Noncommutative surfaces, clusters, and their symmetries

Noncommutative Integrable Systems

Nagoya, March 13, 2024

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- A. Berenstein, V. Retakh, Noncommutative marked surfaces, Adv. Math. 328 (2018).
- A. Berenstein, M. Huang, V. Retakh, Noncommutative marked surfaces II: tagged triangulations, clusters, and their symmetries, *in progress*.

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Noncommutative clusters, informal introduction

A noncommutative cluster structure on a graded $\mathbb{Q}$-algebra $\mathcal{A}$ consists of a certain graded group $Br_{\mathcal{A}}$ together with a collection of homogeneous embeddings ι of a given graded group G into the multiplicative monoid $\mathcal{A}^{\times}$ (these embeddings are referred to as noncommutative clusters) and a faithful homogeneous action $\triangleright_\iota$ of $Br_{\mathcal{A}}$ on G for any ι such that:

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- The extensions $\iota : \mathbb{Q}G \to \mathcal{A}$ are injective and their images generate $\mathcal{A}$ (and $\mathcal{A}$ is a isomorphic to a noncommutative localization of $\mathbb{Q}G$).
- (monomial mutation) For any ι and ι' we expect a (unique) automorphism $\mu_{\iota,\iota'}$ of G which intertwines between ι and ι' as well as between $Br_{\mathcal{A}}$-actions $\triangleright_\iota$ and $\triangleright_{\iota'}$.
- For any cluster homomorphism $f : \mathcal{A} \to \mathcal{A}'$ we expect a unique (up to conjugation) group homomorphism $f_* : G \to G'$ so that the induced homomorphism $Br^f_{\mathcal{A}} := \{T \in Br_{\mathcal{A}} : T(Ker\ f_*) = Ker\ f_*\} \to Br_{\mathcal{A}'}$ is injective.

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In many cases we expect a (noncommutative) Laurent Phenomenon:

- Given a cluster $\iota : G \hookrightarrow \mathcal{A}^\times$, for any cluster $\iota' : G \hookrightarrow \mathcal{A}^\times$ there is a submonoid $M_{\iota'} \subset G$ generating G such that $\iota'(M_{\iota'})$ is in the semiring $\mathbb{Z}_{\geq 0}\iota(G)$, moreover,

$$\iota'(m) = \iota(\mu_{\iota,\iota'}(m)) + \text{lower terms in } \iota(G)$$

for any $m \in M_{\iota'}$.

Examples: Ordinary and quantum cluster structures

The localization $\mathcal{A}$ of a (quantum) cluster algebra $\underline{\mathcal{A}}$, determined by an $m \times n$ exchange matrix $\tilde{B}$ (and compatible $m \times m$ skew-symmetric matrix Λ), by the set X of all of its cluster variables satisfies all of the above requirements with $G \cong \mathbb{Z}^m$ (or its central extension G_q in quantum case) so that $\mathbb{Q}G = \mathbb{Q}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ for a given cluster $\{x_1, \ldots, x_n\}$ in $\mathcal{A}$. The well-known commutative/quantum Laurent Phenomenon asserts that the set of all (quantum) cluster variables belongs to the group algebra $\mathbb{Q}G$ which is an instance of its noncommutative counterpart stated above. In these cases, $Br_{\mathcal{A}}$ is essentially the group of symplectic transvections introduced by B. Shapiro, M. Shapiro, A. Vainshtein, A. Zelevinsky in 2000) and it is always a quotient of an appropriate Artin braid group.

Examples: Rank 2 noncommutative cluster structure

First, fix $r_1, r_2 > 0$ and let $\mathcal{A}_{r_1,r_2}$ be the subalgebra of the free skew field $\mathcal{F}_2 =< y_1, y_2 >$ generated by $z := y_2^{-1} y_1 y_2 y_1^{-1}$, $y_k^{\pm 1}$, $k \in \mathbb{Z}$, where y_k is given by $y_{k+1} z y_{k-1} = \begin{cases} 1 + y_k^{r_1} & \text{if } k \text{ is odd} \\ 1 + y_k^{r_2} & \text{if } k \text{ is even} \end{cases}$. In fact, $y_{k+1} z y_k = y_k y_{k+1}$.

