Linear Algebra II - Quiz 11
Solution

Let $a \in \mathbb{R}$. Find the eigenvectors and eigenvalues of the following matrix, and give the algebraic and geometric multiplicity of each eigenvalue:

$$M = \begin{bmatrix} a & 1 \\ -1 & a+2 \end{bmatrix}.$$

We start by computing the characteristic polynomial:

$$\begin{aligned} f_M(\lambda) &= \det \begin{bmatrix} a-\lambda & 1 \\ -1 & a+2-\lambda \end{bmatrix} = (a-\lambda)(a+2-\lambda)+1 \\ &= \lambda^2 - (2a+2)\lambda + a^2 + 2a + 1 = (\lambda - a - 1)^2. \end{aligned}$$

So M has a unique eigenvalue $a+1$ with algebraic multiplicity 2.

Let us compute the eigenspace E_{a+1}.

$$E_{a-1} = \mathrm{Ker}(M - (a+1)_2) = \mathrm{Ker} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \mathrm{Span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so the eigenvectors of M associated with the eigenvalue $a+1$ are

$$\begin{bmatrix} x \\ x \end{bmatrix}$$

for $x \neq 0$. The geometric multiplicity of $a+1$ is $\dim E_{a+1} = 1$.