4D Wess-Zumino-Witten (WZW) models and a unified theory of integrable systems

Masashi Hamanaka (Nagoya U.)

2023.10.24 @ 杭州

- MH, Shan-Chi Huang, Hiroaki Kanno, 2212.11800
  Prog. Theor. Exp. Phys. (PTEP) 2023-4, 043B03

§1 Introduction

[Donaldson '85]
[Losev-Moore-Nekrasov-Shatashvili '96]
[Inami-Kanno-Ueno-Xiong '96] ...

4-dim WZW (WZW_4) model

- analogue of 2-dim WZW model
- EOM = Yang's eq ≡ Anti-Self-Dual Yang-Mills eq. (ASD)
- In the split signature $(+,+,-,-)$, ← Today we focus on
  SFT action of $N=2$ string theory [Ooguri-Vafa '91]

implication
application

We discuss classical soliton sols. of it

Original Motivation:

Ward's conjecture

We've made it!

(τ-fcn?)

4-dim ASDYM	$\longleftrightarrow$	twistor theory (so far, no Wronskian sol.)	
$(+,+,-,-)$ $\big\downarrow$ reduction			
Toda, KdV, NLS, …	$\longleftrightarrow$	Sato's theory (τ-fcn $\equiv$ Wronskian sol.)	

[Ward '85, Mason-Woodhouse], …

Original Motivation :

NC = extension to Noncommutative sp.

NC Ward's conjecture

We've made it!

4-dim NC ASDYM	$\longleftrightarrow$ twistor theory (so far, no Wronskian sol.)

$(\underbrace{+,+,-,-}) \downarrow$ reduction

NC $\leftrightarrow$ bkg. B-field

$\downarrow$ reduction

NC Toda, NC KdV, NC NLS, ...	$\longleftrightarrow$ Sato's theory (τ-fcn $\equiv$ Wronskian sol.)

(τ-fcn?)

Quasideterminant

[Ward '85, Mason-Woodhouse], ...
(NC) [H-Toda '02, H '06, ...]

Reduction to KdV from ASDYM $(G = SL(2,\mathbb{C}))$

ASDYM : $F_{zw} = 0,\ F_{\tilde{z}\tilde{w}} = 0,\ F_{z\tilde{z}} - F_{w\tilde{w}} = 0$

① $\partial_w - \partial_{\tilde{w}} = 0,\ \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ \frac{u}{2} & 0 \end{pmatrix},\ A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}$

$A_z = \frac{1}{4}\begin{pmatrix} u' & -2u \\ u'' + 2u^2 & -u' \end{pmatrix}$

$u = u(z, x)$, $x = w + \tilde{w}$

$u' = \partial_x u$

$u_z - u_{xxx} - \frac{3}{2} u u_x = 0$: **KdV eq.!**

cf. p9

$z = t$

(t, x) are real $\Rightarrow$ not $(++++)$ but $(++--)$

What is Lagrangian for ASDYM?

$\Downarrow$

Candidates: (i) Yang-Mills action $\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$

↳ ASDYM = BPS eq.

(ii) 4D WZW action

↳ ASDYM = EOM (natural?)

$\Downarrow$ Ward conj

4D WZW model with the split signature $(+ + - -)$

would be suitable? $\rightsquigarrow$ application to $N=2$ strings

A Unified theory of integrable systems

6d meromorphic Chern-Simons (CS)

[Costello]
[Bittleston-Skinner]

$\swarrow$ 4d CS

$\searrow$ 4d WZW (++−−)

← duality? →

4d CS $\downarrow$ [Costello-Yamazaki(-Witten)]

various solvable models [Delduc-Lacroix-Magro-Vicedo], [Yoshida(K), Sakamoto, Fukushima, ⋯] ⋯
(spin chains, PCM, ⋯)

4d WZW $\downarrow$ [Ward] [Mason-Woodhouse]

various integrable eqs.
(KdV NLS, Toda, ⋯)

Nagoya Math-Phys Seminar Online has start!