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$G = F_2 = \langle y_1, y_2 \rangle$ the free group of rank 2, a cluster $\iota_k : G \hookrightarrow \mathcal{A}_{r_1,r_2}$ is given by $y_1 \mapsto y_k, y_2 \mapsto y_{k+1}$. The $Br_{\mathcal{A}_{r_1,r_2}} =< T_1, T_2 >$-action $\triangleright_{\iota_1}$ on G is given by

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where $m = \begin{cases} 3 & \text{if } r_1 r_2 = 1 \\ 4 & \text{if } r_1 r_2 = 2 \\ 6 & \text{if } r_1 r_2 = 3 \\ 0 & \text{if } r_1 r_2 > 3 \end{cases}$, which justifies the name.

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The noncommutative Laurent Phenomenon is the embeddings $\iota_k : \mathbb{Q}G \hookrightarrow \mathcal{A}_{r_1,r_2}$ for all k whose image contains all y_n.

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$G \cong F_{3n-4}$ is free, we identify a noncommutative cluster ι_Δ for any Δ with its isomorphic image $\mathbb{T}_\Delta =< t_{ij}, (ij) \in \Delta >$ subject to

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$Br_{\mathcal{A}_n}$ is the ordinary braid group Br_{n-2} on $n - 2$ strands which acts on each $\mathbb{T}_\Delta$ by $\bar{\cdot}$-equivariant automorphisms via

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$$T_{ik}(t_\gamma) = \begin{cases} t_{ij}t_{kj}^{-1}t_{kl}t_{il}^{-1}t_\gamma & \text{if } \gamma = (ik) \\ t_\gamma t_{li}^{-1}t_{lk}t_{jk}^{-1}t_{ji} & \text{if } \gamma = (ki) \\ t_\gamma & \text{otherwise} \end{cases}$$

for any internal edge (ik) of Δ where $(ijkl)$ is a cyclic quadrilateral containing (ik) as a diagonal.

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Theorem

The group $Br_{\mathcal{A}_n} = Br_{n-2}$ has a presentation for each triangulation Δ of the n-gon: generators $T_{ik} = T_{ki}$ for all internal edges $(ik) \in \Delta$, relations:

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$$\begin{cases} T_{ij}T_{k\ell}T_{ij} = T_{k\ell}T_{ij}T_{k\ell} & \textit{if } (ij) \textit{ and } (k\ell) \textit{ are sides of a triangle} \\ T_{ij}T_{jk}T_{ki}T_{ij} = T_{jk}T_{ki}T_{ij}T_{jk} & \textit{if } (ijk) \textit{ is an internal triangle} \\ T_{ij}T_{k\ell} = T_{k\ell}T_{ij} & \textit{otherwise} \end{cases}$$

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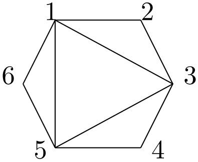
If Δ is a triangulation of the hexagon as in the picture



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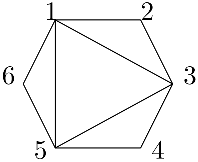

then $Br_{\mathcal{A}_6} = Br_4$ is generated by T_{13}, T_{15}, and T_{35} subject to $T_{13}T_{35}T_{13} = T_{35}T_{13}T_{35}, T_{35}T_{15}T_{35} = T_{15}T_{35}T_{15}, T_{35}T_{15}T_{35} = T_{15}T_{35}T_{15}$ and $T_{13}T_{15}T_{35}T_{13} = T_{15}T_{35}T_{13}T_{15} = T_{35}T_{13}T_{15}T_{35}$.

