THE MOTIVIC GALOIS GROUP, THE GROTHENDIECK-TEICHMÜLLER GROUP AND THE DOUBLE SHUFFLE GROUP

HIDEKAZU FURUSHO

1. The motivic Galois group

We recall the motivic Galois group of the category of mixed Tate motives over $\mathbf{Z}$ [DG] in this section. This is related with the Drinfel'd's Grothendieck-Teichmüller group ([Dr91]) in §2 and the Racinet's double shuffle group ([R]) in §3.

Let k be a field with characteristic 0. Levine [L2] and Voevodsky [V] constructed a triangulated category of mixed motives over k. Levine [L2] showed an equivalence of these two categories. This category denoted by $DM(k)_{\mathbf{Q}}$ has Tate objects $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$). Let $DMT(k)_{\mathbf{Q}}$ be the triangulated sub-category of $DM(k)_{\mathbf{Q}}$ generated by $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$). Levine [L1] extracted a neutral tannakian $\mathbf{Q}$-category $MT(k)_{\mathbf{Q}}$ of mixed Tate motives over k from $DMT(k)_{\mathbf{Q}}$ by taking a heart with respect to a t-structure under the Beilinson-Soulé vanishing conjecture which says $gr_i^{\gamma} K_n(k) = 0$ for $n > 2i$. Here LHS is the graded quotient of the algebraic K-theory for k with respect to γ-filtration.

Assume that k is a number field. In this case the Beilinson-Soulé vanishing conjecture holds and we have $MT(k)_{\mathbf{Q}}$. This category satisfies the following expected properties: Each object M has an increasing filtration of subobjects called weight filtration, $W : \cdots \subseteq W_{m-1}M \subseteq W_m M \subseteq W_{m+1}M \subseteq \cdots$, whose intersection is 0 and union is M. The quotient $Gr^W_{2m+1}M := W_{2m+1}M/W_{2m}M$ is trivial and $Gr^W_{2m}M := W_{2m}M/W_{2m+1}M$ is a direct sum of finite copies of $\mathbf{Q}(m)$ for each $m \in \mathbf{Z}$. Morphisms of $MT(k)_{\mathbf{Q}}$ are strictly compatible with weight filtration. The extension group is related to K-theory as follows

$$Ext^i_{MT(k)_{\mathbf{Q}}}(\mathbf{Q}(0), \mathbf{Q}(m)) = \begin{cases} K_{2m-i}(k)_{\mathbf{Q}} & \text{for } i = 1, \\ 0 & \text{for } i > 1. \end{cases}$$

There are realization fiber functors ([L2] and [H]) corresponding to usual cohomology theories.

Let S be a finite set of finite places of k. Let $\mathcal{O}_S$ be the ring of S-integers in k. Deligne and Goncharov [DG] defined the full subcategory $MT(\mathcal{O}_S)$ of mixed Tate motives over $\mathcal{O}_S$, whose objects are mixed Tate motives M in $MT(k)_{\mathbf{Q}}$ such that for each subquotient E of M which is an extension of $\mathbf{Q}(n)$ by $\mathbf{Q}(n+1)$ for $n \in \mathbf{Z}$, the extension class of E in $Ext^1_{MT(k)_{\mathbf{Q}}}(\mathbf{Q}(n), \mathbf{Q}(n+1)) = Ext^1_{MT(k)_{\mathbf{Q}}}(\mathbf{Q}(0), \mathbf{Q}(1)) = k^{\times}_{\mathbf{Q}}$

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lies in $\mathcal{O}_S^\times \otimes \mathbf{Q}$. In this category the following hold:

$$Ext^1_{MT(\mathcal{O}_S)}(\mathbf{Q}(0), \mathbf{Q}(m)) = \begin{cases} 0 & \text{for } m < 1, \\ \mathcal{O}_S^\times \otimes \mathbf{Q} & \text{for } m = 1, \\ K_{2m-1}(k)_{\mathbf{Q}} & \text{for } m > 1, \end{cases}$$
$$Ext^2_{MT(\mathcal{O}_S)}(\mathbf{Q}(0), \mathbf{Q}(m)) = 0.$$

Let $\omega_{\mathrm{can}} : MT(\mathcal{O}_S) \to Vect_{\mathbf{Q}}$ ($Vect_{\mathbf{Q}}$: the category of $\mathbf{Q}$-vector spaces) be the fiber functor which sends each motive M to $\oplus_n Hom(\mathbf{Q}(n), Gr^W_{-2n} M)$. Define the *motivic Galois group* to be the pro-algebraic group $\mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S) := \underline{Aut}^{\otimes}(MT(\mathcal{O}_S) : \omega_{\mathrm{can}})$. The action of $\mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S)$ on $\omega_{\mathrm{can}}(\mathbf{Q}(1)) = \mathbf{Q}$ defines a surjection $\mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S) \to \mathbf{G}_m$ and its kernel $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S)$ is the unipotent radical of $\mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S)$. There is a canonical splitting $\tau : \mathbf{G}_m \to \mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S)$ which gives a negative grading on the Lie algebra $Lie\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathcal{O}_S)$ (consult [D] §8 for the full story). The above computations of Ext-groups follows

Proposition 1 ([D] §8, [DG] §2). *The graded Lie algebra $Lie\mathcal{U}Gal^{\mathcal{M}}(\mathcal{O}_S)$ is free and its degree n-part of $\left(Lie\mathcal{U}Gal^{\mathcal{M}}(\mathcal{O}_S)\right)^{\mathrm{ab}} = \mathcal{U}Gal^{\mathcal{M}}(\mathcal{O}_S)^{\mathrm{ab}}$ is isomorphic to the dual of $Ext^1_{MT(\mathcal{O}_S)}(\mathbf{Q}(0), \mathbf{Q}(-n))$.*

Let us restrict in the case of $k = \mathbf{Q}$, $S = \emptyset$, $\mathcal{O}_S = \mathbf{Z}$. By Proposition 1 the Lie algebra $Lie\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ of the unipotent part $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ of $\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})$ should be a graded free Lie algebra generated by one element in each degree $-m$ ($m \geqslant 3$: odd).

