

# Pentagon and Hexagons

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## reference

Pentagon and hexagon equations arXiv:math/0702128.

## Plan

§1. Pentagon

§2. Hexagons

§3. GT

§4. Main Theorem

# § I. Pentagon

LL

Def  $(\mathcal{C}, \otimes, I, a, l, r)$

$\mathcal{C}$ : category

$\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  : tensor product

$I \in \mathcal{C}$  : unit

$a_{UVW}: (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$  : associativity constraint

$l_V: I \otimes V \rightarrow V$  : left unit constraint

$r_V: V \otimes I \rightarrow V$  : right unit constraint

forms a **monoidal category** if and only if it satisfies

$$\begin{array}{ccccc}
 & & ((U \otimes V) \otimes W) \otimes X & & \\
 & \swarrow a \otimes id & & \searrow a & \\
 (U \otimes (V \otimes W)) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\
 & \downarrow a & \textcircled{2} & & \downarrow a \\
 U \otimes ((V \otimes W) \otimes X) & \xrightarrow{id \otimes a} & U \otimes (V \otimes (W \otimes X))
 \end{array}$$

and

$$\begin{array}{ccc}
 & V \otimes V & \\
 \nearrow r_{VV} & \textcircled{2} & \searrow id_V \\
 (V \otimes I) \otimes W & \xrightarrow{a} & V \otimes (I \otimes W)
 \end{array}$$

Let  $\mathbb{k}$  be a unitary commutative ring  
and  $A$  be a  $\mathbb{k}$ -algebra.

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Prop (left  $A$ -mod): monoidal cat  $\iff A$  : quasi-bialgebra

Def (Drinfel'd '89)

$(A, \Delta, \varepsilon, \Phi, \ell, r)$

$A$ :  $\mathbb{k}$ -alg

$\Delta: A \rightarrow A \otimes_{\mathbb{k}} A$  : coproduct  $\leftarrow \mathbb{k}\text{-alg from}$

$\varepsilon: A \rightarrow \mathbb{k}$  : counit  $\leftarrow \mathbb{k}\text{-alg from}$

$\Phi \in (A \otimes_{\mathbb{k}} A \otimes_{\mathbb{k}} A)^{\times}$  : Drinfel'd associator

$\ell \in A^{\times}$  : left unit

$r \in A^{\times}$  : right unit

forms a **quasi-bialgebra** if and only if it satisfies

$$(\text{id} \otimes \Delta)(\Delta(a)) = \Phi((\Delta \otimes \text{id})(\Delta(a))) \Phi^{-1} : \text{quasi-coassociativity}$$

$$(\varepsilon \otimes \text{id})(\Delta(a)) = l^{-1}a\ell, \quad (\text{id} \otimes \varepsilon)(\Delta(a)) = r^{-1}ar : \text{quasi-counit}$$

$$(\text{id} \otimes \text{id} \otimes \Delta)(\Phi) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\Phi) = \Phi_{234} (\text{id} \otimes \Delta \otimes \text{id})(\Phi) \cdot \Phi_{123}$$

$$(\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) = r \otimes l^{-1}.$$

Particularly it forms **quasi-Hopf algebra** if and only if it equips with a bijective 'antipode'.

## §2. Hexagons

Def (Joyal-Street)  $(\mathcal{C}, \otimes, I, a, l, r, c)$ :

$(\mathcal{C}, \otimes, I, a, l, r)$ : monoidal category

$C_{VW}: V \otimes W \rightarrow W \otimes V$ : commutativity constraint  
(braiding, R-matrix)

forms braided monoidal category if and only if it satisfies  
(quasi-tensored category)

$$\begin{array}{ccccc}
 & \text{c} & & & \\
 & \swarrow & \searrow & & \\
 J \otimes (V \otimes W) & \xrightarrow{\quad c \quad} & (V \otimes W) \otimes T & & \\
 \downarrow a & & \downarrow a & & \\
 (J \otimes V) \otimes W & & & & V \otimes (W \otimes T) \\
 \downarrow \text{id} & & \circlearrowleft & & \downarrow \text{id} \otimes c \\
 (V \otimes T) \otimes W & \xrightarrow{\quad a \quad} & V \otimes (T \otimes W) & & 
 \end{array}$$