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A *noncommutative angle* $T_i^{jk} \in \mathcal{A}_n$ in a triangle (ijk) at the vertex $i \in [1, n]$ is defined by $T_i^{jk} = x_{ji}^{-1} x_{jk} x_{ik}^{-1}$

This gives a new presentation of $\mathcal{A}_n$:

- (Triangle relations) $T_i^{jk} = T_i^{kj}$ for distinct i, j, k.

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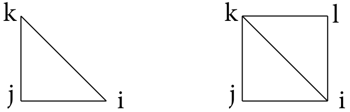

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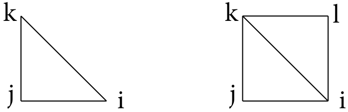

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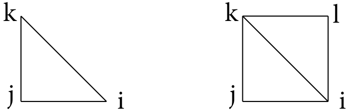

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All T_i are in the image $\mathbb{Q}\iota_\Delta(\mathbb{T}_\Delta)$ for any triangulation Δ of the n-gon, where $\iota_\Delta : \mathbb{T}_\Delta \to \mathcal{A}_n^\times$ is given by $t_\gamma \mapsto x_\gamma$, $\gamma \in \Delta$.

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$\iota_\Delta : \mathbb{Q}\mathbb{T}_\Delta \to \mathcal{A}_n$ is injective for any triangulation Δ of the n-gon and $x_{ij} \in \mathbb{Q}\iota_\Delta(\mathbb{T}_\Delta)$ for any distinct $i, j \in [1, n]$. More precisely,

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- *If $\mathbf{p} := (i_k, i_{k+1}) \cap (i, j) \neq \emptyset$ and $\mathbf{q} := (i_\ell, i_{\ell+1}) \cap (ij) \neq \emptyset$ for some $k < \ell$, then the point $\mathbf{p}$ of (ij) is closer to i than $\mathbf{q}$.*

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and $x_{\mathbf{i}} := x_{i_1,i_2} x_{i_3,i_2}^{-1} x_{i_3,i_4} \cdots x_{i_{2m-1},i_{2m-2}}^{-1} x_{i_{2m-1},i_{2m}} \in \iota_\Delta(\mathbb{T}_\Delta)$

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Example

(a) If $n = 5$ and $\Delta = \{(1,3), (3,1), (1,4), (4,1); (i, i \pm 1) | i \in [1,5]\}$, then

$$x_{21}^{-1} x_{25} x_{15}^{-1} = x_{21}^{-1} x_{23} x_{13}^{-1} + x_{31}^{-1} x_{34} x_{14}^{-1} + x_{41}^{-1} x_{45} x_{15}^{-1}.$$

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(a) If $n = 5$ and $\Delta = \{(1,3),(3,1),(1,4),(4,1);(i,i\pm 1)|i \in [1,5]\}$, then

$$x_{21}^{-1}x_{25}x_{15}^{-1} = x_{21}^{-1}x_{23}x_{13}^{-1} + x_{31}^{-1}x_{34}x_{14}^{-1} + x_{41}^{-1}x_{45}x_{15}^{-1}.$$

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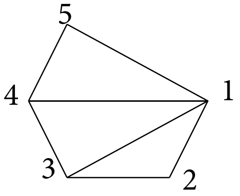

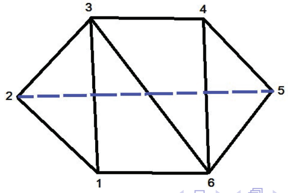

Main example: noncommutative polygon

Monomial mutation $\mu_{\Delta,\Delta'}$ is an isomorphism $\mathbb{T}_{\Delta'} \to \mathbb{T}_\Delta$ given by

$$t_{ij} \mapsto t_{\mathbf{i}^{left}}$$

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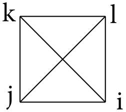
In particular, if Δ' is obtained from Δ by flipping (ik) to (jl) in a clockwise quadrilateral $(ijkl)$, then $\mu_{\Delta,\Delta'}(t_\gamma) = \begin{cases} t_{jk}t_{ik}^{-1}t_{il} & \text{if } \gamma = (jl) \\ t_{li}t_{ki}^{-1}t_{kj} & \text{if } \gamma = (lj) \\ t_\gamma & \text{otherwise} \end{cases}$



