

保型形式・L函数・周期の研究 2006.1.23-27. RIMS

$p$ -adic multiple zeta values and  
double shuffle relations

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references

[F1] H. Furusho :  $p$ -adic multiple zeta values I

Inv Math Vol 155, No.2 (2004)

[F2] , :  $p$ -adic multiple zeta values II

arXiv: math.NT/0506117

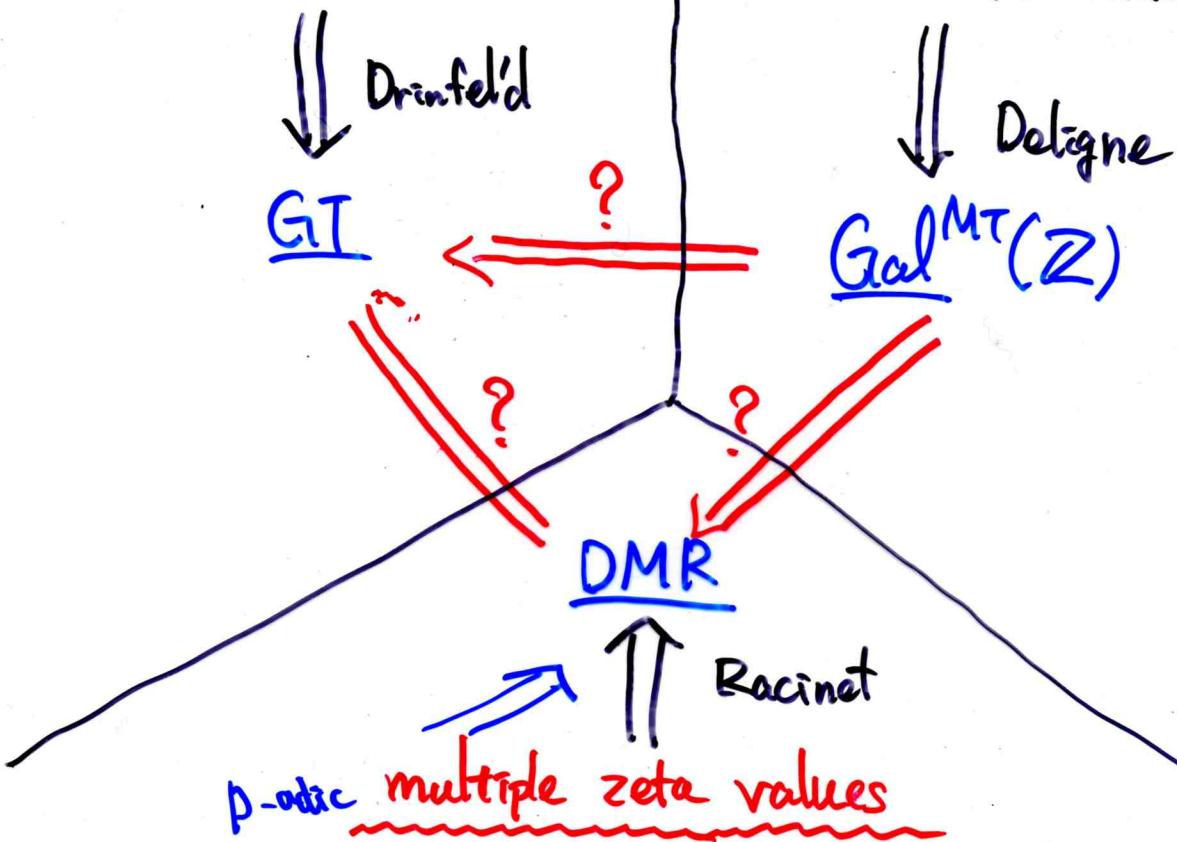
[BF] A. Besser, H. Furusho : The double shuffle relations  
for  $p$ -adic multiple zeta values : arXiv: math.NT/0310177

[FJ] H. Furusho, A. Jafari : Regularizations and generalized  
double shuffle relations for  $p$ -adic multiple zeta values,  
arXiv: math.AG/0510681.