7

Nagoya Math-Phys Seminar

supported by the Ichihara International scholarship foundation

2023 Autumn/2024 Winter

Date [Place]	Speaker	Title	Comment
Dec.21 (Thu) 5pm (JST) [Zoom]	Meer Ashwinkumar (Bern)	TBA (Seminar)	Zoom link will be shown here by 3pm on the seminar day.
Oct.19 (Thu) 5pm (JST) [Zoom]	Francis Howard (Benin)	Particles and p－Adic Integrals of Spin(1/2): Spin Lie Group, R(ρ, q)－gamma and R(ρ, q)－ beta Functions, Ghost and Applications (Seminar)	Abstract, Slide, Video

2023 Spring/Summer

Date [Place]	Speaker	Title	Comment
Aug.30 (Wed)～ Sep.29 (Fri) [Zoom]	7 Plenary Speakers and 18 Parallel Speakers	Topological solitons (International Seminar-Type Online Workshop)	Workshop website
Sep.14 (Thu) 5pm (JST) [Zoom]	Frank Nijhoff (Leeds)	Lagrangian multiforms, the Darboux-KP system and Chern-Simons theory in infinite-dimensional space (Seminar)	Abstract, Slide, Video
July 21 (Fri) 9:30am (JST) [Zoom]	Atul Sharma (Harvard)	Burns holography (Seminar)	Abstract, Slide, Video
June 16 (Fri) 9:30am (JST) [Zoom]	Roland Bittleston (Perimeter)	Classical and Quantum Integrability in Self-Dual Gravity (Seminar)	Abstract, Slide, Video
June 15 (Thu) 9:30am (JST) [Zoom]	Roland Bittleston (Perimeter)	Overview of Classical and Quantum Integrability in Four Dimensions (Overview Seminar)	Abstract, Slide, Video

Plan of Talk (simple discussion)

§1 Introduction (15 min)

§2 Soliton Solutions of Yang's eq (7 min)

§3 4d WZW model (13 min)

§4 Conclusion & Discussion (5 min)

§2 Soliton Solutions of Yang's eq

Yang's eq. (on $\mathbb{C}^4$: complexified space-time)

$$\partial_{\tilde{z}}\left((\partial_z a)a^{-1}\right) - \partial_{\tilde{w}}\left((\partial_w a)a^{-1}\right) = 0$$

$a \in G = GL(N,\mathbb{C})$

※ Real slice

$(z, w, \tilde{z}, \tilde{w}) \in \mathbb{C}^4$, $\quad ds^2 = dz\,d\tilde{z} - dw\,d\tilde{w}$

① $z = x^1 + x^3$, $w = x^2 + x^4$

$\tilde{z} = x^1 - x^3$, $\tilde{w} = x^4 - x^2$

$\downarrow$

$\mathbb{R}^4(+,+,-,-)$

Ultrahyperbolic sp. $\mathbb{U}$

② $z = x^1 + ix^2$, $w = x^3 + ix^4$

$\tilde{z} = \bar{z}$, $\tilde{w} = -\bar{w}$

$\downarrow$

$\mathbb{R}^4(++++)$

Euclid sp. $\mathbb{E}$

Lax representation :

N×N const matrix

(right action)

$$(*)\begin{cases} Lf = \sigma\partial_w(\sigma^{-1}f) - (\partial_{\tilde{z}}f)\zeta = 0 \\ Mf = \sigma\partial_z(\sigma^{-1}f) - (\partial_{\tilde{w}}f)\zeta = 0 \end{cases}$$

compatible condition ⇒ Yang's eq.

$L(M\phi) - M(L\phi) = 0$

Darboux trf. [Nimmo-Gilson-Ohta '00] [Gilson-H-Huang-Nimmo '20]

$$(D)\begin{cases} \tilde{f} = f\zeta - \theta\Lambda\theta^{-1}f \\ \tilde{\sigma} = -\theta\Lambda\theta^{-1}\sigma \end{cases}$$

$\underset{N\times N}{\theta}$: special sol. for Λ

special value

Under the Darboux trf. (*) is form invariant (i.e. $\tilde{L}\tilde{f} = 0$, $\tilde{M}\tilde{f} = 0$)

n-iterations of (D) from a trivial seed sol. ($\sigma=1$)

$$\underset{N\times N}{\sigma_n} = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$N\times N$ $\quad$ $N\times N$

$$\theta_k^{(\ell)} := \theta_k \Lambda_k^{\ell}$$

$$(\theta_i, \Lambda_i): \quad \partial_w \theta_i = \partial_{\tilde{z}} \theta_i \Lambda_i$$
$$\partial_z \theta_i = \partial_{\tilde{w}} \theta_i \Lambda_i$$

Wronskian-type!