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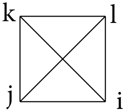

Theorem

In this case, $\mu_{\Delta,\Delta'} \circ \mu_{\Delta',\Delta}$ *is an automorphism of* $\mathbb{T}_\Delta$ *equal* $T_{ik} \in Br_{n-2}$*.*
Otherwise, $\mu_{\Delta,\Delta''} = \mu_{\Delta,\Delta'} \circ \mu_{\Delta',\Delta''}$ *if*
$dist(\Delta, \Delta'') = dist(\Delta, \Delta') + dist(\Delta', \Delta'')$

Noncommutative marked surfaces

To any such surface Σ (each boundary component must have at least one marked point, some orbifold points are allowed) we assign, in a functorial way, an algebra $\mathcal{A}_\Sigma$ generated by x_γ, where γ runs over isotopy classes of directed curves between marked points, subject to

- Triangle relations in any cyclic triangle $(\gamma_1, \gamma_2, \gamma_3)$
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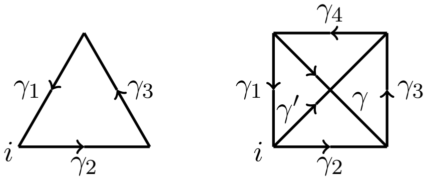

Similarly to $\mathcal{A}_n$ (assigned to the unpunctured disk with n marked boundary points), define a noncommutative angle $T_i^{\gamma_1,\gamma_2} := x_{\gamma_1}^{-1} x_{\overline{\gamma}_3} x_{\gamma_2}^{-1}$ formed by two sides of a cyclic triangle $(\gamma_1, \gamma_2, \gamma_3)$ where γ_1 is incoming to i and γ_2 is outgoing from i (here $\overline{\gamma}$ is the oppositely directed γ).

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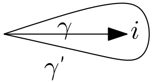

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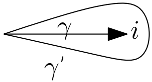

Theorem

For any puncture i the assignments $x_\gamma \mapsto T_i^{\delta_{i,s(\gamma)}} x_\gamma T_i^{\delta_{i,t(\gamma)}}$ define an involutive automorphism φ_i of $\mathcal{A}_\Sigma$, where $s(\gamma)$ and $t(\gamma)$ are respectively the starting and terminating point of γ.
Moreover, these automorphisms commute so that for any subset P of punctures the composition φ_P of all φ_i, $i \in P$ is well-defined.

Noncommutative marked surfaces

Using this, we can describe all noncommutative clusters in $\mathcal{A}_\Sigma$. First, for any triangulation Δ of Σ, we define the triangle group $\mathbb{T}_\Delta$ generated by t_γ, $\gamma \in \Delta$ subject to the triangle relations, define a natural embedding $\iota_\Delta : \mathbb{T}_\Delta \to \mathcal{A}_\Sigma^\times$, and establish the following

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The extension $\iota_\Delta : \mathbb{Q}\mathbb{T}_\Delta \to \mathcal{A}_\Sigma$ is injective for any triangulation Δ of Σ and all x_γ belong to its image. More precisely, each x_γ can be uniquely expressed as a sum of elements of $\iota_\Delta(\mathbb{T}_\Delta)$.

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If Σ is punctured, twisting ι_Δ with with automorphisms φ_P gives rise to *tagged* noncommutative clusters $\iota_{\Delta^{\bowtie}}$ which are labeled by tagged triangulations $\Delta^{\bowtie}$ of Σ together with the corresponding tagged noncommutative Laurent Phenomenon.

