Linear Algebra II - Quiz 5
Solutions

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 10.

Let $\theta, \varphi \in \mathbb{R}$. Prove that the following matrix is orthogonal:

$$\begin{bmatrix} \cos\theta & \sin\theta\cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & \cos\theta\cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}.$$

Let us call $\vec{u}$, $\vec{v}$, $\vec{w}$ the columns of the matrix. We need to check that $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthonormal basis.

- We have $||\vec{u}||^2 = (\cos\theta)^2 + (-\sin\theta)^2 = 1$;
- We have
$$\begin{aligned} ||\vec{v}||^2 &= (\sin\theta\cos\varphi)^2 + (\cos\theta\cos\varphi)^2 + (-\sin\varphi)^2 \\ &= (\sin\theta^2 + \cos\theta^2)(\cos\varphi)^2 + (\sin\varphi)^2 = (\cos\varphi)^2 + (\sin\varphi)^2 = 1; \end{aligned}$$
- We have
$$\begin{aligned} ||\vec{w}||^2 &= (\sin\theta\sin\varphi)^2 + (\cos\theta\sin\varphi)^2 + (\cos\varphi)^2 \\ &= (\sin\theta^2 + \cos\theta^2)(\sin\varphi)^2 + (\cos\varphi)^2 = (\sin\varphi)^2 + (\cos\varphi)^2 = 1; \end{aligned}$$
- We have
$$\vec{u}\cdot\vec{v} = \cos\theta \times \sin\theta\cos\varphi + (-\sin\theta) \times \cos\theta\cos\varphi = (\cos\theta\sin\theta - \sin\theta\cos\theta)\cos\varphi = 0;$$
- We have
$$\vec{u}\cdot\vec{w} = \cos\theta \times \sin\theta\sin\varphi + (-\sin\theta) \times \cos\theta\sin\varphi = (\cos\theta\sin\theta - \sin\theta\cos\theta)\sin\varphi = 0;$$
- We have
$$\begin{aligned} \vec{v}\cdot\vec{w} &= \sin\theta\cos\varphi \times \sin\theta\sin\varphi + \cos\theta\cos\varphi \times \cos\theta\sin\varphi + (-\sin\varphi) \times \cos\varphi \\ &= ((\sin\theta)^2 + (\cos\theta)^2)\cos\varphi\sin\varphi - \sin\varphi\cos\varphi = 0 \end{aligned}$$

so the matrix is orthogonal.