## Linear Algebra II - Worksheet 13

The aim of this exercise is to study linear systems of the following form. We suppose that we have a certain number of products  $A_1, A_2, \ldots, A_n$  such that at each second a part of each product is transformed into the same quantity of some other of the products. Denote by  $v_i^t$  is the quantity of  $A_i$  at time t.

- 1. Explain why  $\vec{v}^{t+1} = M\vec{v}^t$  where M is a matrix with non-negative entries such that the values in any given column add up to 1.
- 2. Prove that M and  $M^T$  have the same (complex) eigenvalues with the same algebraic and geometric multiplicities.
- 3. Deduce that all eigenvalues of M have modulus at most 1.
- 4. Deduce also that M has eigenvalue 1.
- 5. Suppose now that all entries of M are positive. Prove that the geometric multiplicity of 1 for M is 1.
- 6. Prove that, there exists a basis  $\mathscr{B}$  (over complex numbers) such that  $[M]_{\mathscr{B}}^{\mathscr{B}}$  is upper-triangular.
- 7. Prove that there exists a unique stable equilibrium for the system.