Quasideterminant

squares

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} := \underset{N\times N}{d} - CA^{-1}B \qquad \text{(Schur complement)}$$

n-soliton sols. for $G = SL(2,\mathbb{C})$: [H-Huang.'20] [12]

$$\sigma_n = \begin{vmatrix} \theta_1 & \cdots & \theta_n & 1 \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \cdots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k = \begin{pmatrix} e^{L_k} & e^{-\bar{L}_k} \\ -e^{-L_k} & e^{\bar{L}_k} \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

$$L_k = \lambda_k \alpha_k z + \beta_j \tilde{z} + \lambda_j \beta_j w + \alpha_j \tilde{w}$$

(linear in space-time coord)

Rmk (U) $\mu_k = \bar{\lambda}_k, \ |\mu_k| = 1$

$\Rightarrow G = SU(2)$

(E) $\mu_k = -1/\bar{\lambda}_k, \ |\mu_k| = 1$

$\Rightarrow G = U(2)$

Non-abelian system

&

Calculate the WZW action density of them

§3 4-dim WZW model

$\sigma(x) \in G$ 13

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$$S_\sigma = \frac{i}{4\pi} \int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1} \wedge (\tilde{\partial}\sigma)\sigma^{-1} \right]$$

$$S_{WZ} = -\frac{i}{12\pi} \int_{M_4} A \wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1} \right]^3$$

$(z, w, \tilde{z}, \tilde{w})$: local coords of M_4

w/ $\omega = dA$: Kähler form of M_4

M_4 : flat 4-dim space-time $\quad \omega = \frac{i}{2}(dz \wedge d\tilde{z} - dw \wedge d\tilde{w})$

$$d = \partial + \tilde{\partial}, \quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

EoM: $\tilde{\partial}\left(\omega \wedge (\partial\sigma)\sigma^{-1}\right) = 0 \iff$ Yang's eq.

§3 4-dim WZW model

$\sigma(x) \in G$

Action: $\quad S_{WZW_4} = S_\sigma + S_{WZ}$

$$S_\sigma = \frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1} \wedge (\tilde{\partial}\sigma)\sigma^{-1}\right]$$

$$S_{WZ} = -\frac{i}{12\pi}\int_{M_4\times[0,1]} \omega \wedge \mathrm{Tr}\left[(d\tilde{\sigma})\tilde{\sigma}^{-1}\right]^3 \qquad \begin{array}{l}\tilde{\sigma}(0)=1\\ \tilde{\sigma}(1)=\sigma\\ \text{homotopy}\end{array}$$

w/ $\quad \omega = dA$: Kähler form of M_4

M_4 : flat 4-dim space-time $\quad \omega = \frac{i}{2}\left(dz\wedge d\tilde{z} - dw\wedge d\tilde{w}\right)$

$$d = \partial + \tilde{\partial}, \quad \partial = dw\,\partial_w + dz\,\partial_z, \quad \tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$$

EOM: $\tilde{\partial}\left(\omega\wedge(\partial\sigma)\sigma^{-1}\right) = 0 \iff$ Yang's eq.

