Linear Algebra II - Homework 1 - Answer

Exercise 1-6. No, it's not. Let $\mathrm{GL}_3(\mathbb{R})$ be the set. It's easy to see that $I_3, -I_3 \in \mathrm{GL}_3(\mathbb{R})$, but $I_3 + (-I_3) = 0 \notin \mathrm{GL}_3(\mathbb{R})$. Therefore, it's not a subspace of $M_3(\mathbb{R})$.

Exercise 1-8. Yes, it is. Let U be the set. Since $I_3 \in U$, $U \neq \emptyset$. For any $U_1, U_2 \in U$, let

$$U_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix}, \quad U_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ & & b_{33} \end{pmatrix}.$$

Obviously, for $k_1, k_2 \in \mathbb{R}$,

$$k_1U_1 + k_2U_2 = \begin{pmatrix} k_1a_{11} + k_2b_{11} & k_1a_{12} + k_2b_{12} & k_1a_{13} + k_2b_{13} \\ & k_1a_{22} + k_2b_{22} & k_1a_{23} + k_2b_{23} \\ & & k_1a_{33} + k_2b_{33} \end{pmatrix} \in U.$$

Therefore, U is a subspace of $M_3(\mathbb{R})$, and $\{E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\}$ can be a basis of U, where $E_{ij} \in M_3(\mathbb{R})$ such that every element of E_{ij} is zero except that $a_{ij} = 1$.

Exercise 2. Yes, it is. Let A be the set consisting of all arithmetic real sequences. Let $(v_n)_{n\in\mathbb{N}}$ be the real sequence such that $v_n = 0$ for all $n \in \mathbb{N}$. Obviously, $(v_n)_{n\in\mathbb{N}} \in A$, in which case $a = 0, b = 0$. Hence, $A \neq \emptyset$. For any $(u_n)_{n\in\mathbb{N}}, (u'_n)_{n\in\mathbb{N}} \in A$, let

$$u_n = a + nb, \quad u'_n = a' + nb', \quad a, b, a', b' \in \mathbb{R}.$$

Obviously, for $k_1, k_2 \in \mathbb{R}$,

$$k_1u_n + k_2u'_n = (k_1a + k_2a') + n(k_1b + k_2b'), \quad k_1a + k_2a', k_1b + k_2b' \in \mathbb{R}.$$

It follows that $k_1(u_n)_{n\in\mathbb{N}} + k_2(u'_n)_{n\in\mathbb{N}} = (k_1u_n + k_2u'_n)_{n\in\mathbb{N}} \in A$. Therefore, A is a subspace of real sequences.

Exercise 3-12. Yes, it is a linear map, since for $c_1, c_2 \in \mathbb{R}$ and $k_1, k_2 \in \mathbb{R}$, we have

$$\begin{aligned} T(k_1c_1 + k_2c_2) = (k_1c_1 + k_2c_2)\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} &= k_1c_1\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} + k_2c_2\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \\ &= k_1T(c_1) + k_2T(c_2). \end{aligned}$$

No, it's not an isomorphism, since $\dim\mathbb{R} = 1 \neq 4 = \dim M_2(\mathbb{R})$.

Exercise 3-14. Yes, it is a linear map, since for any $A, B \in M_2(\mathbb{R})$ and $k_1, k_2 \in \mathbb{R}$, we have

$$\begin{aligned} &T(k_1A + k_2B) \\ =&(k_1A + k_2B)\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}(k_1A + k_2B) \\ =&k_1(A\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}A) + k_2(B\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}B) \\ =&k_1T(A) + k_2T(B). \end{aligned}$$

Exercise 3-24. No, it's not a linear map. Since for $f(t) = t^2 \in P_2$, we have

$$T(f + f) = 4 \times 2t^2 = 8t^2,$$

while

$$T(f) + T(f) = 2 \times t^2 + 2 \times t^2 = 4t^2 \neq T(f + f).$$

Exercise 3-26. Yes, it is. For any $f_1(t), f_2(t) \in P_2$, let

$$f_1(t) = a_1t^2 + b_1t + c_1, \quad f_2(t) = a_2t^2 + b_2t + c_2.$$

Obviously, for $k_1, k_2 \in \mathbb{R}$,

$$\begin{aligned} T(f_1 + f_2) &= -(a_1 + a_2)t^2 - (b_1 + b_2)t - (c_1 + c_2) \\ &= -(a_1t^2 + b_1t + c_1) - (a_2t^2 + b_2t + c_2) = T(f_1) + T(f_2). \end{aligned}$$

Therefore, T is a linear map. Yes, it is an isomorphism, since it's easy to see that $T^{-1} = T$.

Exercise 4-1. It is obvious that $\{1\}$ can be a basis of $\mathbb{R}$, $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ can be a basis of $M_2(\mathbb{R})$ and $\{t^2, t, 1\}$ can be a basis of P_2.

Exercise 4-2. For 3-12, we have

$$[T]^{\{E_{11},E_{12},E_{21},E_{12}\}}_{\{1\}} = \begin{pmatrix} T_1^{E_{11}} \\ T_1^{E_{12}} \\ T_1^{E_{21}} \\ T_1^{E_{22}} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

For 3-14, we have

$$\begin{aligned} [T]^{\{E_{11},E_{12},E_{21},E_{22}\}}_{\{E_{11},E_{12},E_{21},E_{22}\}} &= \begin{pmatrix} T^{E_{11}}_{E_{11}} & T^{E_{11}}_{E_{12}} & T^{E_{11}}_{E_{21}} & T^{E_{11}}_{E_{22}} \\ T^{E_{12}}_{E_{11}} & T^{E_{12}}_{E_{12}} & T^{E_{12}}_{E_{21}} & T^{E_{12}}_{E_{22}} \\ T^{E_{21}}_{E_{11}} & T^{E_{21}}_{E_{12}} & T^{E_{21}}_{E_{21}} & T^{E_{21}}_{E_{22}} \\ T^{E_{22}}_{E_{11}} & T^{E_{22}}_{E_{12}} & T^{E_{22}}_{E_{21}} & T^{E_{22}}_{E_{22}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \end{aligned}$$

For 3-26, we have

$$[T]^{\{t^2,t,1\}}_{\{t^2,t,1\}} = \begin{pmatrix} T^{t^2}_{t^2} & T^{t^2}_{t} & T^{t^2}_{1} \\ T^{t}_{t^2} & T^{t}_{t} & T^{t}_{1} \\ T^{1}_{t^2} & T^{1}_{t} & T^{1}_{1} \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$