Modules and homological algebra, autumn term 2019
Solution to Homework assignment 3:2

Show that any submodule of a f.g. projective Λ-module is projective:

a) If $U \subset P_i$ is a submodule then $U = P_j$ for some $j \geq i$.
In particular, every submodule of an indecomposable projective Λ-module is projective.

[Let $j = \min\{m \geq i \mid E_{mm}U \neq 0\}$ (where $E_{mm} = \begin{array}{cc} & m\text{:th column} \\ & \\ \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} & m\text{:th row} \end{array}$).

Then clearly $U \subset P_j$. On the other hand, let $u \in U$ be such that $E_{mm}u \neq 0$.
Then $u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_j \\ \vdots \\ u_n \end{pmatrix} \in k^n$, with $u_j \neq 0$.

Thus, $U \ni \frac{1}{u_j} E_{mm} u = e_j \Rightarrow P_j = \langle e_j \rangle \subset U$. Hence, $P_j = U$]

b) Let $M \in \operatorname{proj}\Lambda$, $M = M_1 \oplus \cdots \oplus M_r$ where $M_1, \ldots, M_r$ are indecomposable, and let $\pi_i : M \to M_i$ be the projection onto M_i.
Assume that $U \subset M$ is a submodule, and set $U_i = \pi_i(U) \subset M_i$.
Then, by (a), U_i is projective, and the restricted map $\pi_i' : U \twoheadrightarrow U_i$ is epi.
Thus, there exists a submodule $X \subset U$ s.th. $U = X \oplus (\ker \pi_i')$.
By the first isomorphism theorem, $X \simeq U/(\ker \pi_i') \simeq U_i$, so X_i is projective.
By induction on $\dim U$, $\ker \pi_i'$ is also projective.

Hence U is projective, Q.E.D.