$\mathcal{N}=2$ string theory

# WS SUSY	Name	Target sp.	field contents
$\mathcal{N}=0$	Bosonic String	$(1+25)$ dim	$g_{\mu\nu}, B_{\mu\nu}, \phi, \cdots$
$\mathcal{N}=1$	Superstring	$(1+9)$ dim	〃 〃
$\mathcal{N}=2$	$\mathcal{N}=2$ string	$(2+2)$ dim	massless scalar only!

open $\mathcal{N}=2$ string

$\sigma = e^{\varphi}$ ← the massless scalar

[Ooguri-Vafa, '91]

$$S_{WZW_4} = \underbrace{\text{(in terms of } \varphi)}_{\substack{\text{III} \\ S_{\mathcal{N}=2 \text{ string}} \\ \text{(SFT)}}} \rightsquigarrow \text{n-pt. fcn of } \varphi$$

(coincides with WS calculations)

One soliton (on $\mathbb{U}$)

※ $\lambda = \bar{\lambda} \Rightarrow \mathcal{L}_\sigma \equiv 0$

$$\sigma = -\theta \Lambda \theta^{-1}, \quad \theta = \begin{pmatrix} e^{L} & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\Downarrow$$

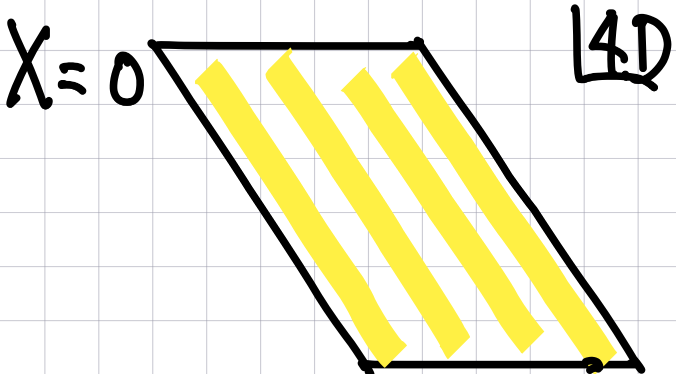
$\propto (\lambda - \bar{\lambda})^3$

$$\mathcal{L}_\sigma = \frac{1}{8\pi} d_{11} \operatorname{sech}^2 X$$

$X := L + \bar{L}$: linear in x^μ

peak

$$\mathcal{L}_{WZ} \equiv 0 \quad \text{(identically)}$$



3-dim hyperplane (codim 1)

not instanton!

Similar!

cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^{X} + e^{-X}\right) \propto \operatorname{sech}^2 X$$

linear in t, x, y

Two Soliton ($\mathcal{L}_\sigma$)

$X_k = L_k + \bar{L}_k, \quad \Theta_{12} = \Theta_1 - \Theta_2$

$i\Theta_k = L_k - \bar{L}_k$

$$\mathcal{L}_\sigma = \frac{\left[A\cosh^2 X_1 + B\cosh^2 X_2 + C_\pm \cosh^2\left(\frac{X_1 + X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2}\right)\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1-X_2) + c\cos\Theta_{12}\right)^2}$$

non-singular

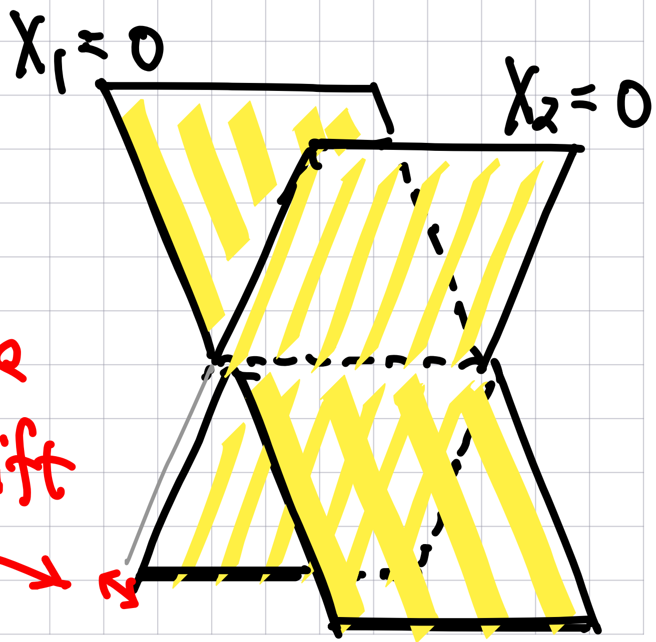
$$\xrightarrow[X_1:\text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_1 \pm \delta_1)$$

$$\xrightarrow[X_2:\text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_2 \pm \delta_2)$$

$$\xrightarrow[\text{otherwise}]{r\to\infty} 0$$

phase shift

(non-linear effect)

