Darboux Transformation & Quasideterminants

H中

'22 3/31

- Darboux trf. : One of solution generating techniques in integrable system (Lax formalism)

- Quasideterminants : One of noncommutative (NC) determinants

- Today : We discuss Darboux trf. for KdV eq. in terms of quasideterminants

- Plan : (§1) KdV eq. (§2) Quasideterminants (§3) Proofs
  (§4) Other applications (ASDYM, Bogomolnyi, Ernst, M2 M5, ⋯)

§1. NC KdV eq.

$u = u(t,x)$: $\mathbb{H}$-valued fcn.

(NC KdV eq.)

$$u_t + u_{xxx} + 3(u_x u + u u_x) = 0 \quad \cdots ①$$

① is derived from compatibility condition of the linear system:

$$[L, M] = 0$$

$$\begin{cases} L\phi = 0 & \leftarrow \mathbb{H}\text{-valued} \\ M\phi = 0 \end{cases} \quad \cdots ② \qquad \begin{cases} L_\zeta := \partial_x^2 + u - \underbrace{\zeta}_{\text{spectral parameter}} \\ M := \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x \end{cases}$$

$$\phi = \phi(t,x;\zeta) \qquad u = u(t,x)$$

(Darboux trf.)

θ : "special" sol. of ② i.e. $\begin{cases} L_\lambda \theta = 0 \\ M \theta = 0 \end{cases}$ ⋯③ ← $\zeta \to \lambda$: special value (fix) $\theta(\lambda) := \phi(\zeta \to \lambda)$

$G_\theta := \theta \partial_x \theta^{-1} = \partial_x - \theta_x \theta^{-1}$ ⋯ ④ ← $\partial_x \theta^{-1} = -\theta^{-1}\theta_x\theta^{-1}$

$\left(\Leftrightarrow \; G_\theta \overset{\forall}{f} = \theta \partial_x (\theta^{-1} f) = \partial_x f - \theta_x \theta^{-1} f \right)$

The following trf. is called the Darboux trf.

$$(D) \begin{cases} L \mapsto \tilde{L} := G_\theta L G_\theta^{-1} \quad (= \partial_x^2 + \tilde{u} - \zeta \; : \text{form invariant}) \\ M \mapsto \tilde{M} := G_\theta M G_\theta^{-1} \\ \phi \mapsto \tilde{\phi} := G_\theta \phi \overset{④}{=} \phi_x - \theta_x \theta^{-1} \phi \end{cases}$$

(D) induces $\tilde{u} = u + 2(\theta_x \theta^{-1})_x$ ⋯ ⑤

∵ (D) $\Leftrightarrow$ $\tilde{L} G_\theta = G_\theta L$, $\tilde{M} G_\theta = G_\theta M$ $\underset{③}{\Rightarrow}$ ⑤ ▨

Strategy [Gilson-Nimmo, 2007]

$$\begin{cases} L_\zeta \phi = (\partial_x^2 - \zeta)\phi = 0 \\ M\phi = (\partial_t + 4\partial_x^3)\phi = 0 \end{cases}$$

$$\boxed{\theta_k := \phi(\zeta \to \lambda_k)}$$

[0] $\left(\overset{0}{\overset{\shortparallel}{u}}, \overset{一般解}{\phi}, \overset{特殊解}{\theta_1}\right)$: initial seed sol.

$\downarrow$ (D) $\phi_{[1]} = \phi' - \overset{\leftarrow\,ビブン\,\rightarrow}{\theta_1' \theta_1^{-1}} \phi$

[1] $\left(\underset{\underset{2(\theta_1'\theta_1^{-1})'}{\shortparallel}}{u_{[1]}}, \phi_{[1]}, \quad \right)$

$\downarrow$

[2]

$\downarrow$

[3]

Strategy [Gilson–Nimmo, 2007]

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$$\boxed{\theta_k := \phi(\zeta \to \lambda_k)}$$

$\downarrow$ (D) $\phi_{[1]} = \phi' - \theta_1' \theta_1^{-1} \phi$ (←ビブン→)

[1] $(u_{[1]},\ \phi_{[1]},\ \theta_{[1]})$, $\quad \phi_{[1]} \xrightarrow{\zeta \to \lambda_2} \theta_{[1]}$