## philosophical background

(quasi-) Quantum Group  $\longleftrightarrow$  Motives

quasi-triangular  $\longleftrightarrow$  unramified mixed Tate



## multiple zeta values (MZV)

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$m, k_1, \dots, k_m \in \mathbb{N}$

$$\zeta(k_1, \dots, k_m) = \sum_{0 < n_1 < \dots < n_m} \frac{1}{n_1^{k_1} \dots n_m^{k_m}} \in \mathbb{R}$$

$$\frac{1}{n_i} = \frac{1}{e^{n_i}} \rightarrow \infty$$

RP  
E

\* converges for  $k_m \geq 1$

\* diverges for  $k_m = 1$

\*  $m=1$   $\zeta(k)$ : Riemann zeta value

$$\zeta = \text{It.} \int_0^1 \underbrace{\frac{dt}{t-t_1} \circ \frac{dt}{t-t_2} \dots \circ \frac{dt}{t-t_m}}_{k_m} \circ \frac{dt}{t-t_1} \dots \circ \frac{dt}{t-t_m} \underbrace{\frac{dt}{t-t_1} \circ \frac{dt}{t-t_2} \dots \circ \frac{dt}{t-t_k}}_{k_1}$$

$$\text{It.} \int_s^t w_1 \circ w_2 := \int_s^t w_1(u) \int_s^u w_2(v) dv$$

\* basic example of MTH(Z)

\* For  $k_m=1$ ,  $\text{Reg} \zeta(k_1, \dots, k_m) \in R[T]$

is defined by regularization.

## Double Shuffle Relation

example

$$\zeta(a)\zeta(b) = \sum_{0 < k} \frac{1}{k^a} \sum_{0 < l} \frac{1}{l^b} = \left( \sum_{0 < k < l} + \sum_{0 < k = l} + \sum_{0 < l < k} \right) \frac{1}{k^a l^b}$$

$$= \zeta(a, b) + \zeta(a+b) + \zeta(b, a)$$

series shuffle product

$$\zeta(a)\zeta(b) = \text{Int} \int_0^1 \underbrace{\frac{dt}{1-t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_a \cdot \text{Int} \int_0^1 \underbrace{\frac{dt}{1-t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_b$$

$$= \sum_{i=0}^{a-1} \binom{b-i+i}{i} \zeta(a-i, b+i) + \sum_{j=0}^{b-1} \binom{a-1+j}{j} \zeta(b-j, a+j)$$

Integral shuffle product

This gives double shuffle relation.

One can extend this into the case for  $k_m=1$  by regularization.

$$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

$$\pi_1^{\text{Be}}(X(\mathbb{C}): \overrightarrow{0}, \overrightarrow{1})(\mathbb{C}) \xrightarrow{\sim} \pi_1^{\text{DR}}(X: \overrightarrow{0}, \overrightarrow{1})(\mathbb{C})$$

↓  
 $b$   
 ↪ real str

↓  
 $d, b$   
 ↪ Deligne

$$\pi_1^{\text{DR}}(X: \overrightarrow{0})(\mathbb{C}) \hookrightarrow \mathbb{C}\langle A, B \rangle$$

$$d'b \longmapsto \Phi_{k2} = 1 + \sum (-1)^m \zeta(k_1, \dots, k_m) A^{k_1-1} B - A^{k_1-1} B + \dots$$

p-adic multiple zeta values by Deligne

$$\Phi \circ \pi_1^{\text{p,rig}}(X_{\mathbb{F}_p}: \overrightarrow{0}, \overrightarrow{1})(\mathbb{Q}_p) \xrightarrow{\sim} \pi_1^{\text{DR}}(X: \overrightarrow{0}, \overrightarrow{1})(\mathbb{Q}_p)$$