In [DG] §4 they constructed the *motivic fundamental group* $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ with $X = \mathbf{P}^1 \backslash \{0, 1, \infty\}$, which is an ind-object of $MT(\mathbf{Z})$. This is an affine group $MT(\mathbf{Z})$-scheme (cf. [DG]). Since all the structure morphism of $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ belong to the set of morphisms of $MT(\mathbf{Z})$ and $\omega_{\mathrm{can}}(\pi_1^{\mathcal{M}}(X : \overrightarrow{01})) = \underline{F_2}$ where $\underline{F_2}$ is the free pro-unipotent algebraic group of rank 2, we have

$$\varphi : \mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \to \underline{Aut F_2}.$$

On this map φ the following is one of the basic problems.

Problem 2. Is φ injective?

This might be said a problem which asks a validity of a unipotent variant of the so-called 'Belyĭ's theorem' in [Be] in the pro-finite setting. Equivalently this asks if the motivic fundamental group $\pi_1^{\mathcal{M}}(X : \overrightarrow{01})$ is a 'generator' of the tannakian category $MT(\mathbf{Z})$. It is related with various conjectures in several realizations (cf. [F07a] note 3.10); Zagier conjecture (partially proved by Terasoma [T] and Deligne-Gonchaov [DG]), Deligne-Ihara conjecture (partially proved by Hain-Matsumoto [HM]) and Furusho-Yamashita conjecture (partially proved by Yamashita [Y]).

2. The Grothendieck-Teichmüller group

In his celebrated papers on quantum groups [Dr86, Dr90, Dr91] Drinfel'd came to the notion of quasitriangular quasi-Hopf quantized universal enveloping algebras. It is a topological algebra which differs from a topological Hopf algebra in the sense that the coassociativity axiom and the cocommutativity axiom is twisted by an associator and an R-matrix satisfying a pentagon axiom and two hexagon axioms. One of the main theorems in [Dr91] is that any quasitriangular quasi-Hopf quantized universal enveloping algebra modulo twists (in other words gauge transformations [Ka]) is obtained as a quantization of a pair (called its classical limit) of a Lie algebra and its symmetric invariant 2-tensor. Quantizations are constructed by 'universal' associators. The set of group-like universal associators forms a pro-algebraic variety, denoted M. The associator set $\underline{M}$ ([Dr91]) is the pro-algebraic variety whose set of k-valued points consists of pairs below (μ,φ) satisfying the *GT-relations*, the Drinfel'd's one *pentagon equation* (1) and his two *hexagon equations* (2) and (3), and M is its open subvariety defined by $\mu\neq 0$. The non-emptiness of $M(k)$ is another of his main theorem (reproved in [Ba]).

The category of representations of a quasitriangular quasi-Hopf quantized universal enveloping algebra forms a quasitensored category [Dr91], in other words, a braided tensor category [JS]; its associativity constraint and its commutativity constraint are subject to one pentagon axiom and two hexagon axioms. The (unipotent part of the graded) *Grothendieck-Teichmüller* (pro-algebraic) *group* GRT_1 is introduced in [Dr91] as a group of deformations of the category which change its associativity constraint keeping all three axioms. It is also closely related to Grothendieck's philosophy of Teichmüller-Lego posed in [Gr]. Its set of k-valued points is defined to be the subset of $\underline{M}$ with $\mu=0$.

Let us fix notation and conventions: Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure and $U\mathfrak{F}_2=k\langle\langle X_0,X_1\rangle\rangle$ a non-commutative formal power series ring with two variables X_0 and X_1. Its element $\varphi=\varphi(X_0,X_1)$ is called *group-like* if it satisfies $\Delta(\varphi)=\varphi\otimes\varphi$ with $\Delta(X_0)=X_0\otimes 1+1\otimes X_0$ and $\Delta(X_1)=X_1\otimes 1+1\otimes X_1$ and its constant term is equal to 1. For a monic monomial W, $c_W(\varphi)$ means the coefficient of W in φ. For any k-algebra homomorphism $\iota : U\mathfrak{F}_2\to S$ the image $\iota(\varphi)\in S$ is denoted by $\varphi(\iota(X_0),\iota(X_1))$. Let $\mathfrak{a}_4$ be the completion (with respect to the natural grading) of the Lie algebra over k with generators t_{ij} $(1\leqslant i,j\leqslant 4)$ and defining relations $t_{ii}=0$, $t_{ij}=t_{ji}$, $[t_{ij},t_{ik}+t_{jk}]=0$ (i,j,k: all distinct) and $[t_{ij},t_{kl}]=0$ (i,j,k,l: all distinct).

Our theorem is on the defining equations of the associator set M (and hence of the Grothendieck-Teichmüller group GRT_1.)