$u_{[1]} = 2(\theta_1' \theta_1^{-1})'$

$$\theta_{[1]} := \phi_{[1]}(\zeta \to \lambda_2) = \underbrace{\phi'(\zeta \to \lambda_2)}_{=\theta_2'} - \theta_1' \theta_1^{-1} \underbrace{\phi(\zeta \to \lambda_2)}_{=\theta_2}$$

$\downarrow$ (D) $\phi_{[2]} = \phi_{[1]}' - \theta_{[1]}' \theta_{[1]}^{-1} \phi_{[1]}$

代入していけば
initial data だけで
解が逐次求まる.

[2] $(u_{[2]},\ \phi_{[2]},\ \quad)$

$\downarrow$

(結果) $\Theta := (\theta_1, \cdots, \theta_n)$: set of sols of ③

$$\Phi_{[n]} = \begin{vmatrix} \Theta & \phi \\ \Theta^{(1)} & \phi^{(1)} \\ \vdots & \vdots \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}$$

[Etingof-Gelfand-Retakh, 1997]

(Quasi-Wronskian)

$$u_{[n]} = 2\left(\sum_{k=1}^{n} \theta^{(1)}_{[k]}\, \theta^{-1}_{[k]}\right)_x = -2 \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}_x$$

Quasideterminant で compact に書ける!

§2 Quasideterminant

$A = (a_{ij})_{1\leq i,j\leq n}$ $a_{ij} \in$ Division Ring (斜体)

(e.g. $\mathbb{H}$) noncommutative

$\varnothing^{-1}$ is assumed to exist.

Def

Let $A = (a_{ij})$ be an $n \times n$ square matrix, and $B = (b_{ij})$ be A^{-1}.

b_{ji}^{-1} is (i,j)-quasideterminant of A and represented:

$$b_{ji}^{-1} =: |A|_{ij} \overset{\text{or}}{\equiv} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \boxed{a_{ij}} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \quad \left(\xrightarrow[\text{limit}]{\text{com.}} (-1)^{i+j} \frac{|A|}{|A^{ij}|} \right)$$

suffix or box

逆行列の公式

Rmk　Block decomposition

square　　　square

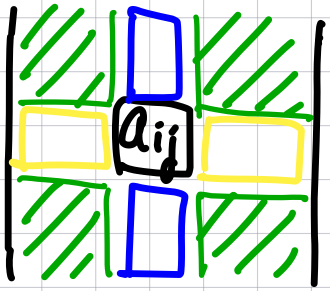
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1}+A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \cdots ⑥$$

これが逆行列の1要素

w/ $S := D - CA^{-1}B$ (Schur compliment)

$$D \equiv a_{ij} \Rightarrow S \equiv |A|_{ij} = a_{ij} - \boxed{\text{i-th row}}\ \left(A'_{ij}\right)^{-1} \boxed{\text{j-th column}}$$

deleting i-th row & j-th column



Ex

$n=1)$ $\quad |A| = a$

$n=2)$ $\quad \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21}, \quad \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}\, a_{21}^{-1}\, a_{22}, \ \cdots$

$$\therefore \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

← all squared

blocked

$n=3)$ $\quad \left|\begin{array}{c|cc} \boxed{a_{11}} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right| = a_{11} - (a_{12}\ a_{13}) \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$

$$= a_{11} - a_{12} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{31}$$

$$- a_{13} \begin{vmatrix} a_{22} & \boxed{a_{23}} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} a_{31}, \quad \cdots$$

Useful Identity

NonCommutative

(NC Jacobi identity)

$$
\begin{array}{c|ccc|}
 & N & 1 & 1 \\
N & A & B & C \\
1 & D & f & g \\
1 & E & h & \boxed{i}
\end{array}
= \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix} \quad \cdots ⑦
$$

$(N+2)\times(N+2)$

気分 "$i - hf^{-1}g$" (Schur comp.)

