Linear Algebra II - Midterm examination

Duration: 90 minutes.

Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1: (8) We consider the following vectors of $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Compute $||\vec{u}||$, $||\vec{v}||$, $\vec{u} \cdot \vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

Problem 2: (14) We consider the following subspace of $\mathbb{R}^4$:

$$V = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right).$$

1. Give an orthonormal basis $\mathscr{B}$ of V.
2. Find a vector $\vec{u} \in \mathbb{R}^4$ such that the set $\mathscr{B}'$ obtained from $\mathscr{B}$ by adding $\vec{u}$ is an orthonormal basis of $\mathbb{R}^4$.
3. Give the matrix $[\mathrm{proj}_V]_{\mathscr{B}'}^{\mathscr{B}'}$.

Problem 3: (8) Which of the following matrices are orthogonal?

$$A = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0 & -0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix} \quad B = \begin{bmatrix} 0.8 & 0 & 0.6 \\ 0 & 1 & 0 \\ 0.6 & 0 & -0.8 \end{bmatrix}.$$

Problem 4: (10) We recall that the set P_2 of polynomials of degree at most 2 form a vector space. We consider three points $M_1 = (x_1, y_1)$, $M_2 = (x_2, y_2)$ and $M_3 = (x_3, y_3)$ of $\mathbb{R}^2$, such that x_1, x_2 and x_3 are distinct. The aim of this exercise is to study elements of P_2 whose graph passes through M_1, M_2 and M_3. All along this exercise, we denote by X the indeterminate (hence the polynomial $p(t) = t^2$ can just be written as X^2).

1. Prove that

$$\mathscr{B} = \{(X - x_1)(X - x_2), (X - x_2)(X - x_3), (X - x_3)(X - x_1)\}$$

   is a basis of P_2.

   hint: you can evaluate a linear combination of these three elements at x_1, x_2 and x_3.

2. We consider the map $T : P_2 \to \mathbb{R}^3$ defined by

$$T(p) = \begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \end{bmatrix}.$$

   Prove that T is a linear map.

3. We consider the standard basis $\mathscr{B}_0 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$. Compute the matrix $[T]_{\mathscr{B}}^{\mathscr{B}_0}$.

4. If the graph of a polynomial $p \in P_2$ passes through M_1, M_2 and M_3, what is the value of $T(p)$?

5. Deduce that there exists $p \in P_2$ such that its graph passes through M_1, M_2 and M_3. Prove that there is in fact a unique choice.

6. Give a formula for p of the previous question.