Theorem 3 ([F07b]). *Let $\varphi=\varphi(X_0,X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies Drinfel'd's pentagon equation:*

$$(1)\quad \varphi(t_{12},t_{23}+t_{24})\varphi(t_{13}+t_{23},t_{34})=\varphi(t_{23},t_{34})\varphi(t_{12}+t_{13},t_{24}+t_{34})\varphi(t_{12},t_{23}).$$

Then there exists an element (unique up to signature) $\mu\in\bar{k}$ such that the pair (μ,φ) satisfies his two hexagon equations:

$$(2)\quad \exp\{\frac{\mu(t_{13}+t_{23})}{2}\}=\varphi(t_{13},t_{12})\exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{13},t_{23})^{-1}\exp\{\frac{\mu t_{23}}{2}\}\varphi(t_{12},t_{23}),$$

(3)
$$\exp\{\frac{\mu(t_{12}+t_{13})}{2}\}=\varphi(t_{23},t_{13})^{-1}\exp\{\frac{\mu t_{13}}{2}\}\varphi(t_{12},t_{13})\exp\{\frac{\mu t_{12}}{2}\}\varphi(t_{12},t_{23})^{-1}.$$

Actually this μ is equal to $\pm(24c_{X_0X_1}(\varphi))^{\frac{1}{2}}$.

It should be noted that we need to use an (actually quadratic) extension of a field k in order to reduce the GT-relations into one pentagon equation. Particularly the theorem claims that the pentagon equation is essentially a single defining equation of the Grothendieck-Teichmüller group.

The proof of theorem 3 is reduced to the following by standard arguments of Lie algebra.

Proposition 4 ([F07b]). *Let $\mathfrak{F}_2$ be the set of Lie-like elements φ in $U\mathfrak{F}_2$ (i.e. $\Delta(\varphi)=\varphi\otimes 1+1\otimes\varphi$). Let φ be an element of $\mathfrak{F}_2$ which is commutator Lie-like* [1] *with $c_{X_0X_1}(\varphi)=0$. Suppose that φ satisfies 5-cycle relation:*

$$\varphi(X_{12},X_{23})+\varphi(X_{34},X_{45})+\varphi(X_{51},X_{12})+\varphi(X_{23},X_{34})+\varphi(X_{45},X_{51})=0$$

in $\hat{\mathfrak{P}}_5$. Then it also satisfies 3- and 2-cycle relation:

$$\varphi(X,Y)+\varphi(Y,Z)+\varphi(Z,X)=0 \text{ with } X+Y+Z=0,$$
$$\varphi(X,Y)+\varphi(Y,X)=0.$$

Here $\mathfrak{P}_5$ stands for the completion (with respect to the natural grading) of the pure sphere braid Lie algebra with 5 strings; the Lie algebra generated by X_{ij} $(1\leqslant i,j\leqslant 5)$ with clear relations $X_{ii}=0$, $X_{ij}=X_{ji}$, $\sum_{j=1}^5 X_{ij}=0$ $(1\leqslant i,j\leqslant 5)$ and $[X_{ij},X_{kl}]=0$ if $\{i,j\}\cap\{k,l\}=\emptyset$.

Proof . There is a projection from $\mathfrak{P}_5$ to the completed free Lie algebra $\mathfrak{F}_2$ generated by X and Y by putting $X_{i5}=0$, $X_{12}=X$ and $X_{23}=Y$. The image of 5-cycle relation gives 2-cycle relation.

For our convenience we denote $\varphi(X_{ij},X_{jk})$ $(1\leqslant i,j,k\leqslant 5)$ by φ_{ijk}. Then the 5-cycle relation can be read as

$$\varphi_{123}+\varphi_{345}+\varphi_{512}+\varphi_{234}+\varphi_{451}=0.$$

We denote LHS by P. Put σ_i $(1\leqslant i\leqslant 12)$ be elements of $\mathfrak{S}_5$ defined as follows: $\sigma_1(12345)=(12345)$, $\sigma_2(12345)=(54231)$, $\sigma_3(12345)=(13425)$, $\sigma_4(12345)=(43125)$, $\sigma_5(12345)=(53421)$, $\sigma_6(12345)=(23514)$, $\sigma_7(12345)=(23415)$, $\sigma_8(12345)=(35214)$, $\sigma_9(12345)=(53124)$, $\sigma_{10}(12345)=(24135)$, $\sigma_{11}(12345)=(52314)$ and $\sigma_{12}(12345)=(23541)$. Then

$$\begin{aligned}\sum_{i=1}^{12}\sigma_i(P)=\ &\varphi_{123}+\varphi_{345}+\varphi_{512}+\varphi_{234}+\varphi_{451}\\
&+\varphi_{542}+\varphi_{231}+\varphi_{154}+\varphi_{423}+\varphi_{315}\\
&+\varphi_{134}+\varphi_{425}+\varphi_{513}+\varphi_{342}+\varphi_{251}\\
&+\varphi_{431}+\varphi_{125}+\varphi_{543}+\varphi_{312}+\varphi_{254}\\
&+\varphi_{534}+\varphi_{421}+\varphi_{153}+\varphi_{342}+\varphi_{215}\\
&+\varphi_{235}+\varphi_{514}+\varphi_{423}+\varphi_{351}+\varphi_{142}\\
&+\varphi_{234}+\varphi_{415}+\varphi_{523}+\varphi_{341}+\varphi_{152}\\
&+\varphi_{352}+\varphi_{214}+\varphi_{435}+\varphi_{521}+\varphi_{143}\\
&+\varphi_{531}+\varphi_{124}+\varphi_{453}+\varphi_{312}+\varphi_{245}\\
&+\varphi_{241}+\varphi_{135}+\varphi_{524}+\varphi_{413}+\varphi_{352}\\
&+\varphi_{523}+\varphi_{314}+\varphi_{452}+\varphi_{231}+\varphi_{145}\\
&+\varphi_{235}+\varphi_{541}+\varphi_{123}+\varphi_{354}+\varphi_{412}.\end{aligned}$$