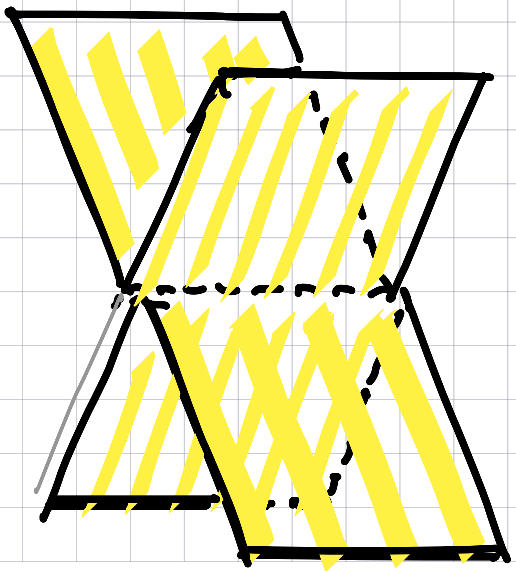
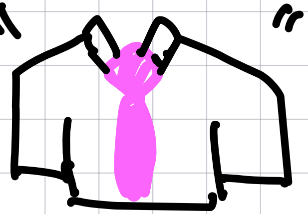


Two Soliton

$\mathcal{L}_{WZ}$ = (very long many terms) non-singular

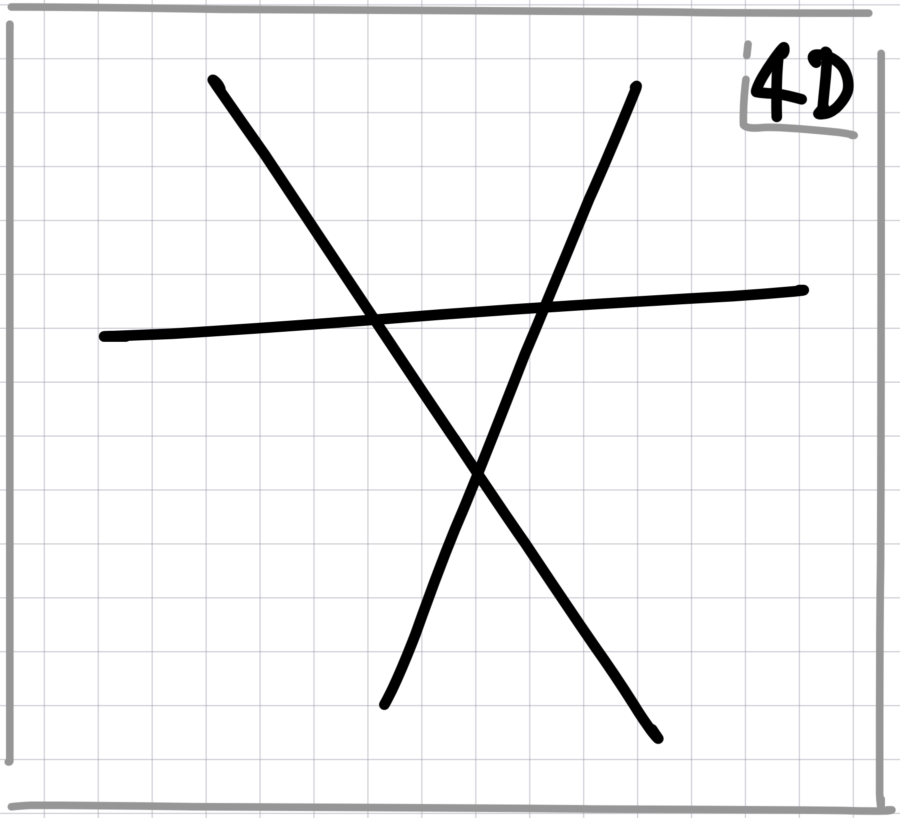
$\xrightarrow{r\to\infty}$ 0 (in any direction)

$\mathcal{L}_{total}$ = $\mathcal{L}_{\alpha}$ + ("dressing" in the middle region)

=  + "  "

(This is an image)

n-soliton sol. = "non-linear superposition of n one solitons" [H-Huang '22]



intersecting n hyperplanes (with phase shifts)

Rmk 1 Reduction to (1+2) dim.