and

$$\begin{array}{ccccc}
 & \text{c} & & & \\
 & \swarrow & \searrow & & \\
 (J \otimes V) \otimes W & \xrightarrow{\quad c \quad} & W \otimes (J \otimes V) & & \\
 \downarrow a^{-1} & & \downarrow a^{-1} & & \\
 J \otimes (V \otimes W) & & & & (W \otimes T) \otimes V \\
 \downarrow \text{id} \otimes c & & \circlearrowleft & & \downarrow \text{id} \\
 J \otimes (W \otimes V) & \xrightarrow{\quad a^{-1} \quad} & (J \otimes W) \otimes V & & 
 \end{array}$$

Particularly it forms symmetric monoidal cat (tensor category)  
if and only if  $C_{VV} \circ C_{VW} = \text{id}$ .

L4

Prop (left A-mod) : braided monoidal category

$\iff A : q + q \text{ bialg}$

$$\alpha(c \otimes w) = \beta(c \otimes \rho(w))$$

$$\Rightarrow \beta = \alpha(\text{id} \otimes \rho)$$

$$\begin{aligned} (\nu \otimes w) \\ = \varepsilon(r \otimes w) \end{aligned}$$

$$\begin{aligned} \ell(\nu \otimes w) \\ = \ell \nu \end{aligned}$$

$$\ell = \ell_1(\text{id}_1)$$

$$\delta(\nu \otimes w) = \nu w$$

$$r = r_2(\text{id}_2)$$

Def (Drinfel'd '89)

$$R = \varepsilon R(\alpha, \beta)$$

$(A, \Delta, \varepsilon, \Phi, \ell, r, R)$ :

$(A, \Delta, \varepsilon, \Phi, \ell, r)$ : quasi-bialgebra

$R \in (A \otimes A)^*$  : R-matrix

forms a quasi-triangular quasi-bialgebra

if and only if it satisfies

$$\Delta^{op}(a) = R \Delta(a) R^{-1} \quad \text{quasi-co-commutativity}$$

$$(\text{id} \otimes \Delta)(R) = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1}$$

$$(\Delta \otimes \text{id})(R) = \Phi_{312}^{-1} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123}$$

Particularly it forms a quasi-triangular quasi-Hopf algebra

if and only if it equips with a bijective 'antipode'.

Def  $k$ : a field with  $\text{char} = 0$

$(A, \Delta, \varepsilon, \Phi, R)$

$A: k[[\hbar]]\text{-alg}$

$\Delta: A \rightarrow A \hat{\otimes}_{k[[\hbar]]} A$ : coproduct

← cont

$\varepsilon: A \rightarrow k[[\hbar]]$ : counit

← cont

$\Phi \in (A \hat{\otimes}_{k[[\hbar]]} A \hat{\otimes}_{k[[\hbar]]} A)^*$ : Drinfeld associator

$R \in (A \hat{\otimes}_{k[[\hbar]]} A)^*$ : R-matrix

forms a quasi-triangular quasi-Hopf Quantized Universal enveloping algebra if and only if

$(A, \Delta, \varepsilon, \Phi, R)$ : qtab alg

$(A, \Delta, \varepsilon, \Phi, R)_{R=0}$ : UEA

$A$ : top free

Rmk The existence of the 'antipode' automatically follows.