By the 2-cycle relation, $\varphi_{ijk}=-\varphi_{kji}$ $(1\leqslant i,j,k\leqslant 5)$. This gives

[1] We call a series $\varphi=\varphi(X_0,X_1)$ commutator Lie-like if it is Lie-like and $c_{X_0}=c_{X_1}=0$, in other words $\varphi\in\mathfrak{F}_2':=[\mathfrak{F}_2,\mathfrak{F}_2]$.

$$\begin{aligned}\sum_{i=1}^{12} \sigma_i(P) =&\varphi_{123} + \varphi_{234}\\ &+ \varphi_{231} + \varphi_{423}\\ &+ \varphi_{342} + \varphi_{312} + \varphi_{342}\\ &+ \varphi_{235} + \varphi_{423}\\ &+ \varphi_{234} + \varphi_{523}\\ &+ \varphi_{352} + \varphi_{312} + \varphi_{352}\\ &+ \varphi_{523} + \varphi_{231}\\ &+ \varphi_{235} + \varphi_{123}\\ =&2(\varphi_{123} + \varphi_{231} + \varphi_{312}) + 2(\varphi_{234} + \varphi_{342} + \varphi_{423}) + 2(\varphi_{235} + \varphi_{352} + \varphi_{523})\\ =&2\Big\{\varphi(X_{12}, X_{23}) + \varphi(X_{23}, X_{31}) + \varphi(X_{31}, X_{12})\Big\}\\ &+ 2\Big\{\varphi(X_{23}, X_{34}) + \varphi(X_{34}, X_{42}) + \varphi(X_{42}, X_{23})\Big\}\\ &+ 2\Big\{\varphi(X_{23}, X_{35}) + \varphi(X_{35}, X_{52}) + \varphi(X_{52}, X_{23})\Big\}.\end{aligned}$$

By $[X_{12}, X_{12}+X_{31}+X_{32}] = [X_{23}, X_{12}+X_{31}+X_{32}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi(X_{12}, X_{23}) = \varphi(-X_{31}-X_{32}, X_{23}) = \varphi(X_{34}+X_{35}, X_{23})$. By $[X_{31}, X_{12}+X_{31}+X_{32}] = [X_{12}, X_{12}+X_{31}+X_{32}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi(X_{31}, X_{12}) = \varphi(X_{31}, -X_{31} - X_{32}) = \varphi(-X_{23} - X_{34} - X_{35}, X_{34} + X_{35})$. By $[X_{34}, X_{42} + X_{23} + X_{34}] = [X_{42}, X_{42} + X_{23} + X_{34}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi(X_{34}, X_{42}) = \varphi(X_{34}, -X_{23} - X_{34})$. By $[X_{23}, X_{42} + X_{23} + X_{34}] = [X_{42}, X_{42} + X_{23} + X_{34}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi(X_{42}, X_{23}) = \varphi(-X_{23} - X_{34}, X_{23})$. By $[X_{35}, X_{52} + X_{23} + X_{35}] = [X_{52}, X_{52} + X_{23} + X_{35}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi(X_{35}, X_{52}) = \varphi(X_{35}, -X_{23} - X_{35})$. By $[X_{23}, X_{52} + X_{23} + X_{35}] = [X_{52}, X_{52} + X_{23} + X_{35}] = 0$ and $\varphi \in \mathfrak{F}_2'$, $\varphi(X_{52}, X_{23}) = \varphi(-X_{23} - X_{35}, X_{23})$.
So it follows

$$\begin{aligned}\sum_{i=1}^{12} \sigma_i(P) =&2\Big\{\varphi(X_{34} + X_{35}, X_{23}) + \varphi(X_{23}, -X_{23} - X_{34} - X_{35})\\ &+ \varphi(-X_{23} - X_{34} - X_{35}, X_{34} + X_{35})\Big\}\\ &+ 2\Big\{\varphi(X_{23}, X_{34}) + \varphi(X_{34}, -X_{23} - X_{34}) + \varphi(-X_{23} - X_{34}, X_{23})\Big\}\\ &+ 2\Big\{\varphi(X_{23}, X_{35}) + \varphi(X_{35}, -X_{23} - X_{35}) + \varphi(-X_{23} - X_{35}, X_{23})\Big\}.\end{aligned}$$

The elements X_{23}, X_{34} and X_{35} generates completed Lie subalgebra $\mathfrak{F}_3$ of $\mathfrak{P}_5$ which is free of rank 3 and it contains $\sum_{i=1}^{12} \sigma_i(P)$. Let $q : \mathfrak{F}_3 \to \mathfrak{F}_2$ be the projection sending $X_{23} \mapsto X$, $X_{34} \mapsto Y$ and $X_{35} \mapsto Y$. Then

$$\begin{aligned}q(\sum_{i=1}^{12} \sigma_i(P)) =&2\Big\{\varphi(2Y, X) + \varphi(X, -X - 2Y) + \varphi(-X - 2Y, 2Y)\Big\}\\ &+ 4\Big\{\varphi(X, Y) + \varphi(Y, -X - Y) + \varphi(-X - Y, X)\Big\}.\end{aligned}$$

By the 2-cycle relation,

$$q(\sum_{i=1}^{12}\sigma_i(P)) = 4\Big\{\varphi(X,Y)+\varphi(Y,-X-Y)+\varphi(-X-Y,X)\Big\}$$
$$-2\Big\{\varphi(X,2Y)+\varphi(2Y,-X-2Y)+\varphi(-X-2Y,X)\Big\}.$$

