Linear Algebra II - Homework 4 - Answer

Exercise 1-1. Projection onto the $\vec{v}_2, \vec{v}_3$-plane.

Exercise 1-2. Projection onto the $\vec{v}_3$-axis.

Exercise 1-3. Axisymmetric of $\vec{v}_1$-axis.

Exercise 1-4. Axisymmetric of $\vec{v}_1, \vec{v}_2$-plane.

Exercise 2. By letting

$$|\lambda I - M| = \begin{vmatrix} \lambda - 2 & 2 & -3 & -1 \\ -1 & \lambda + 1 & & -2 \\ & & \lambda - 3 & 4 \\ & & -2 & \lambda + 3 \end{vmatrix} = 0,$$

we see that the eigenvalues of M are

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = -1.$$

In terms of λ_1, since

$$\lambda_1 I - M = \begin{pmatrix} -2 & 2 & -3 & -1 \\ -1 & 1 & & -2 \\ & & -3 & 4 \\ & & -2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & \frac{3}{2} & \frac{1}{2} \\ & & \frac{3}{2} & -\frac{3}{2} \\ & & -3 & 4 \\ & & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{x_2 \leftrightarrow x_3} \begin{pmatrix} 1 & \frac{3}{2} & -1 & \frac{1}{2} \\ & \frac{3}{2} & & -\frac{3}{2} \\ & -3 & & 4 \\ & -2 & & 3 \end{pmatrix} \to \begin{pmatrix} 1 & & -1 & 2 \\ & 1 & & -1 \\ & & & 1 \\ & & & 1 \end{pmatrix}$$

$$\xrightarrow{x_3 \leftrightarrow x_4} \begin{pmatrix} 1 & & 2 & -1 \\ & 1 & -1 & \\ & & 1 & \\ & & 1 & \end{pmatrix} \to \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & \end{pmatrix},$$

we see that the eigenvectors of λ_1 are

$$\{k \begin{pmatrix} 1 \\ 1 \\ \\ \end{pmatrix} \mid k \in \mathbb{R}, k \neq 0\}.$$

Indeed, the solution corresponding to the last matrix is

$$k \begin{pmatrix} 1 \\ \\ \\ 1 \end{pmatrix}, \quad k \in \mathbb{R},$$

but when we deal with the original matrix $\lambda_1 I - M$ above, we have changed the order of x_2, x_3 and x_3, x_4, so we need to change them back, i.e

$$\begin{pmatrix} 1 \\ \\ \\ 1 \end{pmatrix} \xrightarrow{x_4 \leftrightarrow x_3} \begin{pmatrix} 1 \\ \\ 1 \\ \\ \end{pmatrix} \xrightarrow{x_3 \leftrightarrow x_2} \begin{pmatrix} 1 \\ 1 \\ \\ \\ \end{pmatrix}.$$

Similarly, the eigenvectors of λ_2 are

$$\{k \begin{pmatrix} 2 \\ 1 \\ \\ \\ \end{pmatrix} \mid k \in \mathbb{R}, k \neq 0\},$$

and the eigenvectors of λ_3 are

$$\{k\begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} \mid k \in \mathbb{R}, k \neq 0\}.$$

Exercise 3-1. It's easy to see that

$$m_{t+1} = (1-a)m_t + bn_t \quad \text{and} \quad n_{t+1} = am_t + (1-b)n_t.$$

Exercise 3-2. It's easy to see that

$$m_{t+1} + n_{t+1} = (1-a)m_t + bn_t + am_t + (1-b)n_t = m_t + n_t.$$

Exercise 3-3. It's easy to see that

$$M = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}.$$

Exercise 3-4. It's easy to see that the eigenvalues are

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 1-a-b.$$

The eigenvectors of λ_1 are

$$\{k\begin{pmatrix} b \\ a \end{pmatrix} \mid k \in \mathbb{R}, k \neq 0\},$$

and the eigenvectors of λ_2 are

$$\{k\begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid k \in \mathbb{R}, k \neq 0\}.$$

Exercise 3-5. Yes, it is. It's easy to see that the diagonalization of M is

$$\begin{pmatrix} 1 & \\ & 1-a-b \end{pmatrix}.$$

Exercise 3-6. It's easy to see that

$$\begin{aligned} M^t &= \begin{pmatrix} b & 1 \\ a & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & (1-a-b)^t \end{pmatrix} \begin{pmatrix} \frac{1}{a+b} & \frac{1}{a+b} \\ \frac{a}{a+b} & -\frac{b}{a+b} \end{pmatrix} \\ &= \begin{pmatrix} b & 1 \\ a & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \end{pmatrix} \begin{pmatrix} \frac{1}{a+b} & \frac{1}{a+b} \\ \frac{a}{a+b} & -\frac{b}{a+b} \end{pmatrix} \\ &= \begin{pmatrix} \frac{b}{a+b} & \frac{b}{a+b} \\ \frac{a}{a+b} & \frac{a}{a+b} \end{pmatrix}, t \to \infty. \end{aligned}$$

Therefore,

$$\begin{pmatrix} m_t \\ n_t \end{pmatrix} = M^t \begin{pmatrix} m_0 \\ n_0 \end{pmatrix} = \begin{pmatrix} \frac{b}{a+b} & \frac{b}{a+b} \\ \frac{a}{a+b} & \frac{a}{a+b} \end{pmatrix} \begin{pmatrix} m_0 \\ n_0 \end{pmatrix} = \begin{pmatrix} \frac{b(m_0+n_0)}{a+b} \\ \frac{a(m_0+n_0)}{a+b} \end{pmatrix}, t \to \infty.$$