$(N+1)\times(N+1)$

☺ (S.C. Huang, PhD thesis, 2112.10702)

$$
(\mathrm{LHS}) = i - \begin{pmatrix} E & h \end{pmatrix} \underbrace{\begin{pmatrix} A & B \\ D & f \end{pmatrix}^{-1}}_{⑥\ (\text{ブロック分解})^{-1}} \begin{pmatrix} C \\ g \end{pmatrix} = (\mathrm{RHS}) \quad \blacksquare
$$

(Derivative formula)

derivative

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' \begin{matrix} \updownarrow n \\ \updownarrow 1 \end{matrix} = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & e_k \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & \boxed{(B^k)'} \end{vmatrix}$$

$\underset{n}{\leftrightarrow}$ $\underset{1}{\leftrightarrow}$

k-th row of A

k-th column of A

$$= \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix} \begin{vmatrix} A & B \\ {}^te_k & \boxed{0} \end{vmatrix}$$

${}^te_k = (0 \cdots \overset{\text{k-th}}{1} \cdots 0)$

(☺ $(A \cdot A^{-1})' = 0 \longrightarrow$) $(A^{-1})' = -A^{-1}A'A^{-1} \longrightarrow$

☺ $$\begin{vmatrix} A & B \\ C & d \end{vmatrix}' = (d - CA^{-1}B)' = d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'$$

$\sum_{k=1}^{n} e_k {}^te_k$: unit matrix

$$= d' - C'A^{-1}B + \sum_{k=1}^{n} (CA^{-1}e_k)(\overbrace{{}^te_k A'}^{(A^k)'} A^{-1}B) - \sum_{k=1}^{n} CA^{-1}e_k (B^k)' \quad \blacksquare$$

§3 Proofs : Page 6 の結果の $\phi_{[n]}$, $\theta_{[n]}$ が

$$\phi_{[n+1]} = \phi'_{[n]} - \theta'_{[n]}\,\theta_{[n]}^{-1}\,\phi_{[n]}$$ をみたすか？

$$\phi_{[n]} = \begin{vmatrix} H & \phi \\ \vdots & \vdots \\ H^{(n-1)} & \phi^{(n-1)} \\ H^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}$$

$$\phi'_{[n]} = \begin{vmatrix} H & \phi \\ \vdots & \vdots \\ H^{(n-1)} & \phi^{(n-1)} \\ H^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} + \begin{vmatrix} H & 0 \\ \vdots & \vdots \\ & 0 \\ H^{(n-1)} & 1 \\ H^{(n)} & \boxed{0} \end{vmatrix} \phi_{[n]}$$

only $k=n$ survive

$$\theta_{[n]} = \begin{vmatrix} H & \theta_{n+1} \\ \vdots & \vdots \\ H^{(n-1)} & \theta_{n+1}^{(n-1)} \\ H^{(n)} & \boxed{\theta_{n+1}^{(n)}} \end{vmatrix}$$

$$\theta'_{[n]} = \begin{vmatrix} H & \theta_{n+1} \\ \vdots & \vdots \\ H^{(n-1)} & \theta_{n+1}^{(n-1)} \\ H^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} + \begin{vmatrix} H & 0 \\ \vdots & \vdots \\ & 0 \\ H^{(n-1)} & 1 \\ H^{(n)} & \boxed{0} \end{vmatrix} \theta_{[n]}$$

$$\phi_{[n+1]} = \begin{vmatrix} H & \theta_{n+1} & \phi \\ \vdots & \vdots & \vdots \\ H^{(n-1)} & \theta_{n+1}^{(n-1)} & \phi^{(n-1)} \\ H^{(n)} & \theta_{n+1}^{(n)} & \phi^{(n)} \\ H^{(n+1)} & \theta_{n+1}^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix}$$

$$\therefore\ \phi'_{[n]} - \theta'_{[n]}\,\theta_{[n]}^{-1}\,\phi_{[n]} = \begin{vmatrix} H & \phi \\ \vdots & \vdots \\ H^{(n-1)} & \phi^{(n-1)} \\ H^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} - \begin{vmatrix} H & \theta_{n+1} \\ \vdots & \vdots \\ H^{(n-1)} & \theta_{n+1}^{(n-1)} \\ H^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} \theta_{[n]}^{-1}\,\phi_{[n]}$$

$/\!/$ Jacobi

OK!

