Linear Algebra II - Quiz 4
Solution

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 10.

We consider the following vectors of $\mathbb{R}^4$:

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

1. Justify that $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a basis of the subspace V of equation $x_1+x_2+x_3+x_4 = 0$.

   The map $T : \mathbb{R}^4 \to \mathbb{R}$ defined by $T(\vec{x}) = x_1 + x_2 + x_3 + x_4$ has clearly rank 1 so, by rank-nullity theorem, its kernel V has dimension 3. Moreover, $\{\vec{u}, \vec{v}, \vec{w}\}$ is clearly included in V. If $\lambda\vec{u} + \mu\vec{v} + \nu\vec{w} = 0$, we get $\lambda = 0$, $-\lambda + \mu = 0$, $-\mu + \nu = 0$ and $-\nu = 0$ so clearly $\lambda = \mu = \nu = 0$ so $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent. Finally, it has to be a basis of V.

2. Using the Gram-Schmidt process, compute an orthonormal basis of V.

   The first step of Gram-Schmidt process consist to normalize $\vec{u}$:

$$\vec{u}_1 = \vec{u}/||\vec{u}|| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

   Then, we compute

$$\mathrm{proj}_{\mathrm{Span}(\vec{u}_1)}(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 = \frac{1}{2}(\vec{u} \cdot \vec{v})\vec{u} = \frac{1}{2} \times (-1)\vec{u} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

   so we have

$$\vec{v} - \mathrm{proj}_{\mathrm{Span}(\vec{u}_1)}(\vec{v}) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

   and we take

$$\vec{u}_2 = \frac{\vec{v} - \mathrm{proj}_{\mathrm{Span}(\vec{u}_1)}(\vec{v})}{||\vec{v} - \mathrm{proj}_{\mathrm{Span}(\vec{u}_1)}(\vec{v})||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

In the next step, we need to compute $\text{proj}_{\text{Span}(\vec{u}_1,\vec{u}_2)}(\vec{w})$:

$$\text{proj}_{\text{Span}(\vec{u}_1,\vec{u}_2)}(\vec{w}) = (\vec{u}_1 \cdot \vec{w})\vec{u}_1 + (\vec{u}_2 \cdot \vec{w})\vec{u}_2 = 0\vec{u}_1 - \frac{2}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

so we have

$$\vec{w} - \text{proj}_{\text{Span}(\vec{u}_1,\vec{u}_2)}(\vec{w}) = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

and we take

$$\vec{u}_3 = \frac{\vec{w} - \text{proj}_{\text{Span}(\vec{u}_1,\vec{u}_2)}(\vec{w})}{||\vec{w} - \text{proj}_{\text{Span}(\vec{u}_1,\vec{u}_2)}(\vec{w})||} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

Finally, an orthonormal basis of V is given by

$$\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\}.$$