Linear Algebra II - Homework 3 - Answer

Exercise 1. It's easy to see that

$$\begin{vmatrix} 8 & 4 \\ 1 & 3 \end{vmatrix} = 8 \cdot 3 - 4 \cdot 1 = 20$$

and

$$\begin{vmatrix} 7 & 1 & 2 & \\ 1 & & 3 & 4 \\ & & 2 & 3 \\ 4 & 1 & 8 & \end{vmatrix} = (-1)^{3+3} 2 \begin{vmatrix} 7 & 1 & \\ 1 & & 4 \\ 4 & 1 & \end{vmatrix} + (-1)^{3+4} 3 \begin{vmatrix} 7 & 1 & 2 \\ 1 & & 3 \\ 4 & 1 & 8 \end{vmatrix} = 21.$$

Exercise 2-1. It's easy to see that

$$\begin{vmatrix} a & 3 & d \\ b & 3 & e \\ c & 3 & f \end{vmatrix} = 3 \begin{vmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{vmatrix} = 3 \cdot 7 = 21.$$

Exercise 2-2. It's easy to see that

$$\begin{vmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{vmatrix} = \begin{vmatrix} a & 2 & d \\ b & 4 & e \\ c & 6 & f \end{vmatrix} + \begin{vmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{vmatrix} = 2 \begin{vmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{vmatrix} + 7 = 2 \cdot 11 + 7 = 29.$$

Exercise 3. By letting i-th row minus $(i-1)$-th row from n-th row, we have

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & \cdots & 1 \\ & & 1 & \cdots & 1 \\ & & & \ddots & \vdots \\ & & & & 1 \end{vmatrix} = 1.$$

Exercise 4-1. It's easy to see that

$$\begin{aligned}|M_t| &= (t + m_{11})\tilde{M}_{11} + m_{12}\tilde{M}_{12} + \cdots + m_{1n}\tilde{M}_{1n} \\ &= t\tilde{M}_{11} + m_{11}\tilde{M}_{11} + \cdots + m_{1n}\tilde{M}_{1n} = t\tilde{M}_{11} + |M|.\end{aligned}$$

Exercise 4-2. Assume that the first two situations don't happan. Then, the second situation not happening tells us that there exists t_0 such that $|M_{t_0}| \neq 0$, which implies that $\tilde{M}_{11} \neq 0$ or $|M| \neq 0$. Note that if $\tilde{M}_{11} = 0$ and $|M| \neq 0$, the first situation will always happen, which contradicts to our assumption. Thus, we have $\tilde{M}_{11} \neq 0$. Then, if we regard $|M_t|$ as a function of t, it will be a linear function, which intersect with $y = 0$ for precisely one point.