$$
u_{[n+1]} = -2 \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}_x
$$

$$
\overset{\text{Jacobi}}{=} +2\,\partial_x \left\{ \begin{vmatrix} \ddots & \Theta_n \\ & \vdots \\ & \Theta_n^{(n-2)} \\ \Theta_1^{(n)} \cdots \Theta_{n-1}^{(n)} & \boxed{\Theta_n^{(n)}} \end{vmatrix} \underbrace{\begin{vmatrix} \ddots & \Theta_n \\ & \vdots \\ & \Theta_n^{(n-2)} \\ \Theta_1^{(n-1)} \cdots \Theta_{n-1}^{(n-1)} & \boxed{\Theta_n^{(n-1)}} \end{vmatrix}}_{\text{Quasi Wronskian}}{}^{-1} \right\}
$$

"gapped"

$\downarrow$ commutative limit

$$
= 2\,\partial_x \left\{ Wr(\Theta_1, \cdots, \Theta_n)' \, Wr(\Theta_1, \cdots, \Theta_n)^{-1} \right\}
$$

(実は一般解)

$$
= 2\,\partial_x^2 \log Wr(\Theta_1, \cdots, \Theta_n)
$$

Hirota trf. & Wronskian sol.

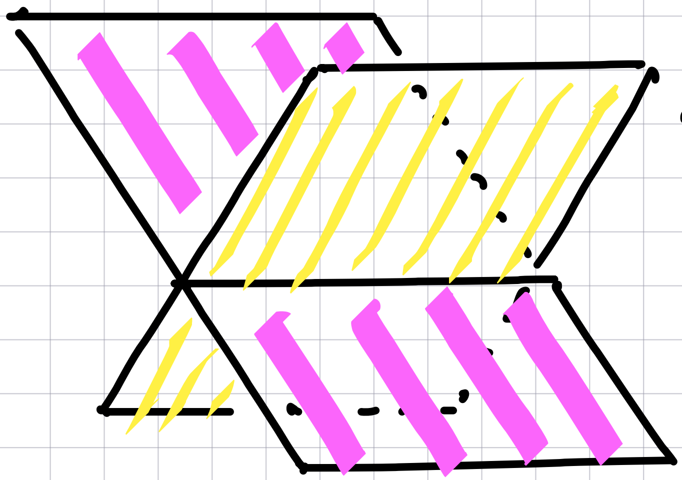
§4 Other Applications

· ASD Yang-Mills: Gilson, Nimmo, Huang との共同研究

$$\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} = C\,(2\,\mathrm{sech}^2 X - 3\,\mathrm{sech}^4 X), \quad X = k_\mu x^\mu$$

$\in \mathbb{R}$

$\mathbb{R}^4$



← codim 1 の超平面上に局在した新しいソリトン

(intersecting 解もある)

w/ 菅野さん, Huangさん

(WZW型作用)

Euclid, Minkowski ⇒ A_μ: エルミート & 反エルミート

Ultrahyperbolic ⇒ $^\forall A_\mu$: 反エルミート ⇔ $G=U(2)$ ⇒ $N=2$ 弦理論で実現!

ASDYM eq. (Yang's form)

$$\partial_{\tilde{z}}(\partial_z J \cdot J^{-1}) - \partial_{\tilde{w}}(\partial_w J \cdot J^{-1}) = 0$$

$\Leftrightarrow$ 両立条件 of

$$\zeta = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}$$: これがミソ

$$\begin{cases} L(\phi) = J\partial_w (J^{-1}\phi) - (\partial_{\tilde{z}}\phi)\zeta = 0 \\ M(\phi) = J\partial_z (J^{-1}\phi) - (\partial_{\tilde{w}}\phi)\zeta = 0 \end{cases}$$

Darboux trf.

$$(D) \begin{cases} \tilde{\phi} = \phi\zeta - \theta\Lambda\theta^{-1}\phi \\ \tilde{J} = -\theta\Lambda\theta^{-1}\phi \end{cases} \Rightarrow \tilde{L}(\tilde{\phi}) = \begin{vmatrix} \theta & L(\phi) \\ \theta\Lambda & \boxed{L(\phi)\zeta} \end{vmatrix}$$: Huang's formula (Ph.D. thesis)

※ $\zeta_1 = \zeta_2$ だと自明な変換