Put $R(X,Y)=\varphi(X,Y)+\varphi(Y,-X-Y)+\varphi(-X-Y,X)$. Then $q(\sum_{i=1}^{12}\sigma_i(P)) = 4R(X,Y)-2R(X,2Y)$. Since $P=0$, it follows $2R(X,Y)=R(X,2Y)$. Expanding this equation in terms of the Hall basis, we see that $R(X,Y)$ must be of the form $\sum_{m=1}^{\infty}a_m(adX)^{m-1}(Y)$ with $a_m\in k$. By the 2-cycle relation, $R(X,Y)=-R(Y,X)$. So $a_1=a_3=a_4=a_5=\cdots=0$. By our assumption $c_{X_0X_1}(\varphi)=0$, a_2 must be 0 either. Therefore $R(X,Y)=0$, which is the 3-cycle relation. This yields the validity of theorem 3. □

We note that the multiplication [2] of GRT_1 is given by

$$\varphi_2\circ\varphi_1 := \varphi_1(\varphi_2X_0\varphi_2^{-1},X_1)\cdot\varphi_2=\varphi_2\cdot\varphi_1(X_0,\varphi_2^{-1}X_1\varphi_2) \tag{4}$$

for φ_1, $\varphi_2\in GRT_1(k)$. By the map sending $X_0\mapsto X_0$ and $X_1\mapsto\varphi X_1\varphi^{-1}$, the group GRT_1 is regarded as a subgroup of $\underline{AutF_2}$. It is known that it contains the motivic Galois image (see for example [A, F07a]), i.e.

Proposition 5. $\varphi(\mathcal{U}Gal^{\mathcal{M}}(\mathbf{Z}))\subset GRT_1$.

In [Ko] Kontsevich raised a mysterious speculation which connects motivic Galois groups and deformation quantizations. His speculation was based on several conjectures and one of which was the following.

Conjecture 6. The map φ might induce the isomorphism $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z})\simeq GRT_1$.

This conjecture is clearly explained in [A] from the viewpoint of motives.

3. The double shuffle group

This section shows that the pentagon equation (1) implies the generalized double shuffle relation (6). As a corollary, we obtain an embedding from the Grothendieck-Teichmüller group GRT_1 to Racinet's double shuffle group DMR_0 ([R]). This realizes the project of Deligne-Terasoma [DT] where a different approach was indicated. Their arguments concerned multiplicative convolutions whereas our methods are based on a bar construction calculus. We also prove that the gamma factorization formula follows from the generalized double shuffle relation. It extends the result in [DT, I] where they show that the GT-relations imply the gamma factorization.

Multiple zeta values $\zeta(k_1,\cdots,k_m)$ are the real numbers defined by the following series

$$\zeta(k_1,\cdots,k_m):=\sum_{0<n_1<\cdots<n_m}\frac{1}{n_1^{k_1}\cdots n_m^{k_m}}$$

for m, $k_1,\dots$, $k_m\in\mathbf{N}(=\mathbf{Z}_{>0})$. This converges if and only if $k_m>1$. They were studied (allegedly) firstly by Euler [E] for $m=1,2$. Several types of relations among multiple zeta values have been discussed. We focus on two types of relations, GT-relations and generalized double shuffle relations. Both of them are described in terms of the Drinfel'd associator [Dr91]

$$\Phi_{KZ}(X_0,X_1)=1+\sum(-1)^m\zeta(k_1,\cdots,k_m)X_0^{k_m-1}X_1\cdots X_0^{k_1-1}X_1+(\text{regularized terms})$$

[2] For our convenience, we change the order of multiplication in the original definition of [Dr91].

which is a non-commutative formal power series in two variables X_0 and X_1. Its coefficients including regularized terms are explicitly calculated to be linear combinations of multiple zeta values in [F03] proposition 3.2.3 by Le-Murakami's method [LM]. The Drinfel'd associator was introduced as the connection matrix of the Knizhnik-Zamolodchikov equation and it was shown in [Dr91] that it is group-like and satisfies the GT-relations with $\mu = \pm 2\pi\sqrt{-1}$, i.e. $(\Phi_{KZ}, \pm 2\pi\sqrt{-1}) \in M(\mathbf{C})$, by using symmetry of the KZ-system on configuration spaces.

The *generalized double shuffle relation* is a kind of combinatorial relation. It arises from two ways of expressing multiple zeta values as iterated integrals and as power series. There are several formulations of the relations (see [IKZ, R]). In particular, they were formulated as (6) (see below) for $\varphi = \Phi_{KZ}$ in [R].

Let us fix notation and conventions: Let $\pi_Y : k\langle\langle X_0, X_1\rangle\rangle \to k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ be the k-linear map between non-commutative formal power series rings that sends all the words ending in X_0 to zero and the word $X_0^{n_m-1}X_1 \cdots X_0^{n_1-1}X_1$ ($n_1, \dots, n_m \in \mathbf{N}$) to $(-1)^m Y_{n_m} \cdots Y_{n_1}$. Define the coproduct Δ_* on $k\langle\langle Y_1, Y_2, \dots\rangle\rangle$ by $\Delta_* Y_n = \sum_{i=0}^{n} Y_i \otimes Y_{n-i}$ with $Y_0 := 1$. For $\varphi = \sum_{W:\mathrm{word}} c_W(\varphi) W \in k\langle\langle X_0, X_1\rangle\rangle$, define the series shuffle regularization $\varphi_* = \varphi_{\mathrm{corr}} \cdot \pi_Y(\varphi)$ with the correction term

$$\varphi_{\mathrm{corr}} = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{X_0^{n-1}X_1}(\varphi) Y_1^n\right). \tag{5}$$

For a group-like series $\varphi \in U\mathfrak{F}_2$ the *generalised double shuffle relation* means the equality

$$\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*. \tag{6}$$

Theorem 7 ([F08]). *Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U\mathfrak{F}_2$. Suppose that φ satisfies Drinfel'd's pentagon equation* (1). *Then it also satisfies the generalized double shuffle relation* (6).