↓  
 $\exists_1 c$   
 s.t.  $\Phi(c) = c$   
 Besser, Vologodsky  
 ↪ Deligne

↓  
 $d, \Phi(d) \neq c$   
 ↪ Deligne

$$\pi_1^{\text{DR}}(X: \overrightarrow{0})(\mathbb{Q}_p) \hookrightarrow \mathbb{Q}_p\langle A, B \rangle$$

$$d' \Phi(d) \longmapsto \Phi_{\text{De}}^p = 1 + \sum (-1)^m \zeta_p(k_1, \dots, k_m) A^{k_1-1} B - A^{k_1-1} B + \dots$$

$$\text{Th } d'c \longmapsto \Xi_{k2}^p = 1 + \sum (-1)^m \zeta_p^T(k_1, \dots, k_m) A^{k_1-1} B - A^{k_1-1} B + \dots$$

Problem (AWS '02, Deligne)

Does  $\zeta_p^{\text{Be}}$  satisfy double shuffle relations?

# Main Th ([BF], [FJ]) Yes!!

Proof.

Step<sup>1</sup> Th-Def ([FI]) Coleman's Public integration ('82)

$\exists \lim_{\varepsilon \rightarrow 0} \text{It} \int_{-\varepsilon}^{1-\varepsilon} \underbrace{\frac{dt}{t-\varepsilon}}_{k_1} \circ \underbrace{\frac{dt}{t}}_{k_2} \circ \dots \circ \underbrace{\frac{dt}{t}}_{k_m} \circ \dots \circ \underbrace{\frac{dt}{t-\varepsilon}}_{k_n} \in \mathbb{Q}_p$

