Linear Algebra II - Quiz 7
Solution

Compute the determinant of

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 & 6 & 7 \\ 1 & 2 & 3 & 5 & 6 & 7 \\ 1 & 2 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}.$$

First of all, recall that row operations of the form $R_i \leftarrow R_i + \lambda R_j$ for $i \neq j$ do not change the determinant. Thus, we start by doing $R_6 \leftarrow R_6 - R_5$:

$$\det M = \det \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 7 \\ 1 & 2 & 3 & 4 & 6 & 7 \\ 1 & 2 & 3 & 5 & 6 & 7 \\ 1 & 2 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then, we do $R_5 \leftarrow R_5 - R_4$ and so on till $R_2 \leftarrow R_2 - R_1$ and we get at the end:

$$\det M = \det \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 7 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The only pattern which is non-zero in this matrix is the one corresponding to the second diagonal. The number of inversions is $5+4+3+2+1 = 15$ (5 with the lower left corner, 4 with the second cell in the diagonal except the lower left corner, ...). So

$$\det M = (-1)^{15} 1 \times 1 \times 1 \times 1 \times 1 \times 7 = -7.$$