## Linear Algebra II - Worksheet 13

The aim of this exercise is to study linear systems of the following form. We suppose that we have a certain number of products $A_{1}, A_{2}, \ldots, A_{n}$ such that at each second a part of each product is transformed into the same quantity of some other of the products. Denote by $v_{i}^{t}$ is the quantity of $A_{i}$ at time $t$.

1. Explain why $\vec{v}^{t+1}=M \vec{v}^{t}$ where $M$ is a matrix with non-negative entries such that the values in any given column add up to 1 .
2. Prove that $M$ and $M^{T}$ have the same (complex) eigenvalues with the same algebraic and geometric multiplicities.
3. Deduce that all eigenvalues of $M$ have modulus at most 1 .
4. Deduce also that $M$ has eigenvalue 1 .
5. Suppose now that all entries of $M$ are positive. Prove that the geometric multiplicity of 1 for $M$ is 1 .
6. Prove that, there exists a basis $\mathscr{B}$ (over complex numbers) such that $[M]_{\mathscr{B}}^{\mathscr{B}}$ is uppertriangular.
7. Prove that there exists a unique stable equilibrium for the system.
