Name: Solution

Linear Algebra II - Quiz 4

Solutions must be written in sufficient detail to make each step clear to the reader. Clarity of presentation will be rewarded. Please continue on a separate piece of paper if the space below is not enough.

The maximal number of points awarded is 10.

Find a closed formula for $x(t)$ (possibly involving some undetermined constants) in the dynamical system

$$x'(t) = Ax(t)\,, \qquad \text{where} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

- $f_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) \Rightarrow$ Eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 1$.

- $A - I = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_1(A) = \ker(A-I) = \operatorname{span}\{\underbrace{e_1}_{v_1}\}$

- $A + I = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathcal{E}_{-1}(A) = \operatorname{span}\left\{\underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{v_2}\right\}$ set $\underline{v} = (v_1, v_2)$.

- Now $A = SDS^{-1}$ where $S = S_{\underline{v}}^{\mathcal{E}} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- Setting $u(t) = S^{-1}x(t)$ gives $u'(t) = S^{-1}x'(t) = S^{-1}Ax(t) = S^{-1}ASu(t) = Du(t)$

$$\begin{cases} u_1'(t) = u(t) \\ u_2'(t) = -u(t) \end{cases} \Rightarrow \begin{cases} u_1(t) = c_1 e^{t} \\ u_2(t) = c_2 e^{-t} \end{cases} \text{ for some } c_1, c_2 \in \mathbb{R}.$$

- $x(t) = Su(t) = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{pmatrix} = \underline{\underline{\begin{pmatrix} c_1 e^{t} + c_2 e^{-t} \\ -2c_2 e^{-t} \end{pmatrix}}}$