By [F07b] lemma 5, theorem 7 is reduced to the following.

Proposition 8 ([F08]). *Let φ be a group-like element of $U\mathfrak{F}_2$ with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0$. Suppose that φ satisfies the 5-cycle relation*

$$\varphi(X_{34}, X_{45})\varphi(X_{51}, X_{12})\varphi(X_{23}, X_{34})\varphi(X_{45}, X_{51})\varphi(X_{12}, X_{23}) = 1$$

in the completed universal enveloping algebra $U\mathfrak{P}_5$ of $\mathfrak{P}_5$. Then it also satisfies the generalized double shuffle relation, i.e. $\Delta_*(\varphi_*) = \varphi_* \widehat{\otimes} \varphi_*$.

Proof . Let $\mathcal{M}_{0,4}$ be the moduli space $\{(x_1, \cdots, x_4) \in (\mathbf{P}^1_k)^4 | x_i \neq x_j (i \neq j)\}/PGL_2(k)$ of 4 different points in $\mathbf{P}^1$. It is identified with $\{z \in \mathbf{P}^1 | z \neq 0, 1, \infty\}$ by sending $[(0, z, 1, \infty)]$ to z. Let $\mathcal{M}_{0,5}$ be the moduli space $\{(x_1, \cdots, x_5) \in (\mathbf{P}^1_k)^5 | x_i \neq x_j (i \neq j)\}/PGL_2(k)$ of 5 different points in $\mathbf{P}^1$. It is identified with $\{(x, y) \in \mathbf{G}_m^2 | x \neq 1, y \neq 1, xy \neq 1\}$ by sending $[(0, xy, y, 1, \infty)]$ to (x, y).

For $\mathcal{M} = \mathcal{M}_{0,4}/k$ or $\mathcal{M}_{0,5}/k$, we consider the Brown's variant $V(\mathcal{M})$ [Br] of the Chen's reduced bar construction [C]. This is a graded Hopf algebra $V(\mathcal{M}) = \oplus_{m=0}^{\infty} V_m$ ($\subset TV_1 = \oplus_{m=0}^{\infty} V_1^{\otimes m}$) over k. Here $V_0 = k$, $V_1 = H^1_{DR}(\mathcal{M})$ and V_m is the totality of linear combinations (finite sums) $\sum_{I=(i_m, \cdots, i_1)} c_I [\omega_{i_m} | \cdots | \omega_{i_1}] \in V_1^{\otimes m}$ ($c_I \in k$, $\omega_{i_j} \in V_1$, $[\omega_{i_m} | \cdots | \omega_{i_1}] := \omega_{i_m} \otimes \cdots \otimes \omega_{i_1}$) satisfying the integrability condition

$$\sum_{I=(i_m, \cdots, i_1)} c_I [\omega_{i_m} | \omega_{i_{m-1}} | \cdots | \omega_{i_{j+1}} \wedge \omega_{i_j} | \cdots | \omega_{i_1}] = 0$$

in $V_1^{\otimes m-j-1} \otimes H^2_{DR}(\mathcal{M}) \otimes V_1^{\otimes j-1}$ for all j $(1 \leqslant j < m)$.

For the moment assume that k is a subfield of $\mathbf{C}$. We have an embedding (called a realisation in [Br]§1.2, §3.6) $\rho : V(\mathcal{M}) \hookrightarrow I_o(\mathcal{M})$ as algebra over k which sends $\sum_{I=(i_m,\cdots,i_1)} c_I[\omega_{i_m}|\cdots|\omega_{i_1}]$ $(c_I \in k)$ to $\sum_I c_I \mathrm{It}\int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$. Here $\sum_I c_I \mathrm{It}\int_o \omega_{i_m} \circ \cdots \circ \omega_{i_1}$ means the iterated integral defined by

$$\sum_I c_I \int_{0<t_1<\cdots<t_{m-1}<t_m<1} \omega_{i_m}(\gamma(t_m)) \cdot \omega_{i_{m-1}}(\gamma(t_{m-1})) \cdots \omega_{i_1}(\gamma(t_1))$$

for all analytic paths $\gamma : (0,1) \to \mathcal{M}(\mathbf{C})$ starting from the tangential basepoint o (defined by $\frac{d}{dz}$ for $\mathcal{M} = \mathcal{M}_{0,4}$ and defined by $\frac{d}{dx}$ and $\frac{d}{dy}$ for $\mathcal{M} = \mathcal{M}_{0,5}$) at the origin in $\mathcal{M}$ (for its treatment see also [D]§15) and $I_o(\mathcal{M})$ denotes the $\mathcal{O}^{\mathrm{an}}_{\mathcal{M}}$-module generated by all such homotopy invariant iterated integrals with $m \geqslant 1$ and holomorphic 1-forms $\omega_{i_1}, \ldots, \omega_{i_m} \in \Omega^1(\mathcal{M})$.

For $\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{Z}^k_{>0}$, its weight and its depth are defined to be $wt(\mathbf{a}) = a_1 + \cdots + a_k$ and $dp(\mathbf{a}) = k$ respectively. Put $z \in \mathbf{C}$ with $|z| < 1$. Consider the following complex function which is called the *one variable multiple polylogarithm*

$$Li_{\mathbf{a}}(z) := \sum_{0<m_1<\cdots<m_k} \frac{z^{m_k}}{m_1^{a_1} \cdots m_k^{a_k}}.$$

It satisfies the recursive differential equations (cf. [BF, F08]) It gives an iterated integral starting from o, which lies on $I_o(\mathcal{M}_{0,4})$. Actually it corresponds to an element of $V(\mathcal{M}_{0,4})$ denoted by $l_{\mathbf{a}}$.

