Linear Algebra II - Homework 2 - Answer

Exercise 1-1. It's easy to see that

$$|\vec{u}| = \sqrt{(\vec{u}, \vec{u})} = \sqrt{1^2 + 1^2} = \sqrt{2}, \quad |\vec{v}| = \sqrt{170}$$

and

$$(\vec{u}, \vec{v}) = 1 \cdot 7 + 1 \cdot 11 = 18.$$

Therefore, we have

$$\langle \vec{u}, \vec{v} \rangle = \arccos \frac{(\vec{u}, \vec{v})}{|\vec{u}||\vec{v}|} = \arccos \frac{18}{\sqrt{2}\sqrt{170}} = \arccos \frac{9\sqrt{85}}{\sqrt{85}}.$$

Exercise 1-2. It's easy to see that

$$|\vec{u}| = \sqrt{10}, \quad |\vec{v}| = 3\sqrt{6}, \quad \langle \vec{u}, \vec{v} \rangle = \arccos\,(-\frac{\sqrt{15}}{30}).$$

Exercise 2-1. It's easy to see that

$$|\vec{u}_1| = |\vec{u}_2| = |\vec{u}_3| = 1, \quad (\vec{u}_1, \vec{u}_2) = (\vec{u}_1, \vec{u}_3) = (\vec{u}_2, \vec{u}_3) = 0.$$

Exercise 2-2. Let $\vec{u}_4 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$. Since $(\vec{u}_i, \vec{u}_4) = 0,\ i = 1, 2, 3$, we have

$$\begin{cases} \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 0 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \\ \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \end{cases}.$$

Then, from

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & & & -1 \\ & 1 & & 1 \\ & & 1 & 1 \end{pmatrix},$$

we see that

$$\vec{u}_4 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad k \in \mathbb{R}.$$

Since $|\vec{u}_4| = 1$, we have $4k^2 = 1$, which implies that $k = \pm\frac{1}{2}$. Therefore,

$$\vec{u}_4 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{or} \quad \vec{u}_4 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Exercise 3. Let

$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix}, \quad \vec{\alpha}_2 = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix}, \quad \vec{\alpha}_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix}.$$

Then, we have

$$\vec{\beta}_1 = \vec{\alpha}_1 = \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix},$$

$$\vec{\beta}_2 = \vec{\alpha}_2 - \frac{(\vec{\alpha}_2, \vec{\beta}_1)}{(\vec{\beta}_1, \vec{\beta}_1)}\vec{\beta}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\vec{\beta}_3 = \vec{\alpha}_3 - \frac{(\vec{\alpha}_3, \vec{\beta}_1)}{(\vec{\beta}_1, \vec{\beta}_1)}\vec{\beta}_1 - \frac{(\vec{\alpha}_3, \vec{\beta}_2)}{(\vec{\beta}_2, \vec{\beta}_2)}\vec{\beta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Exercise 4. The the coefficient matrix of $x_1 + x_2 + x_3 = 0$ is

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

which implies that any vector on the plane can be denoted by

$$\vec{\gamma} = k_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad k_1, k_2 \in \mathbb{R}.$$

It follows from the equation above that we can obtain two linearly independent vectors

$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{\alpha}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

on the plane just by letting $k_2 = 0$ and $k_1 = 0$, respectively. By the Gram-Schmidt process, we have two orthogonal vectors

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{\beta}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

on the plane. Normalizing $\vec{\beta}_1, \vec{\beta}_2$, we can obtain an orthonormal basis $\{\vec{e}_1, \vec{e}_2\}$ of the plane, where

$$\vec{e}_1 = \frac{\vec{\beta}_1}{|\vec{\beta}_1|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

Exercise 3-26. It is easy to see that $c = d = 0$, since

$$a \cdot 0 + c \cdot 1 + e \cdot 0 = 0 \quad \text{and} \quad b \cdot 0 + d \cdot 1 + f \cdot 0 = 0.$$

Moreover, we have

$$\begin{cases} a^2 + b^2 = 1 \\ e^2 + f^2 = 1 \\ ae + bf = 0 \end{cases}.$$

Let $\begin{cases} a = \cos\theta \\ b = \sin\theta \end{cases}$ and $\begin{cases} e = \cos\varphi \\ f = \sin\varphi \end{cases}$. It follows from the third equation of the system of equations above that

$$\cos\theta\cos\varphi + \sin\theta\sin\varphi = 0 \quad \Leftrightarrow \quad \cos(\theta - \varphi) = 0.$$

Hence, we have

$$\theta = \varphi + \frac{2k+1}{2}\pi, \quad k \in \mathbb{Z}.$$

Therefore, the matrix becomes

$$\begin{pmatrix} \cos\theta & \sin\theta & \\ & & 1 \\ \cos\varphi & \sin\varphi & \end{pmatrix} = \begin{pmatrix} \cos(\varphi + \frac{2k+1}{2}\pi) & \sin(\varphi + \frac{2k+1}{2}\pi) & \\ & & 1 \\ \cos\varphi & \sin\varphi & \end{pmatrix}, \quad k \in \mathbb{Z}.$$

When k is even, the matrix has the form

$$\begin{pmatrix} \cos\left(\varphi + \frac{2k+1}{2}\pi\right) & \sin\left(\varphi + \frac{2k+1}{2}\pi\right) & \\ & & 1 \\ \cos\varphi & \sin\varphi & \end{pmatrix} = \begin{pmatrix} -\sin\varphi & \cos\varphi & \\ & & 1 \\ \cos\varphi & \sin\varphi & \end{pmatrix}.$$

When k is odd, the matrix has the form

$$\begin{pmatrix} \cos\left(\varphi + \frac{2k+1}{2}\pi\right) & \sin\left(\varphi + \frac{2k+1}{2}\pi\right) & \\ & & 1 \\ \cos\varphi & \sin\varphi & \end{pmatrix} = \begin{pmatrix} \sin\varphi & -\cos\varphi & \\ & & 1 \\ \cos\varphi & \sin\varphi & \end{pmatrix}.$$