Thus it is really a quasi-Hopf algebra

## § 3. GT

L6

Let  $(A, \Delta, \varepsilon, \Phi, R)$ : qtqH QUE-alg

Put  $(\mathcal{C}, \otimes, I, \alpha, c, l, r) \in {}_{\text{left}}(A, \Delta, \varepsilon, \Phi, R)\text{-Mod}$   
 braided mono-cat of

Let  $\lambda \in k$ ,  $f \in F_2(k)$

Replace  $\alpha$  and  $c$  by

$$\alpha : (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3) \xrightarrow{\text{mms}} \alpha' = af$$

$$c : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1 \xrightarrow{\text{mms}} c' = c^T$$

One get a new data  $\mathcal{C}' = (\mathcal{C}, \otimes, I, \alpha', c', l, r)$

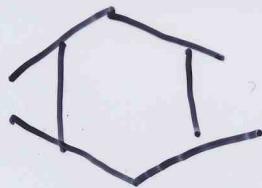
N.B.  $\mathcal{C} = (A, \Delta, \varepsilon, \Phi, R)\text{-Mod}$

$$\xrightarrow{(f, f)} \mathcal{C}' = (A, \Delta, \varepsilon, \Phi', R')\text{-Mod}$$

$$\Phi' = f(\Phi R^{21} R^{12} \Phi^{-1}, R^{32} R^{23}) \Phi$$

$$R' = (R^{12} \cdot R^{21})^m R \quad (m = \frac{d}{2})$$

Prop  $\mathcal{C}'$  forms a braided monoidal category.



$$\text{Pentagon} \quad f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \\ = f(x_{12}, x_{23})f(x_{13}x_{23}, x_{34}) \quad \text{in } K_4(k)$$

$$\text{Hexagon I} \quad f(x_1, x_2)x_1^m f(x_3, x_1)x_3^m f(x_3, x_2)^{-1}x_2^m = 1$$

$$\text{Hexagon II} \quad f(x_2, x_1)^{-1}x_1^m f(x_3, x_1)x_3^m f(x_3, x_2)^{-1}x_2^m = 1$$

$$\text{with } x_1x_2x_3 = 1, m = \frac{2-1}{2}$$

Def (Drinfeld '91)

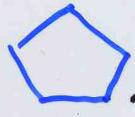
$$GIT(k) = \left\{ (\lambda, f) \in k^\times E(k) \mid \begin{array}{l} (\lambda, f) \text{ satisfies} \\ \text{Pentagon, Hexagon I, Hexagon II} \end{array} \right\} X$$

$$\text{multiplication } (\lambda, f) := (\lambda_1, f_1) \circ (\lambda_2, f_2)$$

$$\left\{ \begin{array}{l} \lambda = \lambda_1 \lambda_2 \\ f = f_1(f_2(x, y)X^{\lambda_2}f_2(x, y)^{-1}, Y^{\lambda_2})f_2(x, y) \end{array} \right.$$

