Solitons in 4-dim. WZW Model

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§1 Introduction

4-dim WZW (WZW_4) model [Donaldson] [Losev-Moore-Nekrasov-Shatashvili]

- analogue of 2-dim WZW model [Inami-Kanno-Ueno-Xiong], ⋯
- EOM = Yang's eq ≡ Anti-Self-Dual Yang-Mills eq. (ASD)
- In the split signature $(-,-,+,+)$, ← Today we focus on
  SFT action of $N=2$ string theory [Ooguri-Vafa]

We discuss soliton sols. of the Yang eq. ↑ implication application

Anather Motivation

6d meromorphic Chern-Simons (CS) (++++−−)

[Bittleston − Skinner]

4d CS (++++)

[Costello-Yamazaki (-Witten)]

← duality? →

4d WZW (++−−)

[Ward] [Mason-Woodhouse]

various solvable models (spin chains, PCM, ⋯)

[Yoshida(K), Sakamoto, Fukushima, ⋯] ⋯

various integrable eqs. (KdV, NLS, Toda, ⋯)

Plan of Talk (simple discussion)

§1 Introduction (5 min)

§2 4dim WZW model (5 min)

§3 Soliton Solutions (10 min)

§4 Reduction to (1+2)-dim. (2 min)

§5 Euclidean case (ε min)

§6 Conclusion & Discussion (3 min)

§2 4-dim WZW model

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$\sigma(x) \in G$ Lie gp

$$S_\sigma = \frac{i}{4\pi}\int_{M_4} \omega \wedge \mathrm{Tr}\left[(\partial\sigma)\sigma^{-1} \wedge (\tilde{\partial}\sigma)\sigma^{-1}\right]$$

$$S_{WZ} = -\frac{i}{12\pi}\int_{M_4} A \wedge \mathrm{Tr}\left[(d\sigma)\sigma^{-1}\right]^3$$

w/ $\omega = dA$: Kähler form of M_4

$(z, w, \tilde{z}, \tilde{w})$: local coords of M_4

M_4 : flat 4-dim space-time

$d = \partial + \tilde{\partial}$, $\partial = dw\,\partial_w + dz\,\partial_z$, $\tilde{\partial} = d\tilde{w}\,\partial_{\tilde{w}} + d\tilde{z}\,\partial_{\tilde{z}}$

EOM: $\tilde{\partial}\left(\omega\wedge(\partial\sigma)\sigma^{-1}\right)=0$

$$\Updownarrow \qquad \omega=\frac{i}{2}\left(dz\wedge d\tilde{z}-dw\wedge d\tilde{w}\right)$$

$$\partial_{\tilde{z}}\left((\partial_z\sigma)\sigma^{-1}\right)-\partial_{\tilde{w}}\left((\partial_w\sigma)\sigma^{-1}\right)=0$$: Yang's eq.

※ Real slice

$(z, w, \tilde{z}, \tilde{w})$ $\quad ds^2=dz\,d\tilde{z}-dw\,d\tilde{w}$

① $z=x^1+x^3,\ w=x^2+x^4$, $\tilde{z}=x^1-x^3,\ \tilde{w}=x^4-x^2$ ↓

$\mathbb{R}^4(+,+,-,-)$ Ultrahyperbolic sp. $\mathbb{U}$

② $z=x^1+ix^2,\ w=x^3+ix^4$, $\tilde{z}=\bar{z},\ \tilde{w}=-\bar{w}$ ↓

$\mathbb{R}^4(++++)$ Euclid sp. $\mathbb{E}$

N=2 string theory

# WS SUSY	Name	Target sp.	field contents
$N=0$	Bosonic String	$(1+25)$ dim	$g_{\mu\nu}, B_{\mu\nu}, \phi, \cdots$
$N=1$	Superstring	$(1+9)$ dim	〃 〃
$N=2$	$N=2$ string	$(2+2)$ dim	massless scalar only!

open $N=2$ string

$$\sigma = e^{\varphi} \longleftarrow \text{massless scalar}$$

[Ooguri-Vafa]

$$S_{WZW_4} = (\text{in terms of } \varphi) \rightsquigarrow \text{n-pt. fcn of } \varphi$$

$$\rotatebox{90}{$\cong$}$$

$$S_{N=2\ \text{string (SFT)}}$$

(coincides with WS calculations)

§3 Soliton Solutions

We have obtained new type of soliton sols.

