Linear Algebra II - Quiz 1
Solutions

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 10.

We consider the set V of 2×2 matrices S satisfying

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S = S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

1. Prove that V is a subspace of $M_2(\mathbb{R})$.

   Let us prove that

   - $0 \in V$. Indeed, we have
   $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
   so $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$.
   - for $M, N \in V$, $M + N \in V$. Let $M, N \in V$. We have
   $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (M+N) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} N = M \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + N \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = (M+N) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
   so $M + N \in V$.
   - for $\lambda \in \mathbb{R}$ and $M \in V$, $\lambda M \in V$. Let $\lambda \in \mathbb{R}$ and $M \in V$. We have
   $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\lambda M) = \lambda \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M \right) = \lambda \left(M \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = (\lambda M) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
   so $\lambda M \in V$.

2. Determine a basis of V.

   Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$. We have
   $$M \in V \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \Leftrightarrow \begin{cases} a = a \\ b = 0 \\ 0 = c \\ 0 = 0 \end{cases}$$
   so
   $$V = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in \mathbb{R} \right\}.$$
   As usual, a basis is determined by putting a parameter to 1 while the others are put to 0, for each parameter. So a basis of V is
   $$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$