Linear Algebra II - Homework 4

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 20. The number of points for each exercise is specified between parenthesis. To hand in June 21 at the beginning of the tutorial.

Exercise 1 : (5) We suppose that $M \in \mathrm{M}_{3,3}(\mathbb{R})$ admits three eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ such that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis. We suppose that the associated eigenvalues are λ_1, λ_2 and λ_3. Give a geometrical interpretation of M in each following cases:

1. $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$;
2. $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$;
3. $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 1$;
4. $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 1$.

Exercise 2 : (5) Find the eigenvectors and the associated eigenvalues of

$$M = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}$$

Exercise 3 : (10) We consider two real numbers $0 < a < 1$ and $0 < b < 1$. We have a solution containing m_0 molecules of a product A and n_0 molecules of a product B. Each second, the proportion a of molecules of A are transformed into molecules of B and the proportion b of molecules of B are transformed into molecules of A.

We denote by m_t the number of molecules of A after t seconds and n_t the number of molecules of B after t seconds.

1. Express m_{t+1} and n_{t+1} as functions of m_t and n_t.
2. Explain why $m_{t+1} + n_{t+1} = m_t + n_t$ and check that it is the case in your answer to the previous question.
3. Find a 2×2 matrix M such that
$$\begin{bmatrix} m_{t+1} \\ n_{t+1} \end{bmatrix} = M \begin{bmatrix} m_t \\ n_t \end{bmatrix}.$$
4. Find the eigenvectors and eigenvalues of M.
5. Is M diagonalizable? If it is the case, diagonalize M.
6. If we wait long enough, which proportion of molecules of A will be contained in our solution?