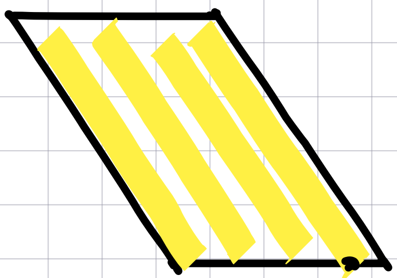
Consider $(x^1, x^2, x^3, x^4) \longrightarrow (\overset{t \text{ (time)}}{\overset{\parallel}{x^1}}, x^3, x^4)$

The soliton sol. $\sigma(\alpha_k = \lambda_k \beta_k)$ solves EoM in (1+2)d

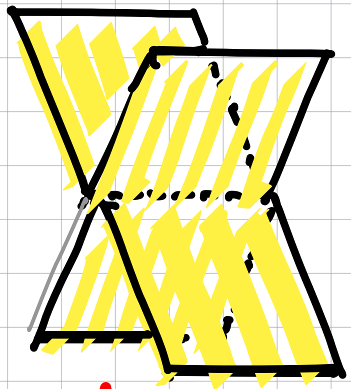
$\because$ $L_k = (\lambda_k \alpha_k + \beta_k) x^1 + \underbrace{(\lambda_k \beta_k - \alpha_k)}_{=0} x^2 + \cdots$ $\blacksquare$

Hamiltonian $\mathcal{H} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} \partial_t \phi_i - \mathcal{L}$ $\quad (\mathcal{H}_{WZ} \equiv 0 \,!)$

One soliton



Two soliton (no dressing)



Energy density has the same peaks as action density.

Rmk 2 Euclidean case $\mathbb{E}$

The soliton sols. : almost the same as in $\mathbb{U}$

Instanton solution (well-known in YM)

(Ex) $G_{YM} = SU(2)$ 't Hooft 1-instanton

$$\mathcal{L}_a \propto \frac{(z\bar{z} + w\bar{w})^3}{(z\bar{z}w\bar{w})^2(1 + z\bar{z} + w\bar{w})^2}$$

$$\mathcal{L}_{wz} \propto \frac{(z\bar{z} + w\bar{w})(z\bar{z} - w\bar{w})^2}{(z\bar{z}w\bar{w})^2(1 + z\bar{z} + w\bar{w})^4}$$

singular

localized at the origin (codim 4)

§4 Conclusion and Discussion

We constructed new-type of codim 1 solitons and calculated action densities of WZW_4 model.

⇝ intersecting 3-branes in the $N=2$ string
(new branes)

There are many things to be seen :

- Solitonic properties (charge, mass, moduli, ⋯)
- Classification of the "soliton planes" cf. [Kodama-Williams '14]
- Reduced systems (YMH, Hitchin system, Ernst eq. ⋯)

A Unified theory of integrable systems

6d meromorphic Chern-Simons (CS)

↙ ↘

4d CS ←duality?→ 4d WZW (++−−)

↓ ↓

various solvable models (spin chains, PCM, ⋯)

various integrable eqs. (KdV NLS, Toda, ⋯)

A Unified theory of NC integrable systems

6d meromorphic
NC Chern-Simons (CS)

NC 4d CS $\longleftarrow$ duality? $\longrightarrow$ NC 4d WZW (++−−)

key: Quasideterminant?

various NC solvable models (spin chains, PCM, ...)

various NC integrable eqs. (KdV, NLS, Toda, ...)

Seminar-Type Online Workshop on NC Integrable Systems will be held

(probably 4 ~ 15 March 2024)

Speakers : V. Retakh, (Reviews on Q-det)
V. Roubtsov,
I. Bobrova, ...

Welcome to Join!

Thank You Very Much!

謝謝