Linear Algebra II - Quiz 3
Solutions

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 10.

We consider the following vectors of $\mathbb{R}^4$:

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -\sqrt{2} \end{bmatrix}$$

1. Compute $||\vec{u}||$, $||\vec{v}||$, $\vec{u} \cdot \vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

$$||\vec{u}|| = \sqrt{2^2 + 3^2 + 1^2 + 0^2} = \sqrt{14}$$
$$||\vec{v}|| = \sqrt{0^2 + 2^2 + 1^2 + \sqrt{2}^2} = \sqrt{7}$$
$$\vec{u} \cdot \vec{v} = 2 \times 0 + 3 \times 2 + 1 \times 1 + 0 \times (-\sqrt{2}) = 7$$

and, by definition, the angle between $\vec{u}$ and $\vec{v}$ is

$$\arccos \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \times ||\vec{v}||} = \arccos \frac{7}{7\sqrt{2}} = \frac{\pi}{4}.$$

2. Give an orthonormal basis of $V = \operatorname{Span}(\vec{u}, \vec{v})$.

We can start by taking $\vec{u}_1 = \vec{u}/||\vec{u}||$. Then, let us consider the line $U = \operatorname{Span}(\vec{u}_1)$.

$$\operatorname{proj}_U(\vec{v}) = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||^2}\vec{u} = \frac{1}{2}\vec{u}.$$

Then, the vector $\vec{v}^\perp = \vec{v} - \operatorname{proj}_U(\vec{v})$ is orthogonal to U so to $\vec{u}_1$. Moreover, we clearly have $V = \operatorname{Span}(\vec{u}_1, \vec{v}^\perp)$ so we can choose $\vec{u}_2 = \vec{v}^\perp/||\vec{v}^\perp||$ to get an orthonormal basis of V. Let us do the computation:

$$\vec{v}^\perp = \vec{v} - \frac{1}{2}\vec{u} = \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \\ -\sqrt{2} \end{bmatrix}$$

so an orthonormal basis of V is

$$\{\vec{u}_1, \vec{u}_2\} = \left\{ \frac{1}{\sqrt{7}} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{14}} \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2\sqrt{2} \end{bmatrix} \right\}.$$