [a, [b, c]] = [[a, b], c] + (-1)^{(|a|+1-n)(|b|+1-n)}[b, [a, c]].$$

These are properties of a higher weak Courant-Dorfman algebras.

Definition (Definition6.2)

A higher weak Courant-Dorfman algebra consists of the following data:

- a graded vector space $\mathcal{E} = (\mathcal{E}^i)$,
- a graded symmetric bilinear form of degree $-n$ $\langle , \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$,
- a map of degree 1 $d : \mathcal{E} \to \mathcal{E}$,
- a Dorfman bracket of degree $1 - n$ $[,] : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$,

which satisfies the following conditions.

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + (-1)^{(|e_1|+1-n)(|e_2|+1-n)}[e_2, [e_1, e_3]],$$

$$[e_1, e_2] + (-1)^{(|e_1|+1-n)(e_2+1-n)}[e_2, e_1] = (-1)^{(|e_1|+1-n)(|e_2|+1-n)}d\langle e_2, e_1\rangle,$$

$$[de_1, e_2] = 0.$$

Definition (Definition6.3)

Let $C = (C^n, d)$ a cochain complex. C is a higher Lie conformal algebra of degree n if it endows with a Λ-bracket $[_\Lambda] : C \otimes C \to C[\Lambda]$ defined by

$$a \otimes b \mapsto [a_\Lambda b] = a_{(0)}b + \Lambda a_{(1)}b$$

satisfying the axioms of higher Lie conformal algebras. C is a higher Poisson vertex algebra if it is a higher LCA and a differential graded-commutative algebra which satisfies

the Leibniz rule

$$[a_\Lambda bc] = [a_\Lambda b]c + (-1)^{(|a|+1-n)|b|}b[a_\Lambda c].$$

$$\{a_\Lambda b\} = [a, b] + (-1)^{|a|}\Lambda\langle a, b\rangle.$$

Substituting this for the axioms of higher Poisson vertex algebras, we can get the axioms of extended higher Courant-Dorfman algebraic structure.

Theorem (Theorem6.1)

The above higher Poisson vertex algebras generated by elements of degree $0 \leq i \leq n-1$ are in one-to-one correspondence with the extended higher Courant-Dorfman algebras.

LCA-like properties:

Lemma (Lemma6.1)

Let $C = (C^n, d_1)$ *be a higher LCA and* (E, d_2) *be a dgca. Then, a tensor product* $C \otimes E$ *of cochain complexes is also a higher LCA by defining a bracket as*
$[a \otimes f_\Lambda b \otimes g] = (-1)^{(|b|+1-n)|f|}[a_{\Lambda+d_2} b] \otimes fg, d(a \otimes f) = d_1 a \otimes f + (-1)^{|a|} a \otimes d_2 f.$

Definition (Definition6.4)

A graded Lie algebra $\mathcal{C}$ of degree $n \in \mathbb{Z}$ is a cochain complex of vector spaces with a bilinear operation $[,] : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ of degree n satisfying:
(1)skew-symmetry:$[a, b] = -(-1)^{(|a|+n)(|b|+n)}[b, a]$,
(2)Jacobi identity:$[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+n)(|b|+n)}[b, [a, c]]$.

Lemma (Lemma6.2)

Let $C = (C^k, d)$ *be a cochain complex which is a higher Lie conformal algebra of degree* n. *Then* $C/\mathrm{Im}d$ *is a naturally a graded Lie algebra of degree* $(1-n)$ *with bracket*

$$[a + dC, b + dC] = [a_\Lambda b]_{\Lambda=0} + dC$$

Lemma (Lemma6.3)

Let L *be a graded Lie algebra of degree* n. *Then,* $L[-n]$ *is a graded Lie algebra with the same bracket.*

For any higher LCA C and dgca E, we put $L(C,E) = C \otimes E/\mathrm{Im}d$ and $\mathrm{Lie}(C,E) = L(C,E)[n-1]$. By the above lemmas, $\mathrm{Lie}(C,E)$ is a graded-Lie algebra via

$$\{a \otimes f, b \otimes g\} = (-1)^{(|b|+1-n)|f|}(a_{(0)}b \otimes fg + (-1)^{|a|}a_{(1)}b \otimes (df)g).$$

Poisson algebraic structure

Next, we discuss the Poisson algebraic structure. Let $C = (C^n, d)$ be a higher PVA. Then, $C \otimes E[n-1]$ is a dgca with products $a \otimes f \cdot b \otimes g = (-1)^{|b||f|} a \cdot b \otimes f \cdot g$, and $\mathrm{Lie}(C, E)$ is a graded Lie algebra. We put $P(C, E) = C \otimes E[n-1]/(Imd) \cdot (C \otimes E[n-1])$.

Theorem (Theorem6.2)

$P(C, E)$ is a graded Poisson algebra with

$$[a \otimes f] \cdot [b \otimes g] = (-1)^{|b||f|}[a \cdot b \otimes fg],$$

$$\{[a \otimes f], [b \otimes g]\} = (-1)^{(|b|+1-n)|f|}(a_{(0)}b \otimes fg + (-1)^{|a|} a_{(1)}b \otimes (df)g).$$

By the above theorem, we get a graded Poisson algebra from a higher PVA and a dgca.