Similarly for $\mathbf{a} = (a_1, \cdots, a_k) \in \mathbf{Z}^k_{>0}$, $\mathbf{b} = (b_1, \cdots, b_l) \in \mathbf{Z}^l_{>0}$ and $x, y \in \mathbf{C}$ with $|x| < 1$ and $|y| < 1$, consider the following complex function which is called the *two variables multiple polylogarithm*

$$Li_{\mathbf{a},\mathbf{b}}(x,y) := \sum_{\substack{0<m_1<\cdots<m_k \\ <n_1<\cdots<n_l}} \frac{x^{m_k} \quad y^{n_l}}{m_1^{a_1} \cdots m_k^{a_k} n_1^{b_1} \cdots n_l^{b_l}}.$$

It also satisfies the recursive differential equations (cf. [BF]§5). They show that the functions $Li_{\mathbf{a},\mathbf{b}}(x,y)$, $Li_{\mathbf{a},\mathbf{b}}(y,x)$, $Li_{\mathbf{a}}(x)$, $Li_{\mathbf{a}}(y)$ and $Li_{\mathbf{a}}(xy)$ give iterated integrals starting from o, which lie on $I_o(\mathcal{M}_{0,5})$. They correspond to elements of $V(\mathcal{M}_{0,5})$ by the map ρ denoted by $l^{x,y}_{\mathbf{a},\mathbf{b}}$, $l^{y,x}_{\mathbf{a},\mathbf{b}}$, $l^x_{\mathbf{a}}$, $l^y_{\mathbf{a}}$ and $l^{xy}_{\mathbf{a}}$ respectively.

The idea of the proof of proposition 8 goes as follows: Recall that multiple polylogarithms satisfy the analytic identity, the series shuffle formula in $I_o(\mathcal{M}_{0,5})$

$$Li_{\mathbf{a}}(x) \cdot Li_{\mathbf{b}}(y) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} Li_{\sigma(\mathbf{a},\mathbf{b})}(\sigma(x,y)).$$

Here $Sh^{\leqslant}(k,l) := \cup_{N=1}^{\infty}\{\sigma : \{1, \cdots, k+l\} \to \{1, \cdots, N\} | \sigma \text{ is onto}, \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)\}$, $\sigma(\mathbf{a},\mathbf{b}) := ((c_1, \cdots, c_j), (c_{j+1}, \cdots, c_N))$ with $\{j, N\} = \{\sigma(k), \sigma(k+l)\}$,

$$c_i = \begin{cases} a_s + b_{t-k} & \text{if } \sigma^{-1}(i) = \{s,t\} \text{ with } s < t, \\ a_s & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s \leqslant k, \\ b_{s-k} & \text{if } \sigma^{-1}(i) = \{s\} \quad \text{with } s > k, \end{cases}$$

$$\text{and } \sigma(x,y) = \begin{cases} xy & \text{if } \sigma^{-1}(N) = k, k+l, \\ (x,y) & \text{if } \sigma^{-1}(N) = k+l, \\ (y,x) & \text{if } \sigma^{-1}(N) = k. \end{cases}$$

Since ρ is an embedding of algebras, the above analytic identity implies the algebraic identity, the series shuffle formula in $V(\mathcal{M}_{0,5})$

$$(7) \qquad l^x_{\mathbf{a}} \cdot l^y_{\mathbf{b}} = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l^{\sigma(x,y)}_{\sigma(\mathbf{a},\mathbf{b})}.$$

Suppose that φ is an element as in proposition 8. Evaluation of the equation (7) at the group-like element $\varphi_{451}\varphi_{123}$ [3] gives the series shuffle formula

$$l_{\mathbf{a}}(\varphi) \cdot l_{\mathbf{b}}(\varphi) = \sum_{\sigma \in Sh^{\leqslant}(k,l)} l_{\sigma(\mathbf{a},\mathbf{b})}(\varphi)$$

for admissible [4] indices $\mathbf{a}$ and $\mathbf{b}$ because of [F08] lemma 4.1. and 4.2.

For non-admissible indices we need a special treatment. The idea is essentially same to the above admissible indices case except that we consider $e^{TX_{51}}\varphi_{451}\varphi_{123}$ (T: a parameter which stands for $\log x$) instead of $\varphi_{451}\varphi_{123}$ (see [F08] in more detail), which completes the proof of theorem 7. $\square$

The *double shuffle group* DMR_0 is a pro-unipotent group introduced by Racinet [R]. Its set of k-valued points consists of group-like series φ which satisfy (6) [5] and $c_{X_0}(\varphi) = c_{X_1}(\varphi) = c_{X_0X_1}(\varphi) = 0$. Its multiplication is given by the equation (4). By the same way to the GRT_1-case, the group DMR_0 is regarded as a subgroup of $\underline{AutF_2}$. This also contains the motivic Galois image.

Proposition 9. *$\varphi(\mathcal{U}Gal^{\mathcal{M}}(\mathbf{Z})) \subset DMR_0$.*

This follows from the result in [Go] and another proof is given in [F07a]. The following is a direct corollary of our theorem 7 since the equations (2) and (3) for (μ, φ) imply $c_{X_0X_1}(\varphi) = \frac{\mu^2}{24}$.

Theorem 10 ([F08])**.** *$GRT_1 \subset DMR_0$.*

As an analogue of conjecture 6, the following conjecture is posed (cf. [R] and see also [A].)