[Gilson-H-Huang-Nimmo][H-Huang]^2

⇓

Calculate the WZW action density of them

Exact solution ($G = SL(2,\mathbb{C})$) of n-solitons

$$\underset{2\times 2}{\sigma} = \begin{vmatrix} \psi_1, & \cdots, & \psi_n & 1 \\ \psi_1^{(1)}, & \cdots, & \psi_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \psi_1^{(n)} & \cdots, & \psi_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\psi_k = \begin{pmatrix} e^{L_k} & e^{-\bar{L}_k} \\ -e^{-L_k} & e^{\bar{L}_k} \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

$$\psi_k^{(i)} := \psi_k \Lambda^i$$

$$L_k = \lambda_k \alpha_k z + \beta_j \tilde{z} + \lambda_j \beta_j w + \alpha_j \tilde{w}$$

← quasi Wronskian

One soliton (on $\mathbb{U}$)

$$\sigma = -\psi\Lambda\psi^{-1}, \quad \psi = \begin{pmatrix} e^{L} & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

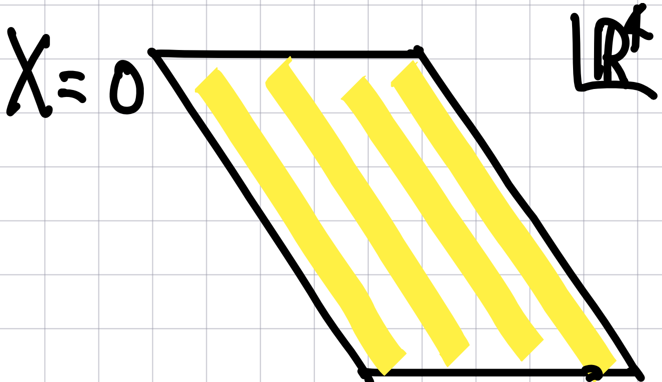
$$\Downarrow$$

$$\mathcal{L}_\sigma = \frac{1}{8\pi} d_{11} \operatorname{sech}^2 X$$

$X := L + \bar{L}$: linear in x^μ

$$\mathcal{L}_{WZ} \equiv 0$$ (identically)

peak



3-dim hyperplane (codim 1)

not instanton!

Similar!

cf. KP soliton

$$u = 2\partial_x^2 \log\left(e^{X} + e^{-X}\right) \propto \operatorname{sech}^2 X$$

linear in t, x, y

Two Soliton

$$X_k = L_k + \bar{L}_k, \quad \Theta_{12} = \Theta_1 - \Theta_2$$
$$i\Theta_k = L_k - \bar{L}_k$$

$$\mathcal{L}_a = \frac{\left[A\cosh^2 X_1 + B\cosh^2 X_2 + C_\pm \cosh^2\left(\frac{X_1 + X_2 \pm i\Theta_{12}}{2}\right) + D_\pm \cosh^2\left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2}\right)\right]}{2\pi\left(a\cosh(X_1+X_2) + b\cosh(X_1 - X_2) + c\cos\Theta_{12}\right)^2}$$