|| and branch disappear for  $k_m \geq 1$

Def  $\mathfrak{S}_p^F(k_1, \dots, k_m) \subset \mathbb{Q}_p$

~~~~~ integral shuffle relations for  $\mathfrak{S}_p^F$

Step<sup>2</sup> ([F2]) tannakian interpretations &

By tangential basept  $Z(l)_{n_2} \rightarrow \mathcal{T}C_l^{Pr_2}(X_{\overline{\mathbb{F}_p}} : \overrightarrow{0})$  in [BF].

One can express Deligne's  $p$ MZVs in terms of my  $p$ MZVs and vice versa.

$\mathfrak{S}_p^D$   $\Leftarrow$   $\Rightarrow$   $\mathfrak{S}_p^F$ .

e.g.  $\mathfrak{S}_p^D(a) = \left(1 - \frac{1}{pa}\right) \mathfrak{S}_p^F(a)$ .

Step 3 Racinet's work implies

$$\begin{array}{ccc} \text{double shuffle} & & \text{double shuffle} \\ \text{relns for } \mathfrak{g}_p^D & \leftrightarrow & \text{relns for } \mathfrak{g}_p^F \end{array}$$

Step 4 when  $k_1, \dots, k_m \in \mathbb{N}, z_1, \dots, z_m \in \mathbb{C}$ ,

$$L_{k_1, \dots, k_m}(z_1, \dots, z_m) = \sum_{0 < n_1 < \dots < n_m} \frac{z_1^{n_1} \cdots z_m^{n_m}}{n_1^{k_1} \cdots n_m^{k_m}} \quad \begin{array}{l} \text{p-adic} \\ : \text{MPL} \\ (\text{multiple polylog}) \end{array}$$

\* converges for  $|z_i| < 1$ .

\*  $m=1$   $L_{k_1}(z) = \sum_n \frac{z^n}{n^k}$  : polylog

\*  $= \text{It} \int_{(0, \dots, 0)}^{(z_1, \dots, z_m)} \text{diff forms} \rightarrow \mathcal{M}_{0, m+3}$

$\mathcal{M}_{0, m+3}$ : the moduli of curves of  $(0, m+3)$ -type

$$= \left\{ (z_1, \dots, z_m) \in A^m \mid \prod_{i=a}^b z_i \neq 0, 1 \quad (1 \leq a \leq b \leq m) \right\}$$

\*  $L_{k_1, \dots, k_m}(z_1, \dots, z_m) \xrightarrow{\cancel{z_i \rightarrow 1}} \mathfrak{g}(k_1, \dots, k_m)$

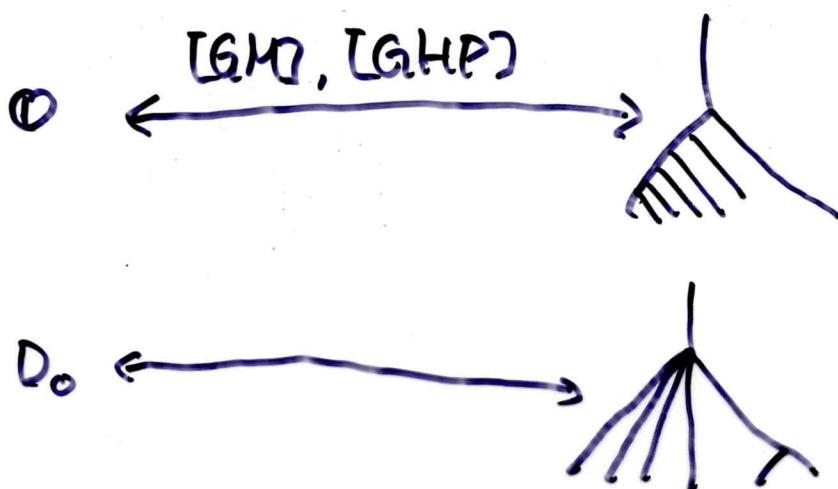
## Step 5

L?

pMPL  $\leadsto$  ana conn to  $M_{0,m+3}(\mathbb{C}_p)$  (Besser's technique)

$\leadsto$  ana conn to  $N_{D_0}(\mathbb{C}_p)$  ( $\forall D_i = \overline{M_{0,m+3}} - M_{0,m+3}$ )

via higher dim'l version of tang'l baspt (with Besser)



$\leadsto \exists$  nice tangent'l line  $L_0 \subset N_{D_0}$

*but not unique*

$$\text{In } ([FJ]) \quad \wp_F(k_1, \dots, k_m) = \left. \text{Li}_{k_1, \dots, k_m}(z_1, \dots, z_m) \right|_{L_0} \in \mathbb{Q}_p \quad \text{if } k_m > 1$$

$$\text{Def In } ([FJ]) \quad \text{Reg } \wp_F(k_1, \dots, k_m) := \left. \text{Li}_{k_1, \dots, k_m}(z_1, \dots, z_m) \right|_{L_0} \quad \text{if } k_m = 1$$

*or*  
*or*  
*or*

(This is apriori not well-defined (branch appear!))

Step 6

$$\begin{aligned}
 L_{ia}(x)L_{ib}(y) &= \sum_{0 \leq k} \frac{x^k}{k^a} \sum_{0 \leq l} \frac{y^l}{l^b} = \left( \sum_{0 \leq k \leq a} + \sum_{0 \leq k=a} + \sum_{0 \leq k > a} \right) \frac{x^k y^l}{k^a l^b} \\
 &= L_{ia,b}(x,y) + L_{iab}(xy) + L_{iba}(y,x) \\
 \downarrow L_0 &\quad \downarrow L_0 &\quad \downarrow L_0 &\quad \downarrow L_0 &\quad \downarrow L_0 \\
 S_p^F(a) \cdot S_p^F(b) &= S_p^F(a,b) + S_p^F(a+b) + S_p^F(b,a)
 \end{aligned}$$

→ ser. shuffle reln's for  $S_p^F$

→ double shuffle reln's for  $S_p^F$

→ double shuffle reln's for  $S_p^D$

//

Cor Reg  $S_p^F(k_1, \dots, k_m)$  is well-defined