Conjecture 11. The map φ might induce the isomorphism $\mathcal{U}\mathrm{Gal}^{\mathcal{M}}(\mathbf{Z}) \simeq DMR_0$.

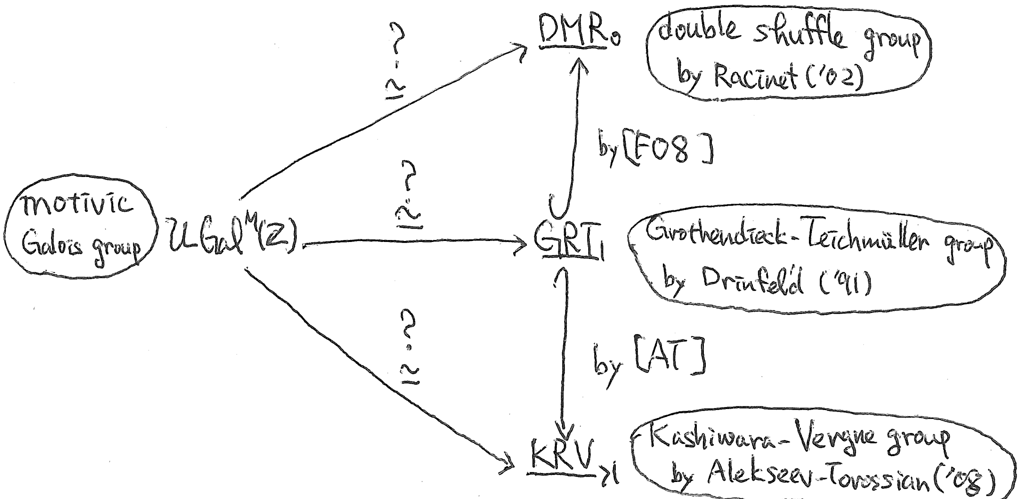
The validities of conjecture 6 and conjecture 11 would imply that GRT_1 might be isomorphic to DMR_0.

Remark 12. Alekseev and Torossian [AT] gave the second proof of the Kashiwara-Vergne (KV) conjecture. It is a conjecture on a property of the Campbell-Baker-Hausdorff formula which was posed in [KV]. Their proof was based on Drinfel'd's theory [Dr91] of the Grothendieck-Teichmüller group. They showed that the set of solutions of the generalized KV-problem admitted a free and transitive action of the (graded) *Kashiwara-Vergne group KRV* (see also [AET] for the definition). It is a subgroup of $\underline{AutF_2}$ and contains GRT_1, i,e, we have an embedding $GRT_1 \hookrightarrow KRV$. They conjectured in [AT]§4 that its degree>1-part $KRV_{>1}$ might be equal to GRT_1.

[3] For simplicity we mean φ_{ijk} for $\varphi(X_{ij}, X_{jk}) \in U\mathfrak{P}_5$.

[4] An index $\mathbf{a} = (a_1, \cdots, a_k)$ is called *admissible* if $a_k > 1$.

[5] For our convenience, we change some signatures in the original definition ([R] definition 3.2.1.))



One of the main defining equations of KRV is the coboundary Jacobian condition (cf. loc.cit.), which is a lift of the gamma factorization formula (8) (see below) to the trace space $\hat{\mathfrak{T}}_2$. The following theorem might be a step to relate KRV with DMR_0.

Theorem 13 ([F08]). *Let φ be a non-commutative formal power series in two variables which is group-like with $c_{X_0}(\varphi) = c_{X_1}(\varphi) = 0$. Suppose that it satisfies the generalized double shuffle relation (6). Then its meta-abelian quotient [6] $B_\varphi(x_0, x_1)$ is gamma-factorisable, i.e. there exists a unique series $\Gamma_\varphi(s)$ in $1 + s^2k[[s]]$ such that*

$$B_\varphi(x_0,x_1) = \frac{\Gamma_\varphi(x_0)\Gamma_\varphi(x_1)}{\Gamma_\varphi(x_0+x_1)}. \tag{8}$$

The gamma element Γ_φ gives the correction term φ_{corr} of the series shuffle regularization (5) by $\varphi_{corr} = \Gamma_\varphi(-Y_1)^{-1}$.

This theorem was proved in [F08] §5. It extends the result in [DT, I] which shows that for any group-like series satisfying (1), (2) and (3) its meta-abelian quotient is gamma factorisable. We note that it was calculated in [Dr91] that especially $\Gamma_\varphi(s) = \exp\{\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} s^n\} = e^{-\gamma s}\Gamma(1-s)$ for $\varphi = \Phi_{KZ}$ where γ is the Euler constant, $\Gamma(s)$ is the classical gamma function and Φ_{KZ} is the Drinfel'd associator.

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[6] It means $(1+\varphi_{X_1}X_1)^{\mathrm{ab}}$ for the unique expression $\varphi = 1 + \varphi_{X_0}X_0 + \varphi_{X_1}X_1$ ($\varphi_{X_0}, \varphi_{X_1} \in k\langle\langle X_0, X_1\rangle\rangle$) and $(\cdot)^{\mathrm{ab}}$ means the image of the abelianization map $k\langle\langle X_0, X_1\rangle\rangle \to k[[x_0, x_1]]$.

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Graduate School of Mathematics, Nagoya University, Chikusa-ku, Furo-cho, Nagoya, 464-8602, Japan

E-mail address: `furusho@math.nagoya-u.ac.jp`

Département de mathématiques et applications, École Normale Supérieure,45, rue d'Ulm, F 75230 Paris cedex 05, France

E-mail address: `Hidekazu.Furusho@ens.fr`