non-singular

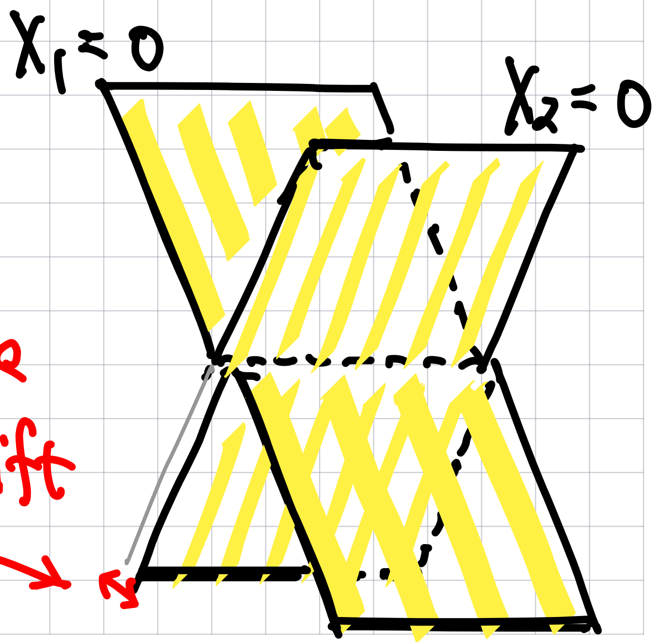
$$\xrightarrow[X_1 : \text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_1 \pm \delta_1)$$

$$\xrightarrow[X_2 : \text{const}]{r\to\infty} \propto \operatorname{sech}^2(X_2 \pm \delta_2)$$

$$\xrightarrow[\text{otherwise}]{r\to\infty} 0$$

phase shift

(non-linear effect)



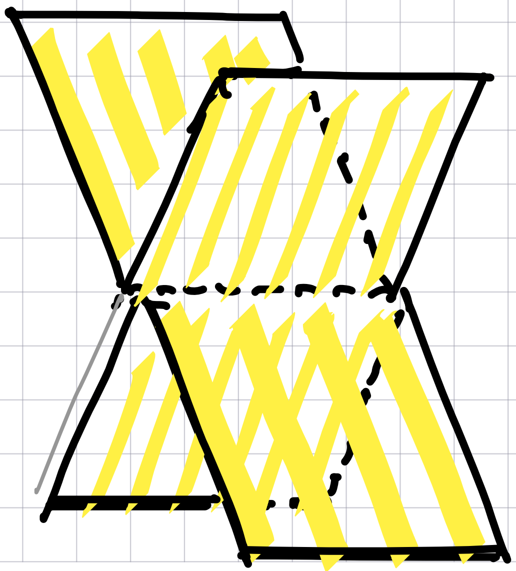
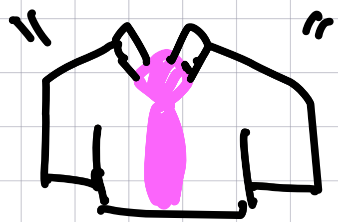
Two Soliton

$\mathcal{L}_{WZ}$ = (very long many terms)

non-singular

$\xrightarrow{r\to\infty}$ 0 (in any direction)

$\mathcal{L}_{total}$ = $\mathcal{L}_{a}$ + ("dressing" in the middle region)

=  + 

(※イメージです)

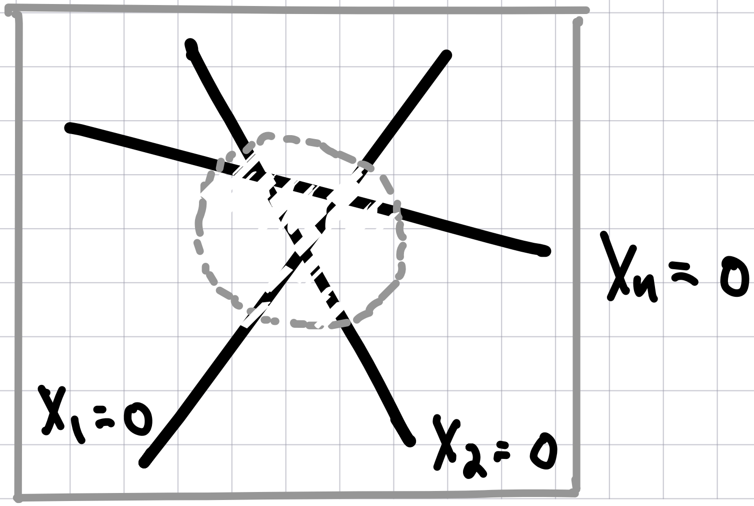
Multi-Soliton (asymptotic analysis)

$$\sigma_{n\text{-soliton}} \xrightarrow[X_k:\,\text{finite}]{r\to\infty} \propto -\hat{\Psi}_k \Lambda_k \hat{\Psi}_k^{-1} \quad \text{(one soliton)}$$

[H-Huang 2022]

= "nonlinear superposition" of n-1solitons

= intersecting n 3-branes (not D-branes)



§4 Reduction to (1+2) dim.