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Denote by $\Sigma_{n,k}$ the k times punctured disk with n boundary points. The following is the list of all tagged and untagged clusters for $\Sigma_{3,1}$.

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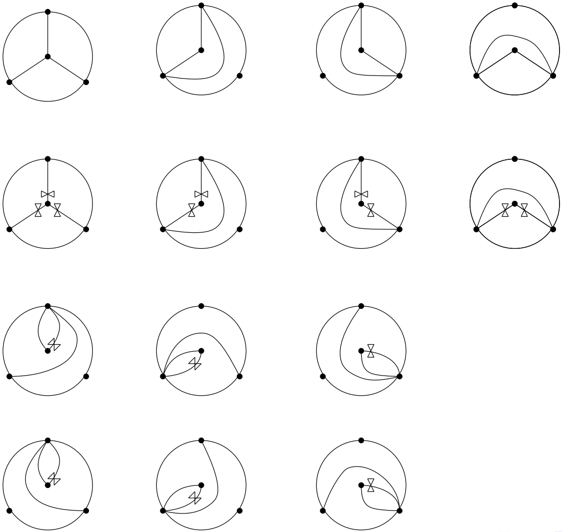

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If Σ is oriented, *monomial mutation* $\mu_{\Delta,\Delta'}$ is an isomorphism $\mathbb{T}_{\Delta'} \to \mathbb{T}_\Delta$ defined similarly to $\mathcal{A}_n = \mathcal{A}_{\Sigma_{n,0}}$, i.e., by assigning to any t_γ, $\gamma \in \Delta'$, the leftmost γ-admissible sequence in Δ. Thus, all these groups are isomorphic to a canonical group $\mathbb{T}_\Sigma$ (it is either free or 1-relator).

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- *Let $\Sigma_p^q \cong \Sigma_q^p$ be the unpunctured cylinder with p points on one boundary and q points on another. Then $Br_{\Sigma_p^q}$ is (a quotient of) the affine braid group $\hat{B}r_{p+q}$.*

Noncommutative marked surfaces

In fact, $\mathbb{T}_\Sigma$ is free iff Σ has a boundary or is a sphere with three punctures.

Example

If Σ is the torus, the Klein bottle, the real projective plane respectively with one, one, two punctures, then $\mathbb{T}_\Sigma$ is generated by a, b, c, d, e subject to, respectively, the following relations:

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We expect that Br_Σ is a free group on 3 generators in these three cases.

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for all $n \geq k+1$, where $D, \overline{D}$, and C_i, $i \in \mathbb{Z}_{>0}$ are free parameters with $C_{n+k-1} = C_{k-1}$ for $n \in \mathbb{Z}_{>0}$.

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Theorem

This recursion has a unique solution in the group algebra $\mathbb{Q}F_{2k+1}$ of the free group F_{2k+1} freely generated by $D, \overline{D}, C_1, \ldots, C_{k-1}$, $U_1, \ldots, U_k$, more precisely, each U_n is a sum of elements of F_{2k+1}.

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Moreover, the elements H_n given by

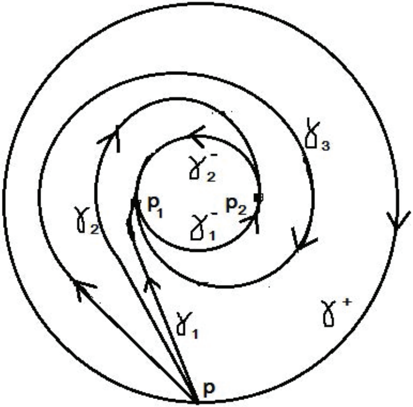
$$H_n := \begin{cases} \overline{D} U_{n+1-k} U_n^{-1} + D U_{n+k-1} U_n^{-1} & \text{if } n \text{ is even} \\ U_n^{-1} U_{n+1-k} D + U_n^{-1} U_{n+k-1} \overline{D} & \text{if } n \text{ is odd} \end{cases} \quad (1)$$

belong to $\mathbb{Z}F_{2k+1}$ and do not depend on n (hence it is a discrete integral)

Noncommutative discrete integrable systems

The first assertion is a noncommutative Laurent Phenomenon for triangulations Δ_n of a cylinder Σ^1_{k-1} obtained by "Dehn twists" one from another.

