Linear Algebra II, spring term 2019.
Solutions to Homework 8

1) Assume that $x \in V$ is such that $F(x) = w$.
Then $F(x) = w = F(v) \Rightarrow F(x-v) = F(x) - F(v) = w - w = 0$
$\Rightarrow x - v \in \ker F$.
Setting $u = x - v$, we get $\underline{x = u + v, \text{ with } u \in \ker F}$.

Assume that $x = u + v$, $u \in \ker F$.
Then $F(x) = F(u+v) = F(u) + F(v) = 0 + F(v) = \underline{w}$.

2.a) $\frac{d}{dt}\left(e^{-at}f(t)\right) \overset{\substack{\text{[product}\\ \text{rule]}}}{=} \frac{d}{dt}(e^{-at})f(t) + e^{-at}\frac{d}{dt}(f(t))$

$= -ae^{-at}f(t) + e^{-at}f'(t) \overset{[f' = g + af]}{=} -ae^{-at}f(t) + e^{-at}(g(t) + af(t))$

$-ae^{-at}f(t) + e^{-at}g(t) + ae^{-at}f(t) = \underline{\underline{e^{-at}g(t)}}$.

b) If $f$ satisfies $f' - af = g$ then, by (a), $e^{-at}f(t)$ is an antiderivative of $e^{-at}g(t)$, and hence
$\underline{f(t) = e^{at}\int e^{-at}g(t)\,dt}$.

Conversely, assume that $f(t) = e^{at}h(t)$, where $h'(t) = e^{-at}g(t)$.
Then
$f'(t) = \frac{d}{dt}(e^{at})h(t) + e^{at}h'(t) = ae^{at}h(t) + e^{at}e^{-at}g(t)$
$= af(t) + g(t)$
$\Longrightarrow \underline{f' - af = g}$.

Thus, the general solution of the equation is $\underline{\underline{f(t) = e^{at}\int e^{-at}g(t)\,dt}}$.

3) Let $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{R}$ be such that $\sum_{i=0}^{m} \lambda_i t^i e^{\lambda t} = 0$ for all $t \in \mathbb{R}$.

Thus, for all $t \in \mathbb{R}$:

$$0 = \left(\sum_{i=0}^{m} \lambda_i t^i e^{\lambda t}\right) \cdot e^{-\lambda t} = \sum_{i=0}^{m} \lambda_i t^i e^{\lambda t} e^{-\lambda t} = \sum_{i=0}^{m} \lambda_i t^i$$

We have seen earlier in the course that the elements $1, t, \dots, t^m$ are linearly independent, so the above equation implies that $\lambda_0 = \cdots = \lambda_m = 0$.

Hence, $e^{\lambda t}, te^{\lambda t}, \dots, t^m e^{\lambda t}$ are linearly independent.