Consider $(x^1, x^2, x^3, x^4) \longrightarrow (\overset{t\ (\text{time})}{\overset{\shortparallel}{x^1}}, x^3, x^4)$

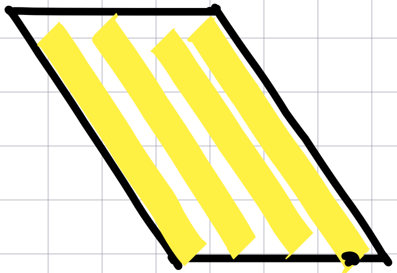
The soliton sol. $\sigma(\alpha_k = \lambda_k \beta_k)$ solves EoM in (1+2)d

$$\Updownarrow$$

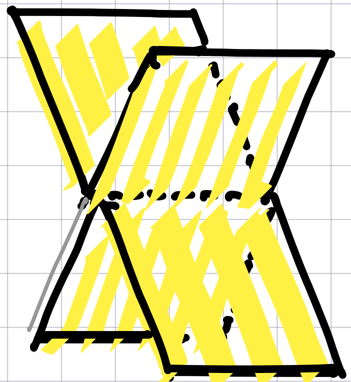
∵ $L_k = (\lambda_k \alpha_k + \beta_k) x^1 + \underbrace{(\lambda_k \beta_k - \alpha_k)}_{=0} x^2 + \cdots$ ■

Hamiltonian $\mathcal{H} = \sum_{i=1}^{3} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i)} \partial_t \phi_i - \mathcal{L}$ $\quad (\mathcal{H}_{WZ} \equiv 0\ !)$

One soliton



Two soliton (no dressing)



Energy density has the same peaks as action density.

§5 Euclidean case $\mathbb{E}$

The soliton sols. : almost the same as in $\mathbb{U}$

Instanton solution (well-known in YM)

(Ex) $G_{YM} = SU(2)$ 't Hooft 1-instanton

$$\mathcal{L}_a \propto \frac{(z\bar{z} + w\bar{w})^3}{(z\bar{z}w\bar{w})^2(1 + z\bar{z} + w\bar{w})^2}$$

$$\mathcal{L}_{wz} \propto \frac{(z\bar{z} + w\bar{w})(z\bar{z} - w\bar{w})^2}{(z\bar{z}w\bar{w})^2(1 + z\bar{z} + w\bar{w})^4}$$

localized at the origin (codim 4)

§6 Conclusion and Discussion

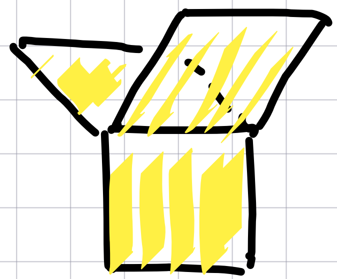
We calculate action densities of WZW_4 model

$\rightsquigarrow$ intersecting 3-branes in the $N=2$ string
(new branes)

Resonance solutions $\rightsquigarrow$ 3-brane reconnections

$\Downarrow$

Classification of the "soliton planes"



cf [Kodama-Williams], [Sato-Sato]

Solitonic properties in the open $N=2$ string

charge, mass, description of φ, ...

Unified theory of integrable system?

6d meromorphic
Chern-Simons (CS)

4d CS ← duality? → 4d WZW

where is the KP?

Nagoya Math-Phys Seminar Online will start! (爆誕)
(welcome to join!) ↔ worldwide!

announced in Sg-l ML list and the researchseminar.org