The second assertion is that H_n is the total angle $T_p \in \mathcal{A}_{\Sigma^1_r}$ at the point p, it is additive and does not depend on Δ_n.



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This recursion has a (unique) solution in the group algebra $\mathbb{Q}T_\infty$ of the free group $\mathbb{T}_\infty$ freely generated by $A_i, \overline{A}_i$, $B_i, \overline{B}_i$, U_{ii}, V_{ii}, $U_{i,i+1}$, $i \in \mathbb{Z}$, more precisely, each U_{ij} and V_{ij} is a sum of elements of the group.

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This recursion has a (unique) solution in the group algebra $\mathbb{Q}T_\infty$ of the free group $\mathbb{T}_\infty$ freely generated by $A_i, \overline{A}_i$, $B_i, \overline{B}_i$, U_{ii}, V_{ii}, $U_{i,i+1}$, $i \in \mathbb{Z}$, more precisely, each U_{ij} and V_{ij} is a sum of elements of the group. Moreover, the elements $H_{ij}^{\pm} \in Frac(\mathbb{Z}\mathbb{T}_\infty)$ given by

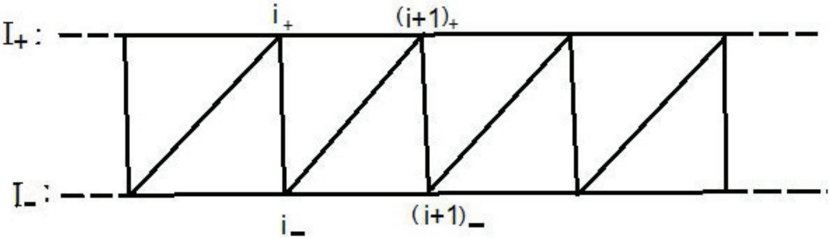
$$H_{ij}^{+} := U_{ji}^{-1}(U_{j,i-1}A_{i-1} + U_{j,i+1}\overline{A}_i), \ H_{ij}^{-} := V_{ji}^{-1}(V_{j,i-1}B_{i-1} + V_{j,i+1}\overline{B}_i^{-1})$$

belong to $\mathbb{Z}\mathbb{T}_\infty$ and do not depend on j (i.e., are discrete integrals).

Noncommutative discrete integrable systems

The first assertion is a noncommutative Laurent Phenomenon for translation-invariant triangulations of an infinite strip Σ_∞.

The second assertion is that $H_{ij}^{\pm}$ are the total angles $T_{i+}, T_{i-} \in \mathcal{A}_{\Sigma_\infty}$ on the upper and lover boundaries, they are additive and do not depend on the triangulations.



Noncommutative discrete integrable systems

These examples suggest the following general approach to constructing noncommutative discrete integrable systems. That is, such a system consists of a marked surface Σ, its automorphism $\tau : \Sigma :\to \Sigma$ permuting marked points, and a triangulation Δ so that the collection $\mathcal{T} = \{x_\gamma \in \mathcal{A}_\Sigma, \gamma \in \bigcup_{k\in\mathbb{Z}} \tau^k(\Delta)\}$ evolves in "discrete time" $k \in \mathbb{Z}$ and for each marked point p of Σ, the total noncommutative angle T_p is a discrete integral. The noncommutative Laurent Phenomenon then guarantees that $\mathcal{T}$ belongs to the algebra isomorphic to the group algebra of $\mathbb{T}_\Delta$.