## Linear Algebra II - Final examination

Duration: 90 minutes.
Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40 . The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1: (8) Compute the determinant of the following matrix:

$$
M=\left[\begin{array}{llllll}
2 & 2 & 2 & 4 & 2 & 2 \\
3 & 3 & 2 & 3 & 3 & 3 \\
5 & 7 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 \\
3 & 4 & 4 & 4 & 4 & 4 \\
8 & 8 & 8 & 8 & 8 & 9
\end{array}\right]
$$

Problem 2: (8) Let

$$
M=\left[\begin{array}{ccc}
a & 3 & b \\
c & 2 & d \\
e & 1 & f
\end{array}\right] \quad \text { and } \quad N=\left[\begin{array}{ccc}
a & 1 & b \\
c & 2 & d \\
e & 3 & f
\end{array}\right] .
$$

We suppose that $x=\operatorname{det} M$ and $y=\operatorname{det} N$. Compute

$$
\operatorname{det}\left(\left[\begin{array}{lll}
a & 1 & b \\
c & 1 & d \\
e & 1 & f
\end{array}\right]\left[\begin{array}{lll}
a & c & e \\
8 & 4 & 0 \\
b & d & f
\end{array}\right]\right)
$$

as a function of $x$ and $y$.
Problem 3: (4) Let $M$ be a $3 \times 3$ matrix admitting and orthonormal eigenbasis $\{\vec{x}, \vec{y}, \vec{z}\}$ with eigenvalues $-1,-1$ and 0 . Describe the transformation of $\mathbb{R}^{3}$ associated to $M$.

Problem 4: (12) Find the eigenvalues and eigenvectors of

$$
M=\left[\begin{array}{ccc}
-5 & 4 & -7 \\
-6 & 5 & -6 \\
1 & -1 & 3
\end{array}\right]
$$

For each eigenvalue, what are the algebraic and geometric multiplicities?
Hint: 2 is an eigenvalue.

Problem 5: (8) Let $M$ be a $4 \times 4$ matrix with real coefficients. We suppose that $f_{M}(\lambda)=\operatorname{det}\left(M-\lambda I_{4}\right)=(a-\lambda)^{4}$ for some $a \in \mathbb{R}$.

In each question, you can use freely results of the previous questions, even in the case you skipped some questions.

1. Justify the existence of a non-zero vector $\vec{v}_{1} \in \mathbb{R}^{4}$ such that $M \vec{v}_{1}=a \vec{v}_{1}$.
2. Explain why there exists a basis $\mathscr{B}=\left\{\vec{v}_{1}, \vec{v}_{2}, \vec{v}_{3}, \vec{v}_{4}\right\}$ of $\mathbb{R}^{4}\left(\vec{v}_{1}\right.$ is an in Question 1$)$.
3. Explain why there are coefficients $m_{i, j}^{\prime} \in \mathbb{R}$ such that

$$
[M]_{\mathscr{B}}^{\mathscr{B}}=\left[\begin{array}{cccc}
a & m_{1,2}^{\prime} & m_{1,3}^{\prime} & m_{1,4}^{\prime} \\
0 & m_{2,2}^{\prime} & m_{2,3}^{\prime} & m_{2,4}^{\prime} \\
0 & m_{3,2}^{\prime} & m_{3,3}^{\prime} & m_{3,4}^{\prime} \\
0 & m_{4,2}^{\prime 2} & m_{4,3}^{\prime} & m_{4,4}^{\prime}
\end{array}\right] .
$$

4. We consider the linear map $T$ from $V=\operatorname{Span}\left\{\vec{v}_{2}, \vec{v}_{3}, \vec{v}_{4}\right\}$ to $W=\operatorname{Span}\left\{\vec{v}_{1}\right\}$ such that

$$
[T]_{\mathscr{B}_{V}}^{\mathscr{B}_{W}}=\left[\begin{array}{lll}
m_{1,2}^{\prime} & m_{1,3}^{\prime} & m_{1,4}^{\prime}
\end{array}\right]
$$

where $\mathscr{B}_{V}=\left\{\vec{v}_{2}, \vec{v}_{3}, \vec{v}_{4}\right\}$ and $\mathscr{B}_{W}=\left\{\vec{v}_{1}\right\}$. Explain why there exists a basis $\mathscr{B}_{V}^{\prime}=$ $\left\{\vec{v}_{2}^{\prime}, \vec{v}_{3}^{\prime}, \vec{v}_{4}^{\prime}\right\}$ of $V$ such that

$$
[T]_{\mathscr{B}_{V}^{\prime}}^{\mathscr{B}_{V}^{W}}=\left[\begin{array}{lll}
\varepsilon_{1} & 0 & 0
\end{array}\right]
$$

where $\varepsilon_{1}=0$ or $\varepsilon_{1}=1$.
5. We consider the basis $\mathscr{B}^{\prime}=\left\{\vec{v}_{1}, \vec{v}_{2}^{\prime}, \vec{v}_{3}^{\prime}, \vec{v}_{4}^{\prime}\right\}$ of $\mathbb{R}^{4}$. Explain why there are coefficients $m_{i, j}^{\prime \prime} \in \mathbb{R}$ such that

$$
[M]_{\mathscr{B}^{\prime}}^{\mathscr{B}^{\prime}}=\left[\begin{array}{cccc}
a & \varepsilon_{1} & 0 & 0 \\
0 & m_{2,2}^{\prime \prime} & m_{2,3}^{\prime \prime} & m_{2,4}^{\prime \prime} \\
0 & m_{3,2}^{\prime \prime} & m_{3,3}^{\prime \prime} & m_{3,4}^{\prime \prime} \\
0 & m_{4,2}^{\prime \prime} & m_{4,3}^{\prime \prime} & m_{4,4}^{\prime \prime}
\end{array}\right] .
$$

6. Let

$$
M^{\prime}=\left[\begin{array}{lll}
m_{2,2}^{\prime \prime} & m_{2,3}^{\prime \prime} & m_{2,4}^{\prime \prime} \\
m_{3,2}^{\prime \prime} & m_{3,3}^{\prime \prime} & m_{3,4}^{\prime \prime} \\
m_{4,2}^{\prime \prime} & m_{4,3}^{\prime \prime} & m_{4,4}^{\prime \prime}
\end{array}\right] .
$$

Prove that $f_{M^{\prime}}(\lambda)=(a-\lambda)^{3}$.
7. Prove that there exists a basis $\tilde{\mathscr{B}}$ of $\mathbb{R}^{4}$ such that

$$
[M]_{\mathscr{B}}^{\tilde{\mathscr{B}}}=\left[\begin{array}{cccc}
a & \varepsilon_{1} & 0 & 0 \\
0 & a & \varepsilon_{2} & 0 \\
0 & 0 & a & \varepsilon_{3} \\
0 & 0 & 0 & a
\end{array}\right]
$$

where each $\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}$ is either 0 or 1 .
8. Let $M$ be a $n \times n$ matrix with real coefficients. We suppose that $f_{M}(\lambda)=\operatorname{det}(M-$ $\left.\lambda I_{n}\right)=(a-\lambda)^{n}$ for some $a \in \mathbb{R}$. Give a result generalizing the one of Question 7 for $n=4$. You do not need to prove it.