# §4. Main Theorem

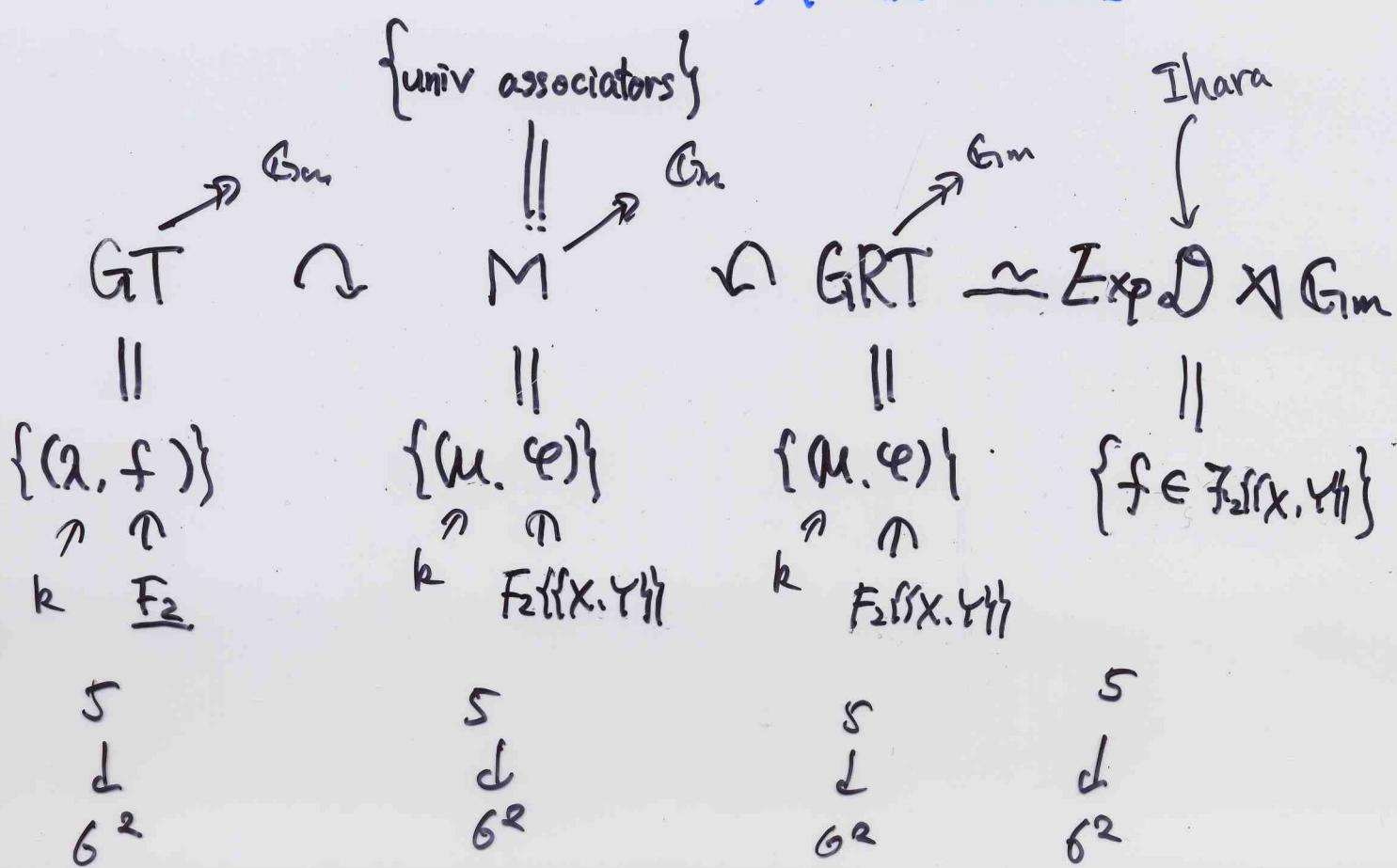
L8

I<sub>h</sub> [F] Suppose that  $f$  satisfies .

Then  $\exists^2 \pm \lambda \in \bar{k}$  s.t.  $(\lambda, f)$  satisfies  .

IDEA: Use Drinfeld's gadgets:

$(A, \Delta, \varepsilon, \Phi, R) / \text{gauge transform} \xrightarrow[\text{quantization by } \varphi: \text{univ associators}]^{h \rightarrow 0 \text{ classical limit}} (g, t)$



lem Let  $f$  be a Lie series in  $k\langle\langle x, y \rangle\rangle$

with  $\deg f > 2$ . Suppose that  $f$  satisfies

$$f(x_{12}, x_{23}) + f(x_{34}, x_{45}) + f(x_{51}, x_{12})$$

$$+ f(x_{23}, x_{34}) + f(x_{45}, x_{51}) = 0 \quad \text{in } R_5(k)$$

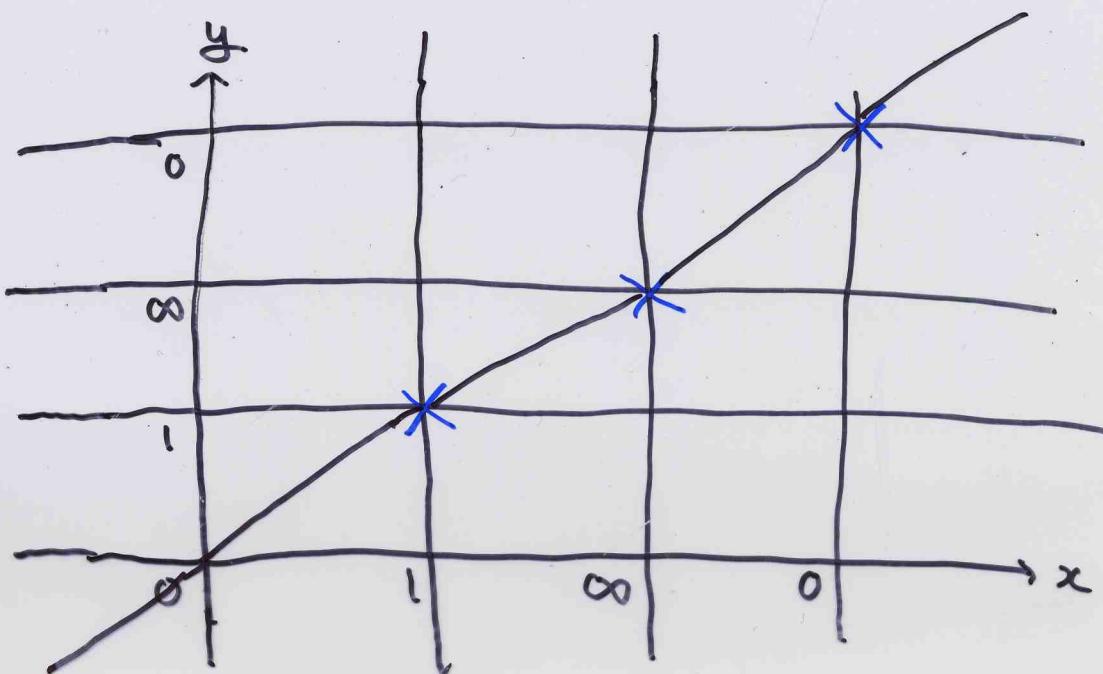
Then it also satisfies

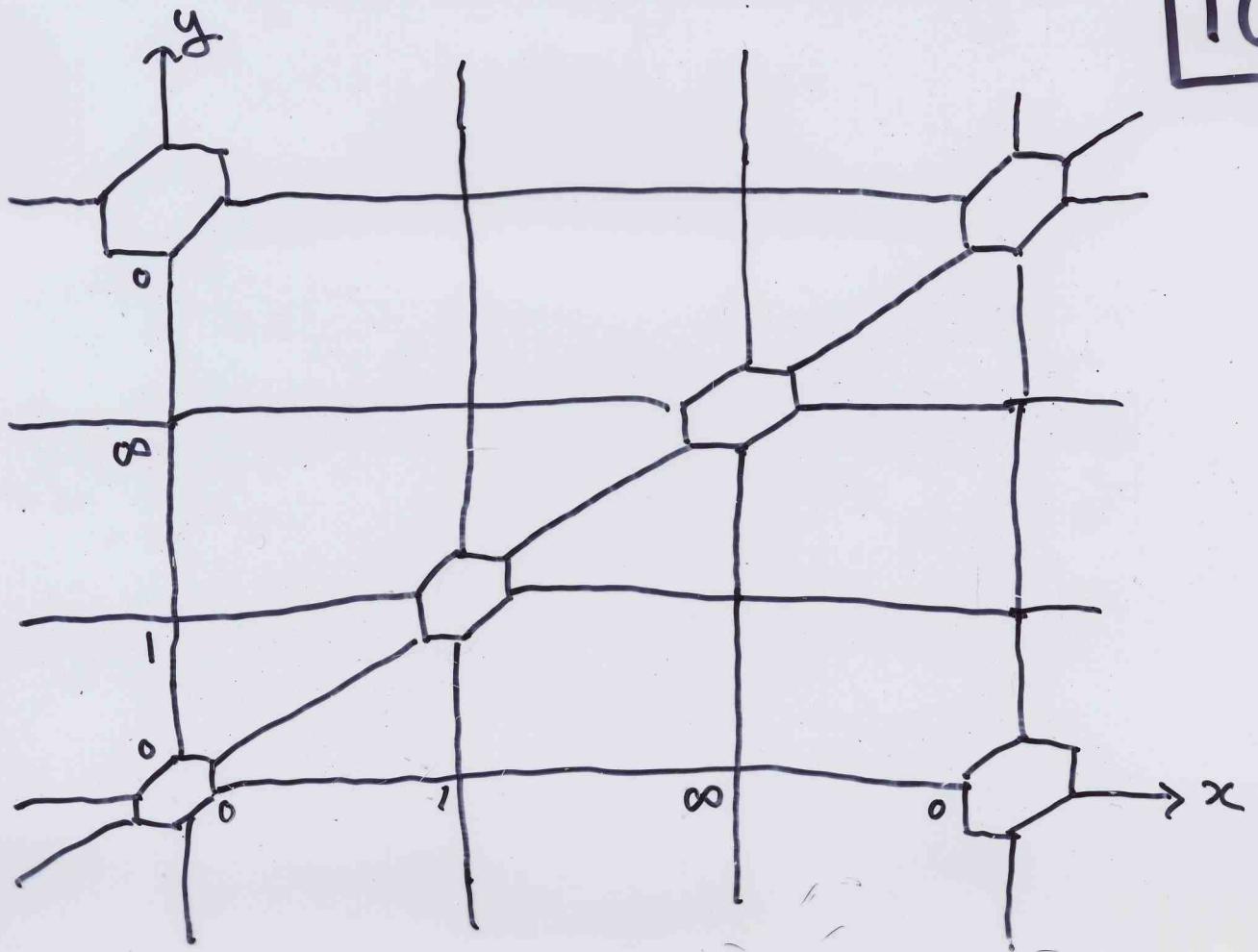
$$f(x, Y) + f(Y, x) = 0$$

$$f(x, Y) + f(Y, z) + f(z, x) = 0 \quad \text{with } X+Y+z=0$$

idea

$$\begin{aligned} M_{05} &= \left\{ (x_1 : x_2 : x_3 : x_4 : x_5) \mid \begin{array}{l} x_i \in \mathbb{P}^1 \\ x_i \neq x_j \ (i \neq j) \end{array} \right\} \\ &= \left\{ (0, 1, \infty, x, y) \mid \dots \right\} \end{aligned} \quad \text{RPCL}$$





L10

+ More Arguments //

Cor Let  $\varphi$  be a group-like series in  $k\langle\langle x, y \rangle\rangle$ .

Suppose that  $\varphi$  satisfies

$$\varphi(t_{23}, t_{34}) \varphi(t_{12} + t_{13}, t_{24} + t_{34}) \varphi(t_{12}, t_{23})$$

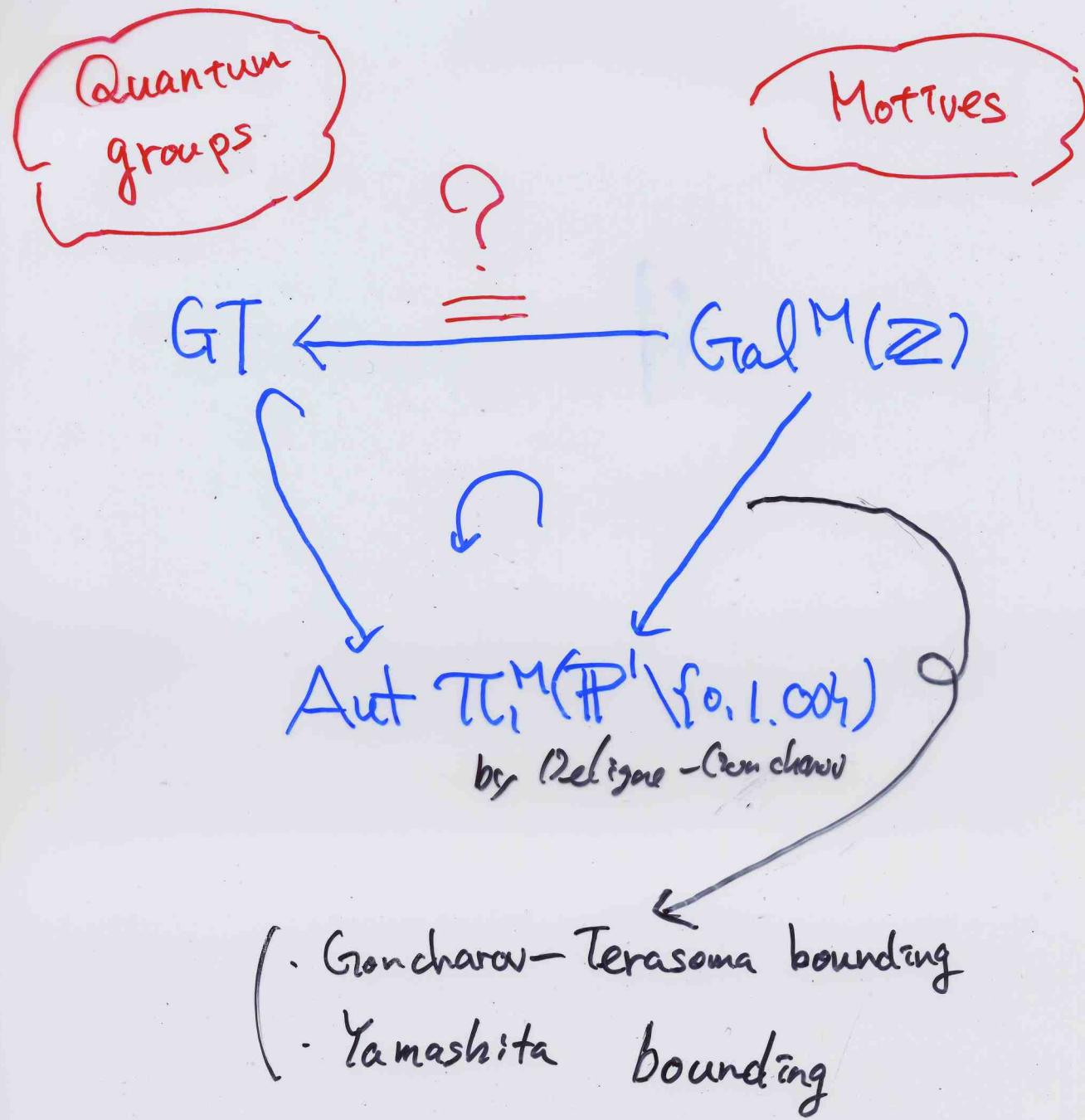
$$= \varphi(t_{12}, t_{23} + t_{24}) \varphi(t_{13} + t_{23}, t_{34}) \quad \text{in } \widehat{\text{UOL}}_4(k)$$

Then  $\exists z \neq \mu \in k$  s.t.  $(\mu, \varphi)$  satisfies

$$\exp \frac{\mu}{2}(t_{13} + t_{23}) = \varphi(t_{13}, t_{12}) \exp\left(\frac{\mu}{2}t_{13}\right) \varphi(t_{13}, t_{23})^{-1} \exp\left(\frac{\mu}{2}t_{23}\right) \varphi(t_{12}, t_{23})$$

$$\exp \frac{\mu}{2}(t_{12} + t_{13}) = \varphi(t_{23}, t_{13})^{-1} \exp\left(\frac{\mu}{2}t_{13}\right) \varphi(t_{12}, t_{13}) \exp\left(\frac{\mu}{2}t_{12}\right) \varphi(t_{12}, t_{23})^{-1}$$

E.g.  $\varphi_{k2} = 1 + \sum (-1)^m \varphi(k_1, \dots, k_m) A^{k_1} B \dots A^{k_{m-1}} B + \dots$



Problem All unramified mixed Tate motives  
are pentagonal?

Thank you for  
your attention.

Let's go to  
the PARTY !!!