Example: formal power series
Define the algebra of formal power series

$$\mathbb{C}[[t_1, t_1^{-1}, ...t_n, t_n^{-1}]][\theta_1, ..., \theta_n]$$

where t_i are even coordinates in degree 0, θ_i are odd coordinates in degree 1.
Define the "de-Rham differential" as

$$df := \sum_i \frac{\partial f}{\partial t^i}\theta_i.$$

Let $C = (C^n, Q)$ be a higher LCA of degree $n+1$.
For $V := C \otimes \mathbb{C}\left[[t_1, t_1^{-1}, ...t_n, t_n^{-1}]][\theta_1, ..., \theta_n\right][n]/((Q\alpha)\otimes f + \alpha \otimes df)$, the bracket

$$[\alpha \otimes t_1^{p_1} \cdots t_n^{p_n} \theta^J, \beta \otimes t_1^{q_1} \cdots t_n^{q_n} \theta^K]$$

$$= (\alpha_{(0)}\beta) t_1^{p_1+q_1} \cdots t_n^{p_n+q_n} \theta^{J\cdot K} + \sum_{k=1}^{n} (\alpha_{(1)}\beta) p_k t_1^{p_1+q_1} \cdots t_k^{p_k+q_k-1} t_n^{p_n+q_n} \theta^{J\cdot\{k\}\cdot K}$$

$$J, K \subset \{1, ..., n\}, J \cdot K = \begin{matrix} \phi & (J \cap K \neq \phi) \\ J \cup K & (J \cap K = \phi) \end{matrix}$$

makes the graded Lie algeraic structure.

$C = (C^n, Q)$:a higher PVA of degree 2
Then $P(C, \mathbb{C}[[t, t^{-1}]][\theta])$ is a graded Poisson algebra via

$$\{\alpha t^m, \beta t^n\} = (\alpha_{(0)}\beta)t^{m+n} + (\alpha_{(1)}\beta)mt^{m+n-1}\theta, \ \{\alpha t^m\theta, \beta t^n\} = (\alpha_{(0)}\beta)t^{m+n}\theta.$$

If we restrict this algebra to degree 0 part, this subalgebra is isomorphic to the Poisson algebra arising from the associated Poisson vertex algebra $C \otimes \mathbb{C}[[t, t^{-1}]]/Im(d + \partial_t) \cdot C \otimes \mathbb{C}[[t, t^{-1}]]$.

$$\alpha \in C^1 \implies \alpha t^m \to \alpha t^m, \ \beta \in C^0 \implies \beta t^m\theta \to \beta t^m$$

Example (Example6.2)

Let $(\mathcal{M}, \omega, Q = \{\Theta, -\})$ be a degree n dg symplectic manifold and $C = C^{n-1}(C^\infty(\mathcal{M})) = \{a \in C^\infty(\mathcal{M}) : |a| \leq n-1\}$ and consider a higher Courant-Dorfman algebra on C. Let Σ_{n-1} be a $n-1$ dimensional manifold and $E = (\Omega^\bullet(\Sigma_{n-1}), D)$ be their de-Rham complex. Then, $P(C, E)$ is equipped with degree 0 Poisson bracket with

$$[a \otimes \epsilon_1, b \otimes \epsilon_2] = \{\{a, \Theta\}, b\} \otimes \epsilon_1 \epsilon_2 + \{a, b\} \otimes (D\epsilon_1)\epsilon_2,$$

where $a, b \in C$ and $\epsilon_1, \epsilon_2 \in E$. This is an algebraic description of BFV current algebras from dg symplectic manifolds.

BFV current algebras on $C^\infty(Map(T[1]\Sigma_{n-1},\mathcal{M}))/I_{\tilde{D}+\tilde{Q}}$ ($T[1]\Sigma_{n-1}$ is the shifted tangent space of Σ_{n-1}.)

$$J_\epsilon\left(a\right)\left(\phi\right)=\int_{T[1]\Sigma_{n-1}}\epsilon\cdot\phi^*(a)(\sigma,\theta)d^{n-1}\sigma d^{n-1}\theta,$$

$$\begin{aligned}&\{J_{\epsilon_1}\left(a\right),J_{\epsilon_2}\left(b\right)\}(\phi)\\&=\int_{T[1]\Sigma_{n-1}}\epsilon_1\epsilon_2\cdot\phi^*(\{\{a,\Theta\},b\})(\sigma,\theta)d^{n-1}\sigma d^{n-1}\theta\\&+\int_{T[1]\Sigma_{n-1}}(D\epsilon_1)\epsilon_2\cdot\phi^*(\{a,b\})(\sigma,\theta)d^{n-1}\sigma d^{n-1}\theta,\end{aligned}$$

where $\epsilon_1,\epsilon_2\in C^\infty(T[1]\Sigma_{n-1})$ are test functions on $T[1]\Sigma_{n-1}$, σ,θ are coordinates on $T[1]\Sigma_{n-1}$ of degree 0 and 1, We gave higher analogs of Lie conformal algebras and Poisson vertex algebras. It is natural to ask whether they have same applications as ordinary Lie conformal algebras and Poisson vertex algebras. For example, our higher PVAs may be used to analyze multi-variable Hamiltonian PDEs. Considering the algebraic description of more general currents would be important. Another interesting problem is the non-commutative analog. Non-commutative version of Courant-Dorfman algebras and Poisson vertex algebras, which are called double Courant-Dorfman algebras and double Poisson vertex algebras, are considered. Their higher generalization would be given using our algebras. Another way taking the non-commutative version is the quantization